

# Semifield Planes of Order 81

Ulrich Dempwolff

## Abstract

In [17] Kantor states "...it is surprising that there has not yet been an enumeration of all semifields of order at most 256 ...". In this note we present such an enumeration for the order 81.

*Mathematics Subject Classification (2000):* 12K10, 51A40

*Key words:* finite semifield, computer enumeration

## 1 Introduction.

We describe the enumeration of semifield planes of order 81 and the computation of the automorphism groups. The methods of the classifications follow our pattern of the enumeration of translation planes of small order [11], [10], and [6]. However for semifield planes such a task is much easier.

In section 2 we introduce some notation, outline the computer aided enumeration and the computation of the automorphism groups.

In section 3 we describe the results and in section 4 we identify known semifields. Also we discuss the embedding of a semifield of order 81 in an infinite series. In section 5 we tabulate the generators of the automorphism groups of some new semifield planes.

More comprehensive files in GAP-format can be obtained from my homepage:

[www.mathematik.uni-kl.de/~dempw/dempw\\_81.semi.html](http://www.mathematik.uni-kl.de/~dempw/dempw_81.semi.html)

## 2 Notations and computations.

**(2.1) Semifields.** First we recall some notations about translation planes [14], [19], [21]. Let  $\mathbf{P} = \mathbf{P}(W, \mathcal{S})$  be a translation plane of order  $p^n$ ,  $p$  a prime. Here  $W$  denotes a  $2n$ -dimensional vectorspace over  $K = \text{GF}(p)$  and  $\mathcal{S}$  is a *spread* of  $W$ , i.e. a collection  $\mathcal{S} = \{V_\infty, V_0, \dots, V_{p^n-1}\}$  of  $n$ -dimensional subspaces of  $W$ , the

components of  $\mathcal{S}$ , such that  $W = \bigcup_i V_i$ . Introducing coordinates one can associate to  $\mathcal{S}$  a set  $\Sigma \subseteq G \cup 0$ ,  $G = \text{GL}(n, p)$ ,  $0 \in \Sigma$ , a *spread set*, such that  $\det(x - y) \neq 0$  for  $x, y \in \Sigma$ ,  $x \neq y$ . We call  $\Sigma$  a *semifield spread set*, short SSS, if  $\Sigma$  is a  $K$ -subspace of  $K^{n \times n}$ . The quasifield  $Q$  associated with  $\mathcal{S}$  ( $\Sigma$ ) satisfies both distributive laws iff  $\Sigma$  is a SSS. In this case  $Q$  is a *pre-semifield*. A pre-semifield is called a *semifield* if  $Q$  has a neutral element with respect to the multiplication or equivalently if  $\mathbf{1} \in \Sigma$ . If  $\Sigma$  is a SSS representing a pre-semifield and  $0 \neq x \in \Sigma$  then  $\Sigma x^{-1}$  represents a semifield and both SSS belong to the same translation plane.

Given a SSS  $\Sigma$  one can define an associated pre-semifield by choosing an arbitrary  $K$ -isomorphism  $\tau : Q = K^n \rightarrow \Sigma$  and by defining the multiplication on  $Q$  by

$$x \circ y = xy^\tau. \quad (*)$$

Conversely, if a pre-semifield  $(Q, +, \circ)$  is given one can use  $(*)$  to define a SSS  $\Sigma$ .

Two pre-semifields  $(Q, +, \circ)$  and  $(Q', +, *)$  are *isotopic* if there exist a triple of isomorphisms  $\alpha, \beta, \gamma : Q \rightarrow Q'$ , called an *isotopism*, such that

$$x\alpha * y\beta = (x \circ y)\gamma$$

for all  $x, y \in Q$ . Two semifield planes are isomorphic iff any two representative pre-semifields are isotopic. If the pre-semifields are associated with the SSS's  $\Sigma$  and  $\Sigma'$  the isotopism  $(\alpha, \beta, \gamma)$  corresponds to a pair  $x, y \in G$  with  $\Sigma' = x^{-1}\Sigma y$ . We call two subsets  $\Delta$  and  $\Delta'$  of  $G \cup 0$  *equivalent* if there exist  $x, y \in G$  with  $\Delta' = x^{-1}\Delta y$ . In particular  $\Sigma$  and  $\Sigma'$  are equivalent iff they represent isotopic pre-semifields. The isotopisms from  $Q$  onto  $Q$  are the *autopisms*. They form the *autopism group*. If an autopism  $(\alpha, \beta, \gamma)$  represented by the pair  $x, y \in G$  then it induces the action  $v \mapsto vx$  on  $V_0$ , the action  $v \mapsto vy$  on  $V_\infty$ , and the action  $t \mapsto x^{-1}ty$  on  $\Sigma$ .

Let  $\Sigma$  and  $Q$  be as before. The set  $\Sigma^d$  of the transposed matrices of  $\Sigma$  is a SSS too and it defines the *dual* pre-semifield  $Q^d$ . Replacing the multiplication  $(x, y) \mapsto x \circ y$  in  $Q$  by the multiplication  $(x, y) \mapsto x * y = y \circ x$  defines the *opposite* pre-semifield  $Q^o$ . We follow the notations of [17]. But note that also the notion "dual" instead of "opposite" and "transposed" instead of "dual" is used [16].

