1. AFFINE VARIETIES

A subset of affine n-space $\mathbb{A}^n$ over a field k is called an algebraic set if it can be written as the zero locus of a set of polynomials. By the Hilbert basis theorem, this set of polynomials can be assumed to be finite. We define the Zariski topology on $\mathbb{A}^n$ (and hence on any subset of $\mathbb{A}^n$) by declaring the algebraic sets to be the closed sets.

Any algebraic set $X \subset \mathbb{A}^n$ has an associated radical ideal $I(X) \subset k[x_1,\dots,x_n]$ that consists of those functions that vanish on X. Conversely, for any radical ideal I there is an associated algebraic set $Z(I)$ which is the common zero locus of all functions in I. If k is algebraically closed, Hilbert's Nullstellensatz states that this gives in fact a one-to-one correspondence between algebraic sets in $\mathbb{A}^n$ and radical ideals in $k[x_1,\dots,x_n]$.

An algebraic set (or more generally any topological space) is called irreducible if it cannot be written as a union of two proper closed subsets. Irreducible algebraic sets in $\mathbb{A}^n$ are called affine varieties. Any algebraic set in $\mathbb{A}^n$ can be decomposed uniquely into a finite union of affine varieties. Under the correspondence mentioned above, affine varieties correspond to prime ideals. The dimension of an algebraic set (or more generally of a topological space) is defined to be the length of the longest chain of irreducible closed subsets minus one.

1.1. **Algebraic sets and the Zariski topology.** We have said in the introduction that we want to consider solutions of polynomial equations in several variables. So let us now make the obvious definitions.

Definition 1.1.1. Let k be a field (recall that you may think of the complex numbers if you wish). We define **affine n-space** over k, denoted $\mathbb{A}^n$, to be the set of all n-tuples of elements of k:

$$\mathbb{A}^n := \{(a_1,\dots,a_n)\ ;\ a_i \in k \text{ for } 1 \le i \le n\}.$$

The elements of the polynomial ring

$$\begin{aligned} k[x_1,\dots,x_n] :=& \{\text{polynomials in the variables } x_1,\dots,x_n \text{ over } k\} \\ =& \{\sum_I a_I x^I\ ;\ a_I \in k\} \end{aligned}$$

(with the sum taken over all multi-indices $I = (i_1,\dots,i_n)$ with $i_j \ge 0$ for all $1 \le j \le n$) define functions on $\mathbb{A}^n$ in the obvious way. For a given set $S \subset k[x_1,\dots,x_n]$ of polynomials, we call

$$\boldsymbol{Z(S)} := \{P \in \mathbb{A}^n\ ;\ f(P) = 0 \text{ for all } f \in S\} \subset \mathbb{A}^n$$

the **zero set** of S. Subsets of $\mathbb{A}^n$ that are of this form for some S are called **algebraic sets**. By abuse of notation, we also write $Z(f_1,\dots,f_i)$ for $Z(S)$ if $S = \{f_1,\dots,f_i\}$.

Example 1.1.2. Here are some simple examples of algebraic sets:

(i) Affine n-space itself is an algebraic set: $\mathbb{A}^n = Z(0)$.
(ii) The empty set is an algebraic set: $\emptyset = Z(1)$.
(iii) Any single point in $\mathbb{A}^n$ is an algebraic set: $(a_1,\dots,a_n) = Z(x_1 - a_1,\dots,x_n - a_n)$.
(iv) Linear subspaces of $\mathbb{A}^n$ are algebraic sets.
(v) All the examples from section 0 are algebraic sets: e.g. the curves of examples 0.1.1 and 0.1.3, and the cubic surface of example 0.1.7.

Remark 1.1.3. Of course, different subsets of $k[x_1,\dots,x_n]$ can give rise to the same algebraic set. Two trivial cases are:

(i) If two polynomials f and g are already in S, then we can also throw in $f+g$ without changing $Z(S)$.
(ii) If f is in S, and g is any polynomial, then we can also throw in $f \cdot g$ without changing $Z(S)$.

Recall that a subset S of a commutative ring R (in our case, $R = k[x_1,\dots,x_n]$) is called an **ideal** if it is closed both under addition and under multiplication with arbitrary ring elements. If $S \subset R$ is any subset, the set

$$(S) = \{f_1g_1 + \cdots + f_mg_m \,;\, f_i \in S, g_i \in R\}$$

is called the **ideal generated by** S; it is obviously an ideal. So what we have just said amounts to stating that $Z(S) = Z((S))$. It is therefore sufficient to only look at the cases where S is an ideal of $k[x_1,\dots,x_n]$.

