Twisted McFarland and Spence Designs and their Automorphisms

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Abstract

In this note we construct symmetric designs which have the parameters of McFarland or Spence designs and whose automorphism groups are isomorphic to some given group.

1 Introduction.

In [K] Kantor considers the problem of designs with prescribed automorphism group. He shows that for any prime power $q > 3$, any finite group G, and $n \geq 50|G|^2$ there exist symmetric and affine designs with the parameters of $\mathrm{PG}(n, q)$ or $\mathrm{AG}(n, q)$ whose automorphism group is precisely G. A simpler approach to this problem is given in [DK]. Here it is shown that there exists an $n \leq 3|G|$ and symmetric and affine designs with parameters of $\mathrm{PG}(n, q)$or $\mathrm{AG}(n, q)$ and automorphism group G. This time the prime powers $q = 2, 3$ in the symmetric case and $q = 3$ in the affine case are included. It follows from these reults that for $q > 3$ and $n \geq 50$ and for $q > 2$ and $n \geq k\ell$; $k > 4$, $\ell > 6$, there exit affine designs with the parameters of $\mathrm{AG}(n, q)$ and with trivial automorphism group.
In this note we use such rigid affine designs to construct symmetric desings with prescribed automorphism groups and parameters of McFarland designs [McF] or Spence designs [S].

Theorem A. *Let $q > 2$ be a prime power and H a finite group whose order divides $r = r_{d,q} = (q^{d+1} - 1)/(q - 1) + 1$. Then there exists a symmetric $(rq^{d+1}, (r - 1)q^d, (q^d - 1)q^d/(q - 1))$-design whose automorphism group is isomorphic to H if one of the following condition holds:*

(a) $q > 3$ *and* $d + 1 \geq 50$.

(b) $d + 1 = k \cdot \ell,\ k > 4,\ \ell > 6$.

Note that for a fixed prime power q the numbers $r_{d,q}$ do not have in general every number coprime to q as a divisor. But choosing q as a suitable prime power of some fixed prime p one has:

Corollary B. *Let H be a finite group and p a prime coprime to $|H|$. Then there exist a p-power q, a number d, and a symmetric $(rq^{d+1}, (r-1)q^d, (q^d-1)q^d/(q-1))$-design, $r = (q^{d+1}-1)/(q-1)+1$, whose automorphism group is isomorphic to H.*

In the case of Spence [S] parameters one has.

Theorem C. *Let H be a finite group whose order is coprime to 3. Then there exist a number d and a symmetric $(3^{d+1}(3^{d+1}-1)/2, 3^d(3^{d+1}+1)/2, 3^d(3^d+1)/2)$-design, whose automorphism group is isomorphic to H.*

2 Twisted McFarland and Spence Designs.

Let E be a $(d+1)$-dimensional F-space, $F = \mathrm{GF}(q)$, and G a group of order

$$r = \frac{q^{d+1}-1}{q-1} + 1.$$

Consider $K = E \times G$ as a multiplicatively written group. Let $\mathcal{U}_d(E)$ be the set of d-dimensional subspaces. We index the elements of this set as $E_g,\ g \in G-1$, using the nontrivial elements of G as subscripts. Choose elements $e_g \in E$ arbitrarily and set $H_g = E_g e_g$ and

$$B_0 = \bigcup_{g \in G-1} H_g g.$$

Then B_0 is a *McFarland difference set* in K (see [McF]). The developement $\mathbf{B} = \{B_0 k \mid k \in K\}$ of B_0 defines a symmetric design $\mathbf{D} = (K, \mathbf{B})$.

The Spence designs are closely related: E be a $(d+1)$-dimensional F-space, $F = \mathrm{GF}(3)$, and G a group of order $(3^{d+1}-1)/2$. Define K and $\mathcal{U}_d(E)$ as above and index the elements of $\mathcal{U}_d(E)$ by the elements of G. Choose elements $e_g \in E$ arbitrarily and set $H_g = E_g e_g$ and

$$B_0 = (E - H_1) \cup \bigcup_{g \in G-1} H_g g.$$

Then B_0 is a *Spence difference set* in K (see [S]). We denote the associated symmetric design again by $\mathbf{D} = (K, \mathbf{B})$.