Knuth shows in [20] that the operations  $Q \mapsto Q^d$  and  $Q \mapsto Q^o$  define an action of the symmetric group  $\text{Sym}(3)$  on the isotopism classes of pre-semifields. We call the orbit the *Knuth class* of  $Q$  (it is called a cubical array by Knuth). A Knuth class consists therefore of 6, 3, 2, or 1 members. It is clear from the definition that all members of a Knuth class have the same autopism group but the action on the spaces  $V_\infty, V_0, \Sigma$  is usually different. A more detailed discussion can be found in [3] and [18].

**(2.2) Enumeration.** In order to classify semifield planes of order 81 it is enough to look for SSS with  $\mathbf{1} \in \Sigma$  and to determine the equivalence classes. A SSS  $\Sigma \subseteq K^{n \times n}$  is uniquely determined by its standard basis  $B_\Sigma = \{s_1 = \mathbf{1}, s_2, \dots, s_n\}$  where the first row of  $s_i$  is the  $i$ -th standard basis vector  $e_i$  of  $K^n$ . In our case  $n = 4$  and  $p = 3$ . Let  $G_1$  be the stabilizer of  $e_1$  in  $G$  and set  $G_i = G_1 x_i$  where  $x_i \in G$  maps  $e_1$  onto  $e_i$ . For a standard basis  $B_\Sigma$  we have  $s_i \in G_i$  for  $1 \leq i \leq 4$ . To determine equivalence classes of SSS is similar to the classification of translation spreads [6], [10], [11], or [22] but the task for SSS's is much easier:

Suppose we have obtained the list  $\mathcal{L}_k = \{X_1, \dots, X_N\}$  of nonequivalent partial standard bases of size  $k$ , i.e.  $\mathbf{1} \in X_i$ ,  $|X_i| = k$ , and  $\langle X_i \rangle_K$  is a subspace contained in  $G \cup 0$  and  $\langle X_i \rangle \not\sim \langle X_j \rangle$  for  $i \neq j$ . Then we compute the list of all extensions  $\mathcal{L}_{k+1}^* = \{Y_1, \dots, Y_M\}$  where  $Y_i$  has the form  $X_{i'} \cup \{y\}$ ,  $y \in G_{k+1}$ , and  $i' \in \{1, \dots, N\}$ . Then we use equivalence and reduce  $\mathcal{L}_{k+1}^*$  to the list  $\mathcal{L}_{k+1}$  of representatives. Starting with  $\mathcal{L}_1 = \{\{\mathbf{1}\}\}$  we are done after three steps.

Equivalence testing of (partial) SSS is the only critical part in these calculations. To minimize the cost of testing an efficient invariant is necessary. Let  $X$  and  $X'$  be two (partial) standard bases and  $\Sigma = \langle X \rangle$ ,  $\Sigma' = \langle X' \rangle$ . Let  $\min(\Sigma)$  be the multiset of minimal polynomials of the elements of  $\Sigma$ . Then  $\Sigma s^{-1}$  can be conjugate to  $\Sigma'$  only if  $\min(\Sigma s^{-1}) = \min(\Sigma')$ . Moreover the requirement  $\min(B_{\Sigma s^{-1}}) = \min(B_{g^{-1}\Sigma s^{-1}g})$  further restricts the choice of  $g$ . With this invariant we obtain a feasible variation of the equivalence test which is described in detail in [6]. The computations of the autopism group is closely related. Again details are described in [6].

Our programs are written in the GAP code [12] which allowed us to use the implemented group and matrix algorithms. The calculations needed a few days on a PC.

**(2.3) Suitable bases.** Let  $\mathcal{S}$  be the spread of a semifield plane. The form of an associated SSS  $\Sigma$  depends on the choice of bases of  $V_0$  and  $V_\infty$ . A suitable selection can greatly improve the usefulness of the SSS. For this purpose we make use of the autopism group  $A$ . This group fixes the decomposition  $W = V_0 \oplus V_\infty$ . We think of  $a \in A$  as a block diagonal matrix  $a = \text{diag}(x, y)$  with  $x = a|_{V_0}$  and  $y = a|_{V_\infty}$ . In our case  $A$  is always a solvable 3'-group and acts therefore completely reducible on both modules. We choose an abelian normal subgroup  $B$  of maximal order. Usually  $V_0$  and  $V_\infty$  decompose into two irreducible  $KB$ -spaces of dimension 2 and  $A$  leaves invariant such a decomposition. Adapting bases to such decompositions lead in most cases to the representations of the semifields displayed in the next section.

### 3 Twelve Knuth classes and 27 semifields.

In this section we present 12 Knuth classes which lead to 27 semifield planes of order 81. We describe one representative of a Knuth class in detail. The others can be found with the operations explained in section 2. First we describe the members of the Knuth class. For instance  $[1, d, do]$  means that the class has 3 members  $\Sigma, \Sigma^d, \Sigma^{do}$  where  $\Sigma$  represents the class.  $OrNuc$  gives the orders of the seminuclei of the representative in the order left nucleus (kernel), middle nucleus, right nucleus. Our conventions agree with the notation of [21], p. 24: the left nucleus is identified with the kernel (in contrast to the conventions of [14] where the right nucleus represents the kernel). Next we give data of the autopism group  $A$ : the order, group theoretic properties, orbits on the autopism triangle (different from the vertices of the triangle). We also give a description by a pre-semifield multiplication in the form  $* : \text{GF}(9)^2 \times \text{GF}(9)^2 \rightarrow \text{GF}(9)^2$  with the exception of the first two cases where we could not find a useful multiplication. Then we present a basis  $B_\Sigma$  for the the SSS  $\Sigma = \{s_0, \dots, s_{80}\}$ , i.e. the spread on  $W = \text{GF}(3)^8$  has the form  $\mathcal{S} = \{V_\infty, V(s_0), \dots, V(s_{80})\}$  with  $V_\infty = \{(0, x) \mid x \in \text{GF}(3)^4\}$  and  $V(s_i) = \{(x, xs_i) \mid x \in \text{GF}(3)^4\}$ . The generators of the autopism group are block diagonal matrices  $\text{diag}(A, B)$  with  $A, B \in \text{GL}(4, 3)$ . For some new semifield we display such generators in chapter 5. In the sequel  $z$  denotes an element in  $\text{GF}(9)$  with minimal polynomial  $X^2 - X - 1$ .