There is a more serious issue though that we will deal with in section 1.2: a function f has the same zero set as any of its powers f^i; so e.g. $Z(x_1) = Z(x_1^2)$ (although the ideals (x_1) and (x_1^2) are different).

We will now address the question whether any algebraic set can be defined by a *finite* number of polynomials. Although this is entirely a question of commutative algebra about the polynomial ring $R = k[x_1,\dots,x_n]$, we will recall here the corresponding definition and proposition.

Lemma and Definition 1.1.4. *Let R be a ring. The following two conditions are equivalent:*

(i) *Every ideal in R can be generated by finitely many elements.*
(ii) *R satisfies the **ascending chain condition**: every (infinite) ascending chain of ideals $I_1 \subset I_2 \subset I_3 \subset \cdots$ is stationary, i.e. we must have $I_m = I_{m+1} = I_{m+2} = \cdots$ for some m.*

*If R satisfies these conditions, it is called **Noetherian**.*

Proof. (i) $\Rightarrow$ (ii): Let $I_1 \subset I_2 \subset \cdots$ be an infinite ascending chain of ideals in R. Then $I := \cup_i I_i$ is an ideal of R as well; so by assumption (i) it can be generated by finitely many elements. These elements must already be contained in one of the I_m, which means that $I_m = I_{m+1} = \cdots$.

(ii) $\Rightarrow$ (i): Assume that there is an ideal I that cannot be generated by finitely many elements. Then we can recursively construct elements f_i in I by picking $f_1 \in I$ arbitrary and $f_{i+1} \in I \backslash (f_1,\dots,f_i)$. It follows that the sequence of ideals

$$(f_1) \subset (f_1,f_2) \subset (f_1,f_2,f_3) \subset \cdots$$

is not stationary. □

Proposition 1.1.5. ***(Hilbert basis theorem)*** *If R is a Noetherian ring then so is R[x]. In particular, $k[x_1,\dots,x_n]$ is Noetherian; so every algebraic set can be defined by finitely many polynomials.*

Proof. Assume that $I \subset R[x]$ is an ideal that is not finitely generated. Then we can define a sequence of elements $f_i \in I$ as follows: let f_0 be a non-zero element of I of minimal degree, and let f_{i+1} be an element of I of minimal degree in $I\backslash(f_0,\dots,f_i)$. Obviously, $\deg f_i \le \deg f_{i+1}$ for all i by construction.

For all i let $a_i \in R$ be the leading coefficient of f_i, and let $I_i = (a_0,\dots,a_i) \subset R$. As R is Noetherian, the chain of ideals $I_0 \subset I_1 \subset \cdots$ in R is stationary. Hence there is an m such that $a_{m+1} \in (a_0,\dots,a_m)$. Let $r_0,\dots,r_m \in R$ such that $a_{m+1} = \sum_{i=0}^m r_i a_i$, and consider the polynomial

$$f = f_{m+1} - \sum_{i=0}^m x^{\deg f_{m+1} - \deg f_i} r_i f_i.$$

We must have $f \in I\backslash(f_0,\dots,f_m)$, as otherwise the above equation would imply that $f_{m+1} \in (f_0,\dots,f_m)$. But by construction the coefficient of f of degree $\deg f_{m+1}$ is zero, so $\deg f < \deg f_{m+1}$, contradicting the choice of f_{m+1}. Hence $R[x]$ is Noetherian.

In particular, as k is trivially Noetherian, it follows by induction that $k[x_1,\dots,x_n]$ is. □

We will now return to the study of algebraic sets and make them into topological spaces.

Lemma 1.1.6.

(i) *If* $S_1 \subset S_2 \subset k[x_1,\dots,x_n]$ *then* $Z(S_2) \subset Z(S_1) \subset \mathbb{A}^n$.
(ii) *If* $\{S_i\}$ *is a family of subsets of* $k[x_1,\dots,x_n]$ *then* $\bigcap_i Z(S_i) = Z(\bigcup_i S_i) \subset \mathbb{A}^n$.
(iii) *If* $S_1, S_2 \subset k[x_1,\dots,x_n]$ *then* $Z(S_1) \cup Z(S_2) = Z(S_1 S_2) \subset \mathbb{A}^n$.

In particular, arbitrary intersections and finite unions of algebraic sets are again algebraic sets.