We now distort for both classes the structure of the designs in the same manner (following the pattern of [DK]): We define on Eg an affine design by taking as blocks the sets Lg, where L ranges over the hyperplanes of the affine geometry $\mathrm{AG}_F(E)$, i.e. the resulting affine design $\mathrm{AG}(Eg)$ is isomorphic to $\mathrm{AG}_F(E)$. Note that the block intersections of the blocks $B \in \mathbf{B}$ induce on the set Eg the design $\mathrm{AG}(Eg)$. Assume that for every $g \in G$ an affine design $\mathbf{A}_g = (Eg, \mathbf{B}_g)$ is given which has Eg as point set and has the same parameters as $\mathrm{AG}_F(E)$. Suppose further that for every $g \in G$ we have a bijection α_g from the block set of $\mathrm{AG}(Eg)$ onto $\mathbf{B}_g$ which preserves parallelism. For $B \in \mathbf{B}$ define

$$B^\alpha = \bigcup_{g \in G} (B \cap Eg)^{\alpha_g}.$$

Set further $\mathbf{B}^\alpha = \{B^\alpha \mid B \in \mathbf{B}\}$ and $\mathbf{D}^\alpha = (K, \mathbf{B}^\alpha)$. The following result is from [DK].

Theorem 2.1. *The incidence structure $\mathbf{D}^\alpha$ is a symmetric design with the same parameters as $\mathbf{D}$.*

Proof. Let $B \in \mathbf{B}$ then $B = \bigcup_{g \in G} (B \cap Eg)$ is a partition. Take $B' \in \mathbf{B}$. For any $g \in G$ we have

$$\begin{aligned} |B \cap B' \cap Eg| &= |(B \cap Eg) \cap (B' \cap Eg)| = |(B \cap Eg)^{\alpha_g} \cap (B' \cap Eg)^{\alpha_g}| \\ &= |B^\alpha \cap (B')^\alpha \cap Eg| \end{aligned}$$

by the definition of α. Hence $|B \cap B'| = |B^\alpha \cap (B')^\alpha|$ and the assertion follows.

Let X, Y be two blocks of a symmetric design. We denote by $[X, Y]$ the set of all blocks which contain $X \cap Y$. This set is the *coline* associated with X and Y (see also [D; p. 65] and [DK]). Note that two colines intersect in at most one block and that $[X, Y] = [X', Y']$ for any pair X', Y' of blocks in $[X, Y]$.

Lemma 2.2. *Let $B_1^\alpha, B_2^\alpha \in \mathbf{B}^\alpha$ be two blocks.*

(a) $|[B_1^\alpha, B_2^\alpha]| = |F|$ *if* $B_2 = B_1 e,\ 1 \neq e \in E$.

(b) $[B_1^\alpha, B_2^\alpha] = \{B_1^\alpha, B_2^\alpha\}$ *if* $B_2 = B_1 k,\ k \in K - E$.

Proof. We only treat the McFarland case: the little modifications in the Spence case are left to the reader. To simplify the notation we may assume that $B_1^\alpha = B_0^\alpha$.

(a) In this case $B_2^\alpha = (B_0e)^\alpha$. Pick $1 \neq g \in G$. Then

$$(B_0 \cap Eg) \cap (B_0e \cap Eg) = \begin{cases} B_0 \cap Eg, & e \in E_g, \\ \emptyset, & e \notin E_g. \end{cases}$$

Hence

$$B_0 \cap B_0e = \bigcup_{g\neq 1,\ e\in E_g} (B_0 \cap Eg)$$

and therefore

$$B_0^\alpha \cap (B_0e)^\alpha = \bigcup_{g\neq 1,\ e\in E_g} (B_0 \cap Eg)^{\alpha_g}.$$

Suppose $(B')^\alpha \in \mathbf{B}^\alpha$ and $(B')^\alpha \cap Eg = B_0^\alpha \cap Eg$. By definition of α we see $B' = B_0e'$ with $e' \in E$ suitable. Moreover if this intersection is nonempty we conclude $e' \in E_g$. Therefore $(B_0e')^\alpha \in [B_0^\alpha, (B_0e)^\alpha]$ iff

$$e' \in \bigcap_{e\in E_g} E_g = Fe.$$

This shows $[B_0^\alpha, (B_0e)^\alpha] = \{(B_0e')^\alpha \,|\, e' \in Fe\}$.