**(I)**  $[1, OrNuc = [3, 3, 3], |A| = 16, A$  is abelian of exponent 4,  
Orbit lengths on  $L_\infty$ : [ 4, 4, 4, 4, 8, 8, 8, 8, 8, 8, 8, 8 ]  
Orbit lengths on  $V_\infty$ : [ 4, 4, 4, 4, 8, 8, 8, 8, 8, 8, 8, 8 ]  
Orbit lengths on  $V(0)$ : [ 4, 4, 4, 4, 8, 8, 8, 8, 8, 8, 8, 8 ]

$$\text{Basis: } \mathbf{1}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 2 & 2 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix},$$

**(II)**  $[1, d, o, do, dod, dodo], OrNuc = [3, 3, 3], |A| = 32, A$  is nilpotent of class 3,  
Orbit lengths on  $L_\infty$ : [ 16, 16, 16, 16, 16 ]  
Orbit lengths on  $V_\infty$ : [ 4, 4, 4, 4, 8, 8, 8, 8, 16, 16 ]  
Orbit lengths on  $V(0)$ : [ 8, 8, 8, 8, 16, 16, 16 ]

$$\text{Basis: } \mathbf{1}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 2 \\ 1 & 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix},$$

**(III)**  $[1, d, o], OrNuc = [3, 3, 3], |A| = 32, A$  is abelian of exponent 16,  
Orbit lengths on  $L_\infty$ : [ 16, 16, 16, 16, 16 ]  
Orbit lengths on  $V_\infty$ : [ 2, 2, 4, 8, 16, 16, 16, 16 ]

Orbit lengths on  $V(0)$ : [ 16, 16, 16, 16, 16 ]

Multiplication:  $(x, y) * (a, b) = (xa + z^6yb^3 + z^3y^3b^3, xb + ya^3 + yb^3 - y^3b^3)$

Basis:  $\mathbf{1}$ ,  $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 2 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$ ,

**(IV)** [1],  $OrNuc = [3, 3, 3]$ ,  $|A| = 64$ ,  $A$  is abelian of exponent 8,

Orbit lengths on  $L_\infty$ : [ 8, 8, 32, 32 ]

Orbit lengths on  $V_\infty$ : [ 8, 8, 32, 32 ]

Orbit lengths on  $V(0)$ : [ 8, 8, 32, 32 ]

Multiplication:  $(x, y) * (a, b) = (xa + z^3yb^3, xb + ya^3 + z^6y^3b^3)$

Basis:  $\mathbf{1}$ ,  $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 2 & 2 \\ 2 & 0 & 0 & 1 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 2 & 2 & 0 & 1 \\ 2 & 1 & 2 & 0 \end{pmatrix}$ ,

**(V)** [1,  $d$ ,  $o$ ],  $OrNuc = [3, 3, 3]$ ,  $|A| = 64$ ,  $A$  is nilpotent of class 3,

Orbit lengths on  $L_\infty$ : [ 16, 16, 16, 16, 16 ]

Orbit lengths on  $V_\infty$ : [ 8, 8, 32, 32 ]

Orbit lengths on  $V(0)$ : [ 16, 16, 16, 16, 16 ]

Multiplication:  $(x, y) * (a, b) = (xa + z^2yb^3 + z^5y^3a^3 + zy^3a, xb + ya^3 + z^3y^3b^3 + zy^3b)$

Basis:  $\mathbf{1}$ ,  $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 2 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{pmatrix}$ ,

**(VI)** [1],  $OrNuc = [3, 3, 3]$ ,  $|A| = 128$ ,  $A$  is nilpotent of class 3,

Orbit lengths on  $L_\infty$ : [ 8, 8, 32, 32 ]

Orbit lengths on  $V_\infty$ : [ 8, 8, 32, 32 ]

Orbit lengths on  $V(0)$ : [ 8, 8, 32, 32 ]

Multiplication:  $(x, y) * (a, b) = (xa + yb^3, xb + ya^3 + z^6y^3b^3)$

Basis:  $\mathbf{1}$ ,  $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 2 & 0 \end{pmatrix}$ ,

**(VII)** [1,  $o$ ,  $dod$ ],  $OrNuc = [9, 3, 3]$ ,  $|A| = 256$ ,  $A$  is nilpotent of class 4,

Orbit lengths on  $L_\infty$ : [ 4, 4, 8, 16, 16, 32 ]

Orbit lengths on  $V_\infty$ : [ 16, 64 ]

Orbit lengths on  $V(0)$ : [ 16, 64 ]