Proof. (i) and (ii) are obvious, so let us prove (iii). "$\subset$": If $P \in Z(S_1) \cup Z(S_2)$ then $P \in Z(S_1)$ or $P \in Z(S_2)$. In particular, for any $f_1 \in S_1$, $f_2 \in S_2$, we have $f_1(P) = 0$ or $f_2(P) = 0$, so $f_1 f_2(P) = 0$. "$\supset$": If $P \notin Z(S_1) \cup Z(S_2)$ then $P \notin Z(S_1)$ and $P \notin Z(S_2)$. So there are functions $f_1 \in S_1$ and $f_2 \in S_2$ that do not vanish at P. Hence $f_1 f_2(P) \neq 0$, so $P \notin Z(S_1 S_2)$. □

Remark 1.1.7. Recall that a **topology** on any set X can be defined by specifying which subsets of X are to be considered closed sets, provided that the following conditions hold:

(i) The empty set $\emptyset$ and the whole space X are closed.
(ii) Arbitrary intersections of closed sets are closed.
(iii) Finite unions of closed sets are closed.

Note that the standard definition of closed subsets of $\mathbb{R}^n$ that you know from real analysis satisfies these conditions.

A subset Y of X is then called open if its complement $X \backslash Y$ is closed. If X is a topological space and $Y \subset X$ any subset, Y inherits an **induced subspace topology** by declaring the sets of the form $Y \cap Z$ to be closed whenever Z is closed in X. A map $f : X \to Y$ is called *continuous* if inverse images of closed subsets are closed. (For the standard topology of $\mathbb{R}^n$ from real analysis and the standard definition of continuous functions, it is a theorem that a function is continuous if and only if inverse images of closed subsets are closed.)

Definition 1.1.8. We define the **Zariski topology** on $\mathbb{A}^n$ to be the topology whose closed sets are the algebraic sets (lemma 1.1.6 tells us that this gives in fact a topology). Moreover, any subset X of $\mathbb{A}^n$ will be equipped with the topology induced by the Zariski topology on $\mathbb{A}^n$. This will be called the Zariski topology on X.

Remark 1.1.9. In particular, using the induced subspace topology, this defines the Zariski topology on any algebraic set $X \subset \mathbb{A}^n$: the closed subsets of X are just the algebraic sets $Y \subset \mathbb{A}^n$ contained in X.

The Zariski topology is the standard topology in algebraic geometry. So whenever we use topological concepts in what follows we refer to this topology (unless we specify otherwise).

Remark 1.1.10. The Zariski topology is quite different from the usual ones. For example, on $\mathbb{A}^n$, a closed subset that is not equal to $\mathbb{A}^n$ satisfies at least one non-trivial polynomial equation and has therefore necessarily dimension less than n. So the closed subsets in the Zariski topology are in a sense "very small". It follows from this that any two non-empty open subsets of $\mathbb{A}^n$ have a non-empty intersection, which is also unfamiliar from the standard topology of real analysis.

Example 1.1.11. Here is another example that shows that the Zariski topology is "unusual". The closed subsets of $\mathbb{A}^1$ besides the whole space and the empty set are exactly the finite sets. In particular, if $f : \mathbb{A}^1 \to \mathbb{A}^1$ is *any* bijection, then f is a homeomorphism. (This last statement is essentially useless however, as we will not define morphisms between algebraic sets as just being continuous maps with respect to the Zariski topology. In fact, this example gives us a strong hint that we should not do so.)

1.2. **Hilbert's Nullstellensatz.** We now want to establish the precise connection between algebraic sets in $\mathbb{A}^n$ and ideals in $k[x_1,\dots,x_n]$, hence between geometry and algebra. We have already introduced the operation $Z(\cdot)$ that takes an ideal (or any subset of $k[x_1,\dots,x_n]$) to an algebraic set. Here is an operation that does the opposite job.

Definition 1.2.1. For a subset $X \subset \mathbb{A}^n$, we call

$$\boldsymbol{I(X)} := \{f \in k[x_1,\dots,x_n] \; ; \; f(P) = 0 \text{ for all } P \in X\} \subset k[x_1,\dots,x_n]$$

the **ideal** of X (note that this is in fact an ideal).

Remark 1.2.2. We have thus defined a two-way correspondence

$$\left\{ \begin{array}{c} \text{algebraic sets} \\ \text{in } \mathbb{A}^n \end{array} \right\} \underset{Z}{\overset{I}{\rightleftarrows}} \left\{ \begin{array}{c} \text{ideals in} \\ k[x_1,\dots,x_n] \end{array} \right\}.$$

We will now study to what extent these two maps are inverses of each other.

Remark 1.2.3. Let us start with the easiest case of algebraic sets and look at points in $\mathbb{A}^n$. Points are minimal algebraic sets, so by lemma 1.1.6 (i) they should correspond to maximal ideals. In fact, the point $(a_1,\dots,a_n) \in \mathbb{A}^n$ is the zero locus of the ideal $I = (x_1 - a_1,\dots,x_n - a_n)$. Recall from commutative algebra that an ideal I of a ring R is maximal if and only if R/I is a field. So in our case I is indeed maximal, as $k[x_1,\dots,x_n]/I \cong k$. However, for general k there are also maximal ideals that are not of this form, e.g. $(x^2+1) \subset \mathbb{R}[x]$ (where $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$). The following proposition shows that this cannot happen if k is *algebraically closed*, i.e. if every non-constant polynomial in $k[x]$ has a zero.