(b) In this case $B_2 = B_0k,\ k \in K - E$. Let k be in Eg_0. Suppose $(B_0k')^\alpha \in [B_0^\alpha, (B_0k)^\alpha]$ and $k' \in Eg'$. We know $B_0^\alpha \cap (B_0k)^\alpha \cap Eg \neq \emptyset$ iff $g \neq 1, g_0$. Assume $g' \neq 1, g_0$, then $(B_0k')^\alpha \cap Eg' = \emptyset \neq B_0^\alpha \cap (B_0k)^\alpha \cap Eg'$, contradiction. Hence $g' = 1$ or g_0.

We claim that $k' = 1$ or k: Suppose $k' = ek,\ 1 \neq e \in E$. Then

$$[B_0^\alpha, (B_0k)^\alpha] = [(B_0ke)^\alpha, (B_0k)^\alpha] = \{(B_0ke')^\alpha \,|\, e' \in Fe\} \not\ni B_0^\alpha$$

by (a), a contradiction. By symmetry the claim follows.

We define a partition $\mathcal{P} = \{Eg \,|\, g \in G\}$ of the set of points and a partition $\mathcal{B} = \{\mathbf{B}(g) \,|\, g \in G\}$, $\mathbf{B}(g) = \{(B_0eg)^\alpha \,|\, e \in E\}$, of the set $\mathbf{B}^\alpha$ of blocks. We also assume from now on $q > 2$ in the McFarland case.

Proposition 2.3. *$\mathcal{P}$ and $\mathcal{B}$ are invariant under the automorphism group of $\mathbf{D}^\alpha$.*

Proof. We define a graph by taking $\mathbf{B}^\alpha$ as the set of vertices. Two blocks are connected iff the corresponding coline has size $|F|$. Then the sets $\mathbf{B}(g)$ are the components of this graph. This implies the invariance of $\mathcal{B}$.
Assume that we are in the McFarland case. Observe that any block from $\mathbf{B}(g)$ has trivial intersection with Eg whereas any point in Eg', $g' \neq g$, lies in a block from $\mathbf{B}(g)$. This characterizes the partition $\mathcal{P}$ too.
In the Spence case a point in E is incident with twice as much blocks from $\mathbf{B}(1)$ as a point in $K - E$ is incident with blocks from $\mathbf{B}(1)$. This characterizes the point set E and similarly the other members from $\mathcal{P}$ are characterized.

3 The Proofs.

We first treat theorem A and corollary B (McFarland case). Let $d+1, q$ be numbers which satisfy condition (a) or (b) of theorem A. By [K] or [DK] there exist two nonisomorphic affine designs $\mathbf{A}^i = (V^i, \mathbf{B}^i)$, $i = 1, 2$, such that these designs have the parameters of $\mathrm{AG}(d+1, q)$ and both have trivial automorphism groups.

Let G be a group of order $r = \frac{q^{d+1}-1}{q-1} + 1$ and denote by $\mathbf{D} = (K, \mathbf{B})$ some associated McFarland design (where K is defined as before). Let H be any subgroup of G and let $G = \bigcup_{s \in S} sH$ be the decomposition into left cosets of H and we assume for convenience $1 \in S$. Let $\mathbf{A}_1 = (E, \mathbf{B}_1)$ be an affine design on the point set E which is isomorphic to $\mathbf{A}^1$ and let $\mathbf{A}_s = (Es, \mathbf{B}_s)$, $1 \neq s \in S$, be an affine design on the point set Es which is isomorphic to $\mathbf{A}^2$. For $s \in S$, $1 \neq h \in H$, we define an incidence structure $\mathbf{A}_{sh} = (Esh, \mathbf{B}_{sh})$ on the point set Esh by taking $\mathbf{B}_{sh} = \{Ch \,|\, C \in \mathbf{B}_s\}$ as the set of blocks. Clearly, $es \mapsto esh$, $C \mapsto Ch$, defines an isomorphism between $\mathbf{A}_s$ and $\mathbf{A}_{sh}$.
Let α_s be a bijection of the blocks of $\mathrm{AG}(Es)$ onto the blocks $\mathbf{B}_s$ of $\mathbf{A}_s$, $s \in S$, which preserves parallelism. For $1 \neq h \in H$ we define a bijection α_{sh} of the blocks of $\mathrm{AG}(Esh)$ onto the blocks $\mathbf{B}_{sh}$ of $\mathbf{A}_{sh}$ by $(Ch)^{\alpha_{sh}} = (C^{\alpha_s})h$, where C is a block in $\mathrm{AG}(Es)$. Observe that for $h_1 \in H$ we get
(*)