Multiplication:  $(x, y) * (a, b) = (xa + z^6yb^3, xb + z^6ya^3 + z^7ya)$

Basis:  $\mathbf{1}$ ,  $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$ ,

**(VIII)** [1, *o, dod*],  $OrNuc = [9, 3, 3]$ ,  $|A| = 512$ ,  $A$  is nilpotent of class 4,

Orbit lengths on  $L_\infty$ : [ 8, 8, 32, 32 ]

Orbit lengths on  $V_\infty$ : [ 16, 64 ]

Orbit lengths on  $V(0)$ : [ 16, 64 ]

Multiplication:  $(x, y) * (a, b) = (xa + z^7yb, xb + ya^3)$

Basis:  $\mathbf{1}$ ,  $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ ,

**(IX)** [1, *o, dod*],  $OrNuc = [9, 3, 3]$ ,  $|A| = 640$ ,  $A$  is solvable but not nilpotent,

Orbit lengths on  $L_\infty$ : [ 40, 40 ]

Orbit lengths on  $V_\infty$ : [ 80 ]

Orbit lengths on  $V(0)$ : [ 80 ]

Multiplication:  $(x, y) * (a, b) = (xa + ya^3 - ya + z^3yb, xb + z^6ya^3 + z^7ya + zyb^3 + yb)$

Basis:  $\mathbf{1}$ ,  $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 0 & 1 & 2 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 2 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix}$ ,

**(X)** [1],  $OrNuc = [9, 9, 9]$ ,  $|A| = 1024$ ,  $A$  is nilpotent of class 4,

Orbit lengths on  $L_\infty$ : [ 16, 64 ]

Orbit lengths on  $V_\infty$ : [ 16, 64 ]

Orbit lengths on  $V(0)$ : [ 16, 64 ]

Multiplication:  $(x, y) * (a, b) = (xa + z^5yb^3, xb + ya^3)$

Basis:  $\mathbf{1}$ ,  $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix}$ ,

(**XI**) [1],  $OrNuc = [9, 9, 9]$ ,  $|A| = 2048$ ,  $A$  is nilpotent of class 4,  
 Orbit lengths on  $L_\infty$ : [ 16, 64 ]  
 Orbit lengths on  $V_\infty$ : [ 16, 64 ]  
 Orbit lengths on  $V(0)$ : [ 16, 64 ]

Multiplication:  $(x, y) * (a, b) = (xa + z^6yb^3, xb + ya^3)$

Basis:  $\mathbf{1}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix},$

(**XII**) [1],  $OrNuc = [81, 81, 81]$ ,  $|A| = 25600$ ,  $A$  is solvable but not nilpotent,  
 Orbit lengths on  $L_\infty$ : [ 80 ]  
 Orbit lengths on  $V_\infty$ : [ 80 ]  
 Orbit lengths on  $V(0)$ : [ 80 ]

Multiplication:  $(x, y) * (a, b) = (xa + z^3yb, xb + ya + z^6y3b)$

Basis:  $\mathbf{1}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 2 & 2 \\ 2 & 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 2 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix},$

**Theorem.** *Every semifield plane of order 81 is isomorphic to one of the semifield planes  $\mathbf{P}(\Sigma)$  where  $\Sigma$  belongs to one of the Knuth classes I–XII.*

**(3.1) Multiplication rules and the Oyama representation.** We describe how in the classes III–XII the multiplication rules are obtained from the spread sets: Let  $K = \text{GF}(q)$  and  $F = \text{GF}(q^2)$ . Following the method of Oyama [24] we identify vectors in  $K^2$  with vectors  $[b] := (b, b^q)$ ,  $b \in F$ , and matrices in  $K^{2 \times 2}$  with  $[a_0, a_1]^t := \begin{pmatrix} a_0 & a_1^q \\ a_1 & a_0^q \end{pmatrix}$ ,  $a_0, a_1 \in F$ .

Turning to the case  $q = 3$  and inspecting the Knuth classes III–XII one observes that a typical SSS is represented by  $2 \times 2$ -block matrices of the form

$$\begin{pmatrix} A & B \\ f_1(A) + g_1(B) & f_2(A) + g_2(B) \end{pmatrix}, \quad A, B \in L.$$

Here  $L \simeq \text{GF}(9)$  is the field generated by  $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$  and the functions  $f_i, g_i$  are additive mappings from  $L$  into  $K^{2 \times 2}$ . Using the Oyama representation for blocks

one obtains matrices of the form

$$\begin{pmatrix} [a, 0]^t & [b, 0]^t \\ \widehat{f}_1(a) + \widehat{g}_1(b) & \widehat{f}_2(a) + \widehat{g}_2(b) \end{pmatrix}, \quad a, b \in F.$$

The additive functions  $\widehat{f}_i, \widehat{g}_i$  have the form  $x \mapsto [c_{00}x + c_{01}x^q, c_{10}x + c_{11}x^q]^t$ ,  $c_{ij} \in F$ . Representing vectors of  $K^4$  as  $(x, y) := ([x], [y])$ ,  $x, y \in F$ , one obtains the multiplication rules displayed above.