Proposition 1.2.4. ***(Hilbert's Nullstellensatz** ("theorem of the zeros")) Assume that k is algebraically closed (e.g. $k = \mathbb{C}$). Then the maximal ideals of $k[x_1,\dots,x_n]$ are exactly the ideals of the form $(x_1 - a_1,\dots,x_n - a_n)$ for some $a_i \in k$.*

Proof. Again this is entirely a statement of commutative algebra, so you can just take it on faith if you wish (in fact, many textbooks on algebraic geometry do so). For the sake of completeness we will give a short proof here in the case $k = \mathbb{C}$ that uses only some basic algebra; but feel free to ignore it if it uses concepts that you do not know. A proof of the general case can be found e.g. in [Ha] proposition 5.18.

So assume that $k = \mathbb{C}$. From the discussion above we see that it only remains to show that any maximal ideal $\mathfrak{m}$ is contained in an ideal of the form $(x_1 - a_1,\dots,x_n - a_n)$.

As $\mathbb{C}[x_1,\dots,x_n]$ is Noetherian, we can write $\mathfrak{m} = (f_1,\dots,f_r)$ for some $f_i \in \mathbb{C}[x_1,\dots,x_n]$. Let K be the subfield of $\mathbb{C}$ obtained by adjoining to $\mathbb{Q}$ all coefficients of the f_i. We will now restrict coefficients to this subfield K, so let $\mathfrak{m}_0 = \mathfrak{m} \cap K[x_1,\dots,x_n]$. Note that then $\mathfrak{m} = \mathfrak{m}_0 \cdot \mathbb{C}[x_1,\dots,x_n]$, as the generators f_i of $\mathfrak{m}$ lie in $\mathfrak{m}_0$.

Note that $\mathfrak{m}_0 \subset K[x_1,\dots,x_n]$ is a maximal ideal too, because if we had an inclusion $\mathfrak{m}_0 \subsetneq \mathfrak{m}_0' \subsetneq K[x_1,\dots,x_n]$ of ideals, this would give us an inclusion $\mathfrak{m} \subsetneq \mathfrak{m}' \subsetneq \mathbb{C}[x_1,\dots,x_n]$ by taking the product with $\mathbb{C}[x_1,\dots,x_n]$. (This last inclusion has to be strict as intersecting it with $K[x_1,\dots,x_n]$ gives the old ideals $\mathfrak{m}_0 \subsetneq \mathfrak{m}_0'$ back again.)

So $K[x_1,\dots,x_n]/\mathfrak{m}_0$ is a field. We claim that there is an embedding $K[x_1,\dots,x_n]/\mathfrak{m}_0 \hookrightarrow \mathbb{C}$. To see this, split the field extension $K[x_1,\dots,x_n]/\mathfrak{m}_0 : \mathbb{Q}$ into a purely transcendental part $L : \mathbb{Q}$ and an algebraic part $K[x_1,\dots,x_n]/\mathfrak{m}_0 : L$. As $K[x_1,\dots,x_n]/\mathfrak{m}_0$ and hence L is finitely generated over $\mathbb{Q}$ whereas $\mathbb{C}$ is of infinite transcendence degree over $\mathbb{Q}$, there is an embedding $L \subset \mathbb{C}$. Finally, as $K[x_1,\dots,x_n]/\mathfrak{m}_0 : L$ is algebraic and $\mathbb{C}$ algebraically closed, this embedding can be extended to give an embedding $K[x_1,\dots,x_n]/\mathfrak{m}_0 \subset \mathbb{C}$.

Let a_i be the images of the x_i under this embedding. Then $f_i(a_1,\dots,a_n) = 0$ for all i by construction, so $f_i \in (x_1 - a_1,\dots,x_n - a_n)$ and hence $\mathfrak{m} \subset (x_1 - a_1,\dots,x_n - a_n)$. □

Remark 1.2.5. The same method of proof can be used for any algebraically closed field k that has infinite transcendence degree over the prime field $\mathbb{Q}$ or $\mathbb{F}_p$.

Corollary 1.2.6. *Assume that k is algebraically closed.*

(i) *There is a 1:1 correspondence*

$$\{\text{points in } \mathbb{A}^n\} \longleftrightarrow \{\text{maximal ideals of } k[x_1,\dots,x_n]\}$$

given by $(a_1,\dots,a_n) \longleftrightarrow (x_1 - a_1,\dots,x_n - a_n)$.