$$(Chh_1)^{\alpha_{shh_1}} = (C^{\alpha_s})hh_1 = (Ch)^{\alpha_{sh}} h_1.$$

Let $\mathbf{D}^\alpha = (K, \mathbf{B}^\alpha)$ be the twisted McFarland designs obtained from these bijections and take $B \in \mathbf{B}$. The corresponding block in $\mathbf{B}^\alpha$ has the form

$$B^\alpha = \bigcup_{g \in G} (B \cap Eg)^{\alpha_g} = \bigcup_{s \in S} \bigcup_{h \in H} (B \cap Esh)^{\alpha_{sh}}.$$

Take $k \in H$. Using (*) we obtain

$$\begin{aligned} (B^\alpha)k &= \bigcup_{s \in S} \bigcup_{h \in H} (B \cap Esh)^{\alpha_{sh}} k \\ &= \bigcup_{s \in S} \bigcup_{h \in H} ((B \cap Esh)k)^{\alpha_{shk}} \\ &= \bigcup_{s \in S} \bigcup_{h \in H} (Bk \cap Eshk)^{\alpha_{shk}} \\ &= (Bk)^\alpha. \end{aligned}$$

This shows:

Lemma 3.1. $H \leq \mathrm{Aut}(\mathbf{D}^\alpha)$.

Proof of Theorem A. Take $\sigma \in A = \mathrm{Aut}(\mathbf{D}^\alpha)$. By proposition 2.3 σ induces a permutation of $\mathcal{P}$.

(1) σ fixes the set $\mathcal{P}_1 = \{Eh \mid h \in H\}$.

Assume the opposite. By choosing the notation suitably we may assume $E\sigma = Es$, $1 \neq s \in S$. Assume $e \in E$, $C \in \mathbf{B}^\alpha$. Then $e \in C \cap E$ is equivalent to $e\sigma \in (C \cap E)\sigma = C\sigma \cap Es$. Hence $e \mapsto e\sigma$, $C \cap E \mapsto C\sigma \cap Es$ is an isomorphism of $\mathbf{A}_1$ onto $\mathbf{A}_s$, a contradiction as these designs are not isomorphic. (1) follows.

(2) The stabilizer of E in A is trivial.

Assume that σ fixes E. Then σ induces an automorphism of $\mathbf{A}_1$. By the choice of $\mathbf{A}_1$ we see that $\sigma_E = \mathbf{1}_E$. In particular $(C \cap E)\sigma = C \cap E$ for every block C. But this shows that $C \in \mathbf{B}(g)$ implies $C\sigma \in \mathbf{B}(g)$. Therefore σ induces the identity on $\mathcal{B}$. The argument of the proof of the proposition shows that σ also fixes every set Eg. Again as $\mathbf{A}_g$ has a trivial automorphism group we conclude $\sigma_{Eg} = \mathbf{1}_{Eg}$. Claim (2) follows.

Since H has a regular action on $\mathcal{P}_1$ theorem A follows from (1), (2) and lemma 3.1.

Proof of Corollary B. Choose $q = p^f$ such that the multiplicative group of $F = \mathrm{GF}(q)$ contains a primitive $|H|$-th root of unity. Note that

$$(r, q-1) = (q^d + q^{d-1} + \cdots + q + 2, q-1) = (d+1, q-1).$$

Choosing q and $d+1$ large enough we can apply theorem A using condition (a).

Proof of Theorem C. Choose $k > 4$ large enough such that the multiplicative group of $F = \mathrm{GF}(3^k)$ contains a primitive $2|H|$-th root of unity. Then choose $\ell > 6$ and set $d+1 = k\ell$. By [DK] there exist nonisomorphic affine designs $\mathbf{A}^1$, $\mathbf{A}^2$ with trivial automorphism groups and with the same parameters as $\mathrm{AG}(d+1, 3)$. We then proceed exactly as in the McFarland case.

4 REFERENCES

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