**Remarks.** (a) A spread  $\mathcal{S}$  on  $W$  is *symplectic* if there exists a nondegenerate symplectic form on  $W$  such that all subspaces of the spread are isotropic. In this case there exist a spread set  $\Sigma$  such that  $\mathcal{S} = \mathcal{S}_\Sigma$  and all matrices in  $\Sigma$  are symmetric. Assume  $\mathbf{1} \in \Sigma$ . It is easy to see that  $\Sigma$  is equivalent to a symplectic spread set iff there exists a symmetric matrix  $G$  with  $T^t = G^{-1}TG$  for all  $T \in \Sigma$ . With this criterion one can show that only the SSS in class XII and  $\Sigma^{dod}$  of class VIII are symplectic SSS. Symplectic and commutative semifields are closely related (see [18]).

(b) The invariant fingerprint (see [6], [10], [23]) is not a complete invariant for the semifield planes of order 81. It even does not distinguish the Knuth classes. However Marc Röder has shown the the number of Fano planes incident with an affine point provides a complete invariant.

## 4 Identifications and generalizations.

The identification of the semifield planes of order 81 was done via computer calculation. In this way we obtain an extra control on the computer enumeration. We produced all planes  $\mathbf{P}(Q)$ ,  $Q$  a semifield of order 81, in the following cases:

DK\*  $Q$  belongs to the 5 Dickson-Knuth types. See [20], (7.16), (7.17), [8], p.128, and [17], p. 106. The numbering follows [17].

HK  $Q$  is a Hughes-Kleinfeld semifield [13]. This class is covered by class [DK1] (see [17]).

JJ  $Q$  is obtained by the Jha-Johnson construction [15].

TF  $Q$  is a generalized twisted field [1], [2], [5].

CG  $Q$  is of Cohen-Ganley type [7].

TP  $Q$  is of Thas-Payne type [25].

BL  $Q$  is the sporadic Boerner-Lantz semifield of order 81 [4], [8].

GF  $Q = \text{GF}(81)$ .

Then we used our equivalence test to identify these semifield planes with the members in the list obtained by the enumeration. The result is displayed in the following table. The first column is the Knuth class, the other columns stand for the representatives of the Knuth class. "no" means that this case has no identification with one of the semifield planes described above.

	1	d	o	do	dod	dodo
I	no	-	-	-	-	-
II	no	no	no	no	no	no
III	no	no	no	-	-	-
IV	DK2	-	-	-	-	-
V	no	no	no	-	-	-
VI	DK2	-	-	-	-	-
VII	BL	-	no	-	no	-
VIII	DK1, TP	-	DK1	-	DK1, CG	-
IX	TF	-	TF	-	TF	-
X	DK1 - DK5, JJ, HK	-	-	-	-	-
XI	DK1 - DK5, JJ, HK	-	-	-	-	-
XII	DK1, GF	-	-	-	-	-

Next we like to show that for every odd prime power  $q$  we have semifields of order  $q^4$  belonging to the Knuth classes III – XII. By the above identification this is obvious with the exception of the cases III, V, and VII. Note that the *infinite* series of Boerner-Lantz semifields [4] is only defined for odd characteristic  $\neq 3$ . Moreover we found no reference for the size of the Knuth class of a semifield plane of Boerner-Lantz type. For the remainder of this section  $K \simeq \text{GF}(q)$  is finite field of odd characteristic,  $F = K[\alpha] \simeq \text{GF}(q^2)$  where  $\alpha^2 = t \in F - K^2$ . Also  $tr : F \rightarrow K$  and  $n : F \rightarrow K$  denote the trace and norm maps and  $x \mapsto \bar{x}$  denotes Galois conjugation in  $F$ .

**(4.1) On semifields of Boerner-Lantz type.** The opposite Boerner-Lantz semifield of order 81 (i.e.  $q = 3$ ) has a SSS in the Oyama representation of the form:

$$s(x, y) = \begin{pmatrix} [x, 0]^t & [y, 0]^t \\ [a\bar{y}, b\bar{y}]^t & [\bar{x}, 0]^t \end{pmatrix}, \quad x, y \in F,$$

with  $a = -1$ ,  $b = z^5$ . We replace  $q = 3$  by an arbitrary prime power and choose  $0 \neq a, b \in F$  such that  $X^2 - tr(a)X + n(a) - n(b)$  is an irreducible polynomial over

$K$ . One has  $d_{xy} = \det s(x, y) = n(x)^2 - \text{tr}(a)n(x)n(y) + (n(a) - n(b))n(y)^2$ . Thus  $d_{xy} \neq 0$  if  $x \neq 0 = y$  or  $x = 0 \neq y$ . Suppose  $x \neq 0 \neq y$  and set  $z = x/y$ . Then

$$d_{xy}/n(y)^2 = n(z)^2 - \text{tr}(a)n(z) + n(a) - n(b).$$

By our choice of  $a, b$  we see again  $d_{xy} \neq 0$ . Therefore the Boerner-Lantz type exists also for the orders  $3^{4n}$  and  $2^{4n}$ .

Specializing first  $y = 0$  and then  $x = 0$  a straightforward computation shows that matrices commuting with the matrices  $s(x, y)$  have the form  $\text{diag}([u, 0], [u, 0])$ ,  $u \in K$ , i.e. the kernel of the semifield plane is  $K$ . On the other hand the SSS of the general representative of Knuth class VII is isotopic to

$$\Sigma = \left\{ \left( \begin{array}{cc} y & y \\ \bar{y} & ax + b\bar{x} \end{array} \right) \mid x, y \in F \right\}.$$

This shows that  $\Sigma$  and  $\Sigma^d$  are equivalent. The Knuth class associated with the Boerner-Lantz semifields has therefore the form  $[1, o, \text{dod}]$ .