(ii) *Every ideal* $I \subsetneq k[x_1,\dots,x_n]$ *has a zero in* $\mathbb{A}^n$.

Proof. (i) is obvious from the Nullstellensatz, and (ii) follows in conjunction with lemma 1.1.6 (i) as every ideal is contained in a maximal one. □

Example 1.2.7. We just found a correspondence between points of $\mathbb{A}^n$ and certain ideals of the polynomial ring. Now let us try to extend this correspondence to more complicated algebraic sets than just points. We start with the case of a collection of points in $\mathbb{A}^1$.

(i) Let $X = \{a_1,\dots,a_r\} \subset \mathbb{A}^1$ be a finite algebraic set. Obviously, $I(X)$ is then generated by the function $(x-a_1)\cdots(x-a_r)$, and $Z(I(X)) = X$ again. So Z is an inverse of I.

(ii) Conversely, let $I \subset k[x]$ be an ideal (not equal to (0) or (1)). As $k[x]$ is a principal ideal domain, we have $I = (f)$ for some non-constant monic function $f \in k[x]$. Now for the correspondence to work at all, we have to require that k be algebraically closed: for if f had no zeros, we would have $Z(I) = \emptyset$, and $I(Z(I)) = (1)$ would give us back no information about I at all. But if k is algebraically closed, we can write $f = (x-a_1)^{m_1}\cdots(x-a_r)^{m_r}$ with the a_i distinct and $m_i > 0$. Then $Z(I) = \{a_1,\dots,a_r\}$ and therefore $I(Z(I))$ is generated by $(x-a_1)\cdots(x-a_r)$, i.e. all exponents are reduced to 1. Another way to express this fact is that a function is in $I(Z(I))$ if and only if some power of it lies in I. We write this as $I(Z(I)) = \sqrt{I}$, where we use the following definition.

Definition 1.2.8. For an ideal $I \subset k[x_1,\dots,x_n]$, we define the **radical** of I to be

$$\sqrt{I} := \{f \in k[x_1,\dots,x_n]\ ;\ f^r \in I \text{ for some } r > 0\}.$$

(In fact, this is easily seen to be an ideal.) An ideal I is called radical if $I = \sqrt{I}$. Note that the ideal of an algebraic set is always radical.

The following proposition says that essentially the same happens for $n > 1$. As it can be guessed from the example above, the case $Z(I(X))$ is more or less trivial, whereas the case $I(Z(I))$ is more difficult and needs the assumption that k be algebraically closed.

Proposition 1.2.9.

(i) *If* $X_1 \subset X_2$ *are subsets of* $\mathbb{A}^n$ *then* $I(X_2) \subset I(X_1)$.

(ii) *For any algebraic set* $X \subset \mathbb{A}^n$ *we have* $Z(I(X)) = X$.

(iii) *If k is algebraically closed, then for any ideal* $I \subset k[x_1,\dots,x_n]$ *we have* $I(Z(I)) = \sqrt{I}$.

Proof. (i) is obvious, as well as the "⊃" parts of (ii) and (iii).

(ii) "⊂": By definition $X = Z(I)$ for some I. Hence, by (iii) "⊃" we have $I \subset I(Z(I)) = I(X)$. By 1.1.6 (i) it then follows that $Z(I(X)) \subset Z(I) = X$.

(iii) "$\subset$": (This is sometimes also called Hilbert's Nullstellensatz, as it follows easily from proposition 1.2.4.) Let $f \in I(Z(I))$. Consider the ideal

$$J = I + (ft-1) \subset k[x_1,\dots,x_n,t].$$

This has empty zero locus in $\mathbb{A}^{n+1}$, as f vanishes on $Z(I)$, so if we require $ft = 1$ at the same time, we get no solutions. Hence $J = (1)$ by corollary 1.2.6 (i). In particular, there is a relation

$$1 = (ft-1)g_0 + \sum f_i g_i \in k[x_1,\dots,x_n,t]$$

for some $g_i \in k[x_1,\dots,x_n,t]$ and $f_i \in I$. If t^N is the highest power of t occurring in the g_i, then after multiplying with f^N we can write this as

$$f^N = (ft-1)G_0(x_1,\dots,x_n,ft) + \sum f_i G_i(x_1,\dots,x_n,ft)$$

where $G_i = f^N g_i$ is considered to be a polynomial in $x_1,\dots,x_n,ft$. Modulo $ft-1$ we get

$$f^N = \sum f_i G_i(x_1,\dots,x_n,1) \in k[x_1,\dots,x_n,ft]/(ft-1).$$

But as the map $k[x_1,\dots,x_n] \to k[x_1,\dots,x_n,ft]/(ft-1)$ is injective, this equality holds in fact in $k[x_1,\dots,x_n]$, so $f^N \in I$. □

Corollary 1.2.10. *If k is algebraically closed, there is a one-to-one inclusion-reversing correspondence between algebraic sets in $\mathbb{A}^n$ and radical ideals in $k[x_1,\dots,x_n]$, given by the operations $Z(\cdot)$ and $I(\cdot)$. (This is also sometimes called the Nullstellensatz.)*

Proof. Immediately from proposition 1.2.9 and lemma 1.1.6 (i). □

From now on up to the end of section 4, *we will always assume that the ground field k is algebraically closed.*

Remark 1.2.11. Even though the radical $\sqrt{I}$ of an ideal I was easy to define, it is quite difficult to actually compute $\sqrt{I}$ for any given ideal I. Even worse, it is already quite difficult just to check whether I itself is radical or not. In general, you will need non-trivial methods of computer algebra to solve problems like this.

1.3. **Irreducibility and dimension.** The algebraic set $X = \{x_1x_2 = 0\} \subset \mathbb{A}^2$ can be written as the union of the two coordinate axes $X_1 = \{x_1 = 0\}$ and $X_2 = \{x_2 = 0\}$, which are themselves algebraic sets. However, X_1 and X_2 cannot be decomposed further into finite unions of smaller algebraic sets. We now want to generalize this idea. It turns out that this can be done completely in the language of topological spaces. This has the advantage that it applies to more general cases, i.e. open subsets of algebraic sets.

However, you will want to think only of the Zariski topology here, since the concept of irreducibility as introduced below does not make much sense in classical topologies.

Definition 1.3.1.

(i) A topological space X is said to be **reducible** if it can be written as a union $X = X_1 \cup X_2$, where X_1 and X_2 are (non-empty) closed subsets of X not equal to X. It is called **irreducible** otherwise. An irreducible algebraic set in $\mathbb{A}^n$ is called an **affine variety**.

(ii) A topological space X is called **disconnected** if it can be written as a *disjoint* union $X = X_1 \cup X_2$ of (non-empty) closed subsets of X not equal to X. It is called **connected** otherwise.

Remark 1.3.2. Although we have given this definition for arbitrary topological spaces, you will usually want to apply the notion of irreducibility only in the Zariski topology. For example, in the usual complex topology, the affine line $\mathbb{A}^1$ (i.e. the complex plane) is *reducible* because it can be written e.g. as the union of closed subsets

$$\mathbb{A}^1 = \{z \in \mathbb{C}\,;\, |z| \le 1\} \cup \{z \in \mathbb{C}\,;\, |z| \ge 1\}.$$

In the Zariski topology however, $\mathbb{A}^1$ is irreducible (as it should be).

In contrast, the notion of connectedness can be used in the "usual" topology too and does mean there what you think it should mean.

Remark 1.3.3. Note that there is a slight inconsistency in the existing literature: some authors call a variety what we call an algebraic set, and consequently an irreducible variety what we call an affine variety.

The algebraic characterization of affine varieties is the following.

Lemma 1.3.4. *An algebraic set $X \subset \mathbb{A}^n$ is an affine variety if and only if its ideal $I(X) \subset k[x_1, \dots, x_n]$ is a prime ideal.*

Proof. "$\Leftarrow$": Let $I(X)$ be a prime ideal, and suppose that $X = X_1 \cup X_2$. Then $I(X) = I(X_1) \cap I(X_2)$ by exercise 1.4.1 (i). As $I(X)$ is prime, we may assume $I(X) = I(X_1)$, so $X = X_1$ by proposition 1.2.9 (ii).

"$\Rightarrow$": Let X be irreducible, and let $fg \in I(X)$. Then $X \subset Z(fg) = Z(f) \cup Z(g)$, hence $X = (Z(f) \cap X) \cup (Z(g) \cap X)$ is a union of two algebraic sets. As X is irreducible, we may assume that $X = Z(f) \cap X$, so $f \in I(X)$. □

Example 1.3.5.

(i) $\mathbb{A}^n$ is an affine variety, as $I(\mathbb{A}^n) = (0)$ is prime. If $f \in k[x_1, \dots, x_n]$ is an irreducible polynomial, then $Z(f)$ is an affine variety. A collection of m points in $\mathbb{A}^n$ is irreducible if and only if $m = 1$.