**(4.2)** The representative SSS of Knuth class III has a Oyama representation of the form

$$s(x, y) = \left( \begin{array}{cc} [x, 0]^t & [y, 0]^t \\ [a\bar{y}, b\bar{y}]^t & [\bar{x} + c\bar{y}, d\bar{y}]^t \end{array} \right), \quad x, y \in F,$$

with  $a = z^6, b = z^3, c = 1, d = -1$ . We turn from  $q = 3$  to an arbitrary odd prime power and show that there are always suitable choices of the parameters  $a, \dots, d \in F - \{0\}$  such that  $\Sigma = \{s(x, y) \mid x, y \in F\}$  is a SSS.

Assume for the moment that  $\Sigma$  is a SSS. Then similar computation as in (4.1) shows that the kernel of the semifield plane is  $K$ .

**Lemma.**

(a) Choose  $v = v_1 + v_2\alpha \in F$ ,  $v_1, v_2 \in K$ , such that  $f_v = X^2 + X - tv_2^2 + v_1$  is an irreducible Polynomial over  $K$ . Then  $d_v(x, y) = n(x) + x\bar{y} + vn(y) \neq 0$  for  $(0, 0) \neq (x, y) \in F^2$ .

(b) For  $v = v_1 + v_2\alpha, v_1 \neq 1/4$ , as in (a) one can choose  $a, \dots, d \in F - \{0\}$  such that  $\det s(x, y) = n(d_v(x, y))$  for  $(x, y) \in F^2$ .

*Proof.* (a) The claim is obvious for  $x = 0$  or  $y = 0$ . So assume  $x \neq 0 \neq y$  and set  $z = x/y$ . Dividing by  $n(y)$  we have to show that

$$g(z) = n(z) + z + v \neq 0$$

for  $0 \neq z \in F$ . Write  $z = z_1 + z_2\alpha$ ,  $z_1, z_2 \in K$ . Assuming  $g(z) = 0$  we have  $z_2 = -v_2$  and  $z_1^2 + z_1 - tv_2^2 + v_1 = 0$ , a contradiction.

(b) One has:

$$\begin{aligned} \det s(x, y) &= n(x)^2 + (n(a) - n(b))n(y)^2 + (n(c) - n(d) - \text{tr}(a))n(x)n(y) \\ &\quad + \text{tr}(cx\bar{y})n(x) + \text{tr}(\bar{b}d - \bar{a}c)x\bar{y}n(y) \end{aligned}$$

We choose  $c = 1$ . The requirement  $\det s(x, y) = n(d_v(x, y))$  leads to the equations:

$$n(v) = n(a) - n(b) \quad (1)$$

$$\text{tr}(v) = -n(d) - \text{tr}(a) \quad (2)$$

$$v = b\bar{d} - a \quad (3)$$

Equation (3) implies  $-\text{tr}(a) = \text{tr}(v) - \text{tr}(b\bar{d})$  and  $n(a) = n(b)n(d) + n(v) - \text{tr}(b\bar{d}v)$ . Then equations (1) and (2) become:

$$0 = n(b)n(d) - n(b) - \text{tr}(b\bar{d}v) \quad (1')$$

$$0 = -n(d) - \text{tr}(b\bar{d}) \quad (2')$$

We specialize again  $d = rb$ ,  $0 \neq r \in K$ . Then  $n(d) = r^2n(b)$  and  $b\bar{d} = rn(b)$ . Equations (1') and (2') become:

$$0 = r^2n(b)^2 - n(b) - rn(b)\text{tr}(v) \quad (1'')$$

$$0 = r^2n(b) + 2rn(b) \quad (2'')$$

This forces  $r = -2$  and we have to choose the parameter  $b$  such that  $4v_1 = -r\text{tr}(v) = 1 - 4n(b)$ , i.e.  $n(b) = (1 - 4v_1)/4$ . It is clear that we can choose all parameters  $\neq 0$ .  $\square$

The lemma shows that for any odd prime power  $q$  there exist semifield planes of order  $q^4$  with a kernel of order  $q$  which have the type of the Knuth class III. Computer experiments for small  $q$  show that there are many other choices for the parameters  $a, \dots, d$  which lead to SSS. However it seems difficult to describe in the general case all admissible parameter choices and to decide the equivalence of the resulting SSS's.

Although for  $q^4 = 81$  the Knuth class has Size 3 it is not obvious what the size of the Knuth class is in the general case.

**(4.3)** The representative SSS of Knuth class V is isotopic to a pre-semifield spread set which has in the Oyama representation the form

$$s(x, y) = \begin{pmatrix} [x, 0]^t & [y, 0]^t \\ [a\bar{y}, b\bar{x}]^t & [\bar{x}, c\bar{y}]^t \end{pmatrix}, \quad x, y \in F,$$

with  $a = z^2, b = z^5, c = z^3$ . If one multiplies the matrices  $s(x, y)$  from left with the matrix

$$\begin{pmatrix} [1, 0]^t & [0, 0]^t \\ [0, d]^t & [1, 0]^t \end{pmatrix}, \quad d = z,$$

one obtains the SSS  $\Sigma$  representing the multiplication rule. We turn from  $q = 3$  to an arbitrary odd prime power and show that there are always suitable choices of the parameters  $a, b, c \in F - \{0\}$  which result in a pre-semifield spread set.