(ii) Every affine variety is connected. The union of the n coordinate axes in $\mathbb{A}^n$ is always connected, although it is reducible for $n > 1$. A collection of m points in $\mathbb{A}^n$ is connected if and only if $m = 1$.

As it can be expected, any topological space that satisfies a reasonable finiteness condition can be decomposed uniquely into finitely many irreducible spaces. This is what we want to show next.

Definition 1.3.6. A topological space X is called **Noetherian** if every descending chain $X \supset X_1 \supset X_2 \supset \cdots$ of closed subsets of X is stationary.

Remark 1.3.7. By corollary 1.2.10 the fact that $k[x_1, \dots, x_n]$ is a Noetherian ring (see proposition 1.1.5) translates into the statement that any algebraic set is a Noetherian topological space.

Proposition 1.3.8. *Every Noetherian topological space X can be written as a finite union $X = X_1 \cup \cdots \cup X_r$ of irreducible closed subsets. If one assumes that $X_i \not\subset X_j$ for all $i \neq j$, then the X_i are unique (up to permutation). They are called the* ***irreducible components*** *of X.*

In particular, any algebraic set is a finite union of affine varieties in a unique way.

Proof. To prove existence, assume that there is a topological space X for which the statement is false. In particular, X is reducible, hence $X = X_1 \cup X_1'$. Moreover, the statement of the proposition must be false for at least one of these two subsets, say X_1. Continuing this construction, one arrives at an infinite chain $X \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots$ of closed subsets, which is a contradiction as X is Noetherian.

To show uniqueness, assume that we have two decompositions $X = X_1 \cup \cdots \cup X_r = X_1' \cup \cdots \cup X_s'$. Then $X_1 \subset \bigcup_i X_i'$, so $X_1 = \bigcup (X_1 \cap X_i')$. But X_1 is irreducible, so we can assume $X_1 = X_1 \cap X_1'$, i.e. $X_1 \subset X_1'$. For the same reason, we must have $X_1' \subset X_i$ for some i. So $X_1 \subset X_1' \subset X_i$, which means by assumption that $i = 1$. Hence $X_1 = X_1'$ is contained in both decompositions. Now let $Y = \overline{X \backslash X_1}$. Then $Y = X_2 \cup \cdots \cup X_r = X_2' \cup \cdots \cup X_s'$; so proceeding by induction on r we obtain the uniqueness of the decomposition. □

Remark 1.3.9. It is probably time again for a warning: given an ideal I of the polynomial ring, it is in general not easy to find the irreducible components of $Z(I)$, or even to determine whether $Z(I)$ is irreducible or not. There are algorithms to figure this out, but they are computationally quite involved, so you will in most cases want to use a computer program for the actual calculation.

Remark 1.3.10. In the same way one can show that every algebraic set X is a (disjoint) finite union of *connected* algebraic sets, called the **connected components** of X.

Remark 1.3.11. We have now seen a few examples of the correspondence between geometry and algebra that forms the base of algebraic geometry: points in affine space correspond to maximal ideals in a polynomial ring, affine varieties to prime ideals, algebraic sets to radical ideals. Most concepts in algebraic geometry can be formulated and most proofs can be given both in geometric and in algebraic language. For example, the geometric statement that we have just shown that any algebraic set can be written as a finite union of irreducible components has the equivalent algebraic formulation that every radical ideal can be written uniquely as a finite intersection of prime ideals.

Remark 1.3.12. An application of the notion of irreducibility is the definition of the dimension of an affine variety (or more generally of a topological space; but as in the case of irreducibility above you will only want to apply it to the Zariski topology). Of course, in the case of complex varieties we have a geometric idea what the dimension of an affine variety should be: it is the number of complex coordinates that you need to describe X locally around any point. Although there are algebraic definitions of dimension that mimics this intuitive one, we will give a different definition here that uses only the language of topological spaces. Finally, all these definitions are of course equivalent and describe the intuitive notion of dimension (at least over $\mathbb{C}$), but it is actually quite hard to prove this rigorously.

The idea to define the dimension in algebraic geometry using the Zariski topology is the following: if X is an irreducible topological space, then any closed subset of X not equal to X must have dimension (at least) one smaller. (This is of course an idea that is not valid in the usual topology that you know from real analysis.)

Definition 1.3.13. Let X be a (non-empty) irreducible topological space. The **dimension** of X is the biggest integer n such that there is a chain $\emptyset \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = X$ of irreducible closed subsets of X. If X is any Noetherian topological space, the dimension of X is defined to be the supremum of the dimensions of its irreducible components. A space of dimension 1 is called a **curve**, a space of dimension 2 a **surface**.

Remark 1.3.14. In this definition you should think of X_i as having dimension i. The content of the definition is just that there is "nothing between" varieties of dimension i and $i+1$.