Assume for the moment that  $\Sigma$  is a SSS. By similar computations as in (4.1) one can show that the kernel of the semifield plane is  $K$ .

**Lemma.** *Let  $a, b, c$  be elements in  $F - \{0\}$  and consider the function  $g : F \rightarrow K$  defined by  $g(z) = n(z)^2 + n(a) - n(z)(\text{tr}(a) + n(b) + n(c)) + \text{tr}(c\bar{b}z^2)$ . Then  $g$  has not the value 0 in the following cases.*

$$(a) \quad c = bs, \quad s \in K \text{ and } -sn(b) \text{ is a nonsquare and } a = a_1 + a_2\alpha \text{ with } ta_2^2 + (1 + s)^2a_1n(b) + \frac{1}{4}(1 + s)^4n(b)^2 = 0.$$

$$(b) \quad c = bs, \quad 0 \neq s \in K \text{ and } -sn(b) \text{ is a square and } a = a_1 + a_2\alpha \text{ with } ta_2^2 + (1 - s)^2a_1n(b) + \frac{1}{4}(1 - s)^4n(b)^2 = 0.$$

*Proof.* Let  $z = x + y\alpha$  with  $x, y \in K$ . Then  $n(z) = x^2 - ty^2$  and in both cases  $\text{tr}(c\bar{b}z^2) = 2sn(b)(x^2 + ty^2)$ . Hence

$$g(z) = x^4 - 2tx^2y^2 + t^2y^4 + n(a) - (2a_1 + (1 - s)^2n(b))x^2 + (2a_1 + (1 + s)^2n(b))ty^2.$$

Assume  $g(z) = 0$ .

Case (a). Here  $n(a) = a_1^2 - ta_2^2$ ,  $\text{tr}(a) = 2a_1$  which shows  $n(a) = a_1^2 + a_1(1 + s)^2n(b) + \frac{1}{4}(1 + s)^4n(b)^2 = (a_1 + \frac{1}{2}(1 + s)^2n(b))^2$ . Then we get

$$g(z) = x^2(x^2 - 2ty^2 - 2a_1 - (1 - s)^2n(b)) + (ty^2 + a_1 + \frac{1}{2}(1 + s)^2n(b))^2.$$

There exists  $r \in K$  with  $-r^2 = x^2 - 2ty^2 - 2a_1 - (1 - s)^2n(b)$  and  $xr = ty^2 + a_1 + \frac{1}{2}(1 + s)^2n(b)$ . Multiplying with 2 we obtain

$$\begin{aligned} 2xr &= 2ty^2 + 2a_1 + (1 + s)^2n(b) \\ &= x^2 + r^2 - 2a_1 - (1 - s)^2n(b) + 2a_1 + (1 + s)^2n(b) \\ &= x^2 + r^2 + 4sn(b). \end{aligned}$$

Then  $(x - r)^2 = -4sn(b)$ , a contradiction to our assumption.

Case (b). Now  $n(a) = a_1^2 + a_1(1 - s)^2n(b) + \frac{1}{4}(1 - s)^4n(b)^2 = (a_1 + \frac{1}{2}(1 - s)^2n(b))^2$ . Hence

$$g(z) = (x^2 - (a_1 + \frac{1}{2}(1 - s)^2n(b)))^2 + y^2(t^2y^2 - 2tx^2 + 2ta_1 + (1 + s)^2tn(b)).$$

There exists  $r \in K$  with  $-r^2 = t^2y^2 - 2tx^2 + 2a_1t + (1+s)^2tn(b)$  and  $x^2 - (a_1 + \frac{1}{2}(1-s)^2n(b)) = yr$ . Multiply the last equation with  $2t$ . We obtain

$$\begin{aligned} 2ytr &= 2tx^2 - 2ta_1 - (1-s)^2tn(b) \\ &= t^2y^2 + r^2 + 2a_1t + (1+s)^2tn(b) - 2a_1t - (1-s)^2tn(b) \\ &= t^2y^2 + r^2 + 4stn(b). \end{aligned}$$

Hence  $(ty - r)^2 = -4stn(b)$ , a contradiction as  $-sn(b)$  is a square.  $\square$

We apply the lemma to show that the matrices  $s(x, y)$  define a pre-semifield spread set. We compute

$$\det s(x, y) = n(x)^2 + n(a)n(y)^2 - n(x)n(y)(tr(a) + n(b) + n(c)) + tr(\overline{cb}x^2\overline{y}^2).$$

Clearly,  $\det s(x, y) \neq 0$  if  $(0, 0) \neq (x, y) \in F^2$  but  $x = 0$  or  $y = 0$ . So assume that  $x \neq 0 \neq y$  and that the parameters are chosen as in case (a) or (b) of the lemma. Dividing by  $n(y)^2$  we obtain  $\det s(x, y)/n(y)^2 = g(z)$  with  $z = x/y$ . The lemma shows that  $\det s(x, y) \neq 0$ .

We have shown that for any odd prime power  $q$  there exist semifield planes of order  $q^4$  with a kernel of order  $q$  which have the type of Knuth class V. Computer experiments for small  $q$  show that there are many other choices for the parameters  $a, b, c$  which lead to pre-semifield spread sets. Again it seems difficult to describe in the general case all admissible parameter choices and to decide the equivalence question.

The SSS  $\Sigma^\circ$  is represented by matrices

$$s^\circ(x, y) = \begin{pmatrix} [x, b\overline{y}]^t & [0, y]^t \\ [0, ay]^t & [x, c\overline{y}]^t \end{pmatrix}, \quad x, y \in F$$

and therefore  $\Sigma^{od}$  by matrices

$$s^{od}(x, y) = \begin{pmatrix} [x, \overline{by}]^t & [0, \overline{ay}]^t \\ [0, \overline{y}]^t & [x, \overline{cy}]^t \end{pmatrix}, \quad x, y \in F.$$

If  $b\overline{c}^{-1} \in F^{q-1}$  one can find a matrix

$$T = \begin{pmatrix} [0, 0]^t & [0, v]^t \\ [0, u]^t & [0, 0]^t \end{pmatrix}$$

such that  $(\Sigma^\circ)^T = \Sigma^{od}$ , so that the Knuth class has in this case size at most three. Again it not clear what the size of the Knuth class is in general.

## 5 Generators of automorphism groups

We tabulate the generators of the automorphism groups for the SSS which represent the Knuth classes I, II, III, and IV. The generators have the form  $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ . These matrices are represented as a pair  $A, B$  followed by a semicolon which separates this pair by its successor.

(I)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & 2 & 1 & 0 \\ 2 & 2 & 1 & 1 \end{pmatrix};$$

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 2 & 1 & 1 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & 2 & 2 \end{pmatrix}; \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

(II)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}; \begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 0 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 1 & 1 & 1 \end{pmatrix};$$

$$\begin{pmatrix} 2 & 0 & 0 & 1 \\ 2 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(III)

$$\begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix}; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}; \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

(v)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix};$$

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 2 & 0 & 2 \\ 2 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{pmatrix}; \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

$$\begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix}.$$

## References

- [1] A. A. Albert, Isotopy for generalized twisted fields, *Anais Acad. Brasil. Ci.* **33** (1961), 265-275.
- [2] A. A. Albert, On the collineation groups associated with twisted fields, *Calcutta Math. Soc., Golden Jubilee Commem. Vol. (1958-59), Part 2, (1959)*, 485-497.
- [3] S. Ball and M. Brown, The six semifield planes associated with a semifield flock, *Adv. Math.* **189** (2004) 68-87.
- [4] V. Boerner-Lantz, A class of semifields of order  $q^4$ , *J. Geom.* **27** (1986) 112-118.
- [5] M. Biliotti, V. Jha and N. Johnson, The collineation groups of the generalized twisted field planes, *Geom. Dedicata* **76** (1999) 97-126.
- [6] C. Charnes, U. Dempwolff, Translation planes of order 49 and their automorphism groups, *Math. Comp.* **67** (1998) 1207-1224.
- [7] S. Cohen, M. Ganley, Commutative, semifields, two-dimensional over their middle nuclei, *J. Algebra* **75** (1982) 375-385.
- [8] M. Cordero, G. Wene, A survey of finite semifields, *Discrete Math.* **208/209** (1999) 125-137.
- [9] P. Dembowski, *Finite Geometries*, Springer, 1968.

- [10] U. Dempwolff, Translation planes of order 27, *Des. Codes Cryptogr.* **4** (1994) 104-121.
- [11] U. Dempwolff and A. Reifart, The classification of the translation planes of order 16, I, *Geom. Dedicata* **15** (1983) 137-153.
- [12] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.9; 2006. (<http://www.gap-system.org>)
- [13] D. Hughes and E. Kleinfeld, Seminuclear extensions of Galois fields, *Amer. J. Math.* **82** (1960) 389-392.
- [14] D. Hughes and F. Piper, *Projective Planes*, Springer, 1973.
- [15] V. Jha and N. Johnson, An analog of the Albert-Knuth theorem on the orders of finite semifields, and a complete solution to Cofman’s subplane problem, *Algebras Groups Geom.* **6** (1989) 1-35.
- [16] N. Johnson, V. Jha and M. Biliotti, *Handbook of Finite Translation Planes*, CRC Press, Boca Raton, 2007.
- [17] W. Kantor, Finite Semifields, in: *Finite Geometries, Groups, and Computaion*, A. Hulpke, R. Liebler, T. Penttila, A. Seress Eds., Pingree Park, Col. USA, Sept. 4-9, 2004, W. de Gruyter, 2006, pp. 103-114.
- [18] W. Kantor, Commutative semifields and symplectic spreads, *J. Algebra* **270** (2003) 96-114.
- [19] M. Kallaher, *Affine planes with transitive collineation groups*, North Holland, 1982.
- [20] D. Knuth, Finite semifields and projective planes, *J. Algebra* **2** (1965) 182-217.
- [21] H. Lüneburg, *Translation planes*, Springer, 1980.
- [22] R. Mathon and G. Royle, The translation planes of order 49, *Des. Codes Cryptogr.* **5** (1995) 52-72.
- [23] E. Moorhouse, Two-graphs and skew two-graphs in finite geometries, *Linear Algebra Appl.* **226-228** (1995) 529-551.
- [24] T. Oyama, On quasifields, *Osaka J. Math.* **22** (1985) 35-54.
- [25] J. Thas and S. Payne, Spreads and ovoids in generalized quadrangles, *Geom. Dedicata* **52** (1994) 227-253.
- [26] R. Walker, Determination of division algebras with 32 elements, *Proc. AMS Symp. Applied. Math.* **15** (1962) 83-85.