Example 1.3.15. The dimension of $\mathbb{A}^1$ is 1, as single points are the only irreducible closed subsets of $\mathbb{A}^1$ not equal to $\mathbb{A}^1$. We will see in exercise 1.4.9 that the dimension of $\mathbb{A}^2$ is 2. Of course, the dimension of $\mathbb{A}^n$ is always n, but this is a fact from commutative algebra that we cannot prove at the moment. But we can at least see that the dimension of $\mathbb{A}^n$ is not less than n, because there are sequences of inclusions

$$\mathbb{A}^0 \subsetneq \mathbb{A}^1 \subsetneq \cdots \subsetneq \mathbb{A}^n$$

of linear subspaces of increasing dimension.

Remark 1.3.16. This definition of dimension has the advantage of being short and intuitive, but it has the disadvantage that it is very difficult to apply in actual computations. So for the moment we will continue to use the concept of dimension only in the informal way as we have used it so far. We will study the dimension of varieties rigorously in section 4, after we have developed more powerful techniques in algebraic geometry.

Remark 1.3.17. Here is another application of the notion of irreducibility (that is in fact not much more than a reformulation of the definition). Let X be an irreducible topological space (e.g. an affine variety). Let $U \subset X$ be a non-empty open subset, and let $Y \subsetneq X$ be a closed subset. The fact that X cannot be the union $(X\backslash U) \cup Y$ can be reformulated by saying that U cannot be a subset of Y. In other words, the **closure** of U (i.e. the smallest closed subset of X that contains U) is equal to X itself. Recall that an open subset of a topological space X is called **dense** if its closure is equal to the whole space X. With this wording, we have just shown that in an irreducible topological space every non-empty open subset is dense. Note that this is not true for reducible spaces: let $X = \{x_1x_2 = 0\} \subset \mathbb{A}^2$ be the union of the two coordinate axes, and let $U = \{x_1 \neq 0\} \cap X$ be the open subset of X consisting of the x_1-axis minus the origin. Then the closure of U in X is just the x_1-axis, and not all of X.

1.4. **Exercises.** In all exercises, the ground field k is assumed to be algebraically closed unless stated otherwise.

Exercise 1.4.1. Let $X_1, X_2 \subset \mathbb{A}^n$ be algebraic sets. Show that

(i) $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$,
(ii) $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$.

Show by example that taking the radical in (ii) is in general necessary, i.e. find algebraic sets X_1, X_2 such that $I(X_1 \cap X_2) \neq I(X_1) + I(X_2)$. Can you see geometrically what it means if we have inequality here?

Exercise 1.4.2. Let $X \subset \mathbb{A}^3$ be the union of the three coordinate axes. Determine generators for the ideal $I(X)$. Show that $I(X)$ cannot be generated by fewer than 3 elements, although X has codimension 2 in $\mathbb{A}^3$.

Exercise 1.4.3. In affine 4-dimensional space $\mathbb{A}^4$ with coordinates x, y, z, t let X be the union of the two planes

$$X' = \{x = y = 0\} \qquad \text{and} \qquad X'' = \{z = x - t = 0\}.$$

Compute the ideal $I = I(X) \subset k[x, y, z, t]$. For any $a \in k$ let $I_a \subset k[x, y, z]$ be the ideal obtained by substituting $t = a$ in I, and let $X_a = Z(I_a) \subset \mathbb{A}^3$.

Show that the family of algebraic sets X_a with $a \in k$ describes two skew lines in $\mathbb{A}^3$ approaching each other, until they finally intersect transversely for $a = 0$.

Moreover, show that the ideals I_a are radical for $a \neq 0$, but that I_0 is not. Find the elements in $\sqrt{I_0}\backslash I_0$ and interpret them geometrically.

Exercise 1.4.4. Let $X \subset \mathbb{A}^3$ be the algebraic set given by the equations $x_1^2 - x_2x_3 = x_1x_3 - x_1 = 0$. Find the irreducible components of X. What are their prime ideals? (Don't let the simplicity of this exercise fool you. As mentioned in remark 1.3.9, it is in general *very* difficult to compute the irreducible components of the zero locus of given equations, or even to determine if it is irreducible or not.)

Exercise 1.4.5. Let $\mathbb{A}^3$ be the 3-dimensional affine space over a field k with coordinates x, y, z. Find ideals describing the following algebraic sets and determine the minimal number of generators for these ideals.

(i) The union of the (x, y)-plane with the z-axis.
(ii) The union of the 3 coordinate axes.
(iii) The image of the map $\mathbb{A}^1 \to \mathbb{A}^3$ given by $t \mapsto (t^3, t^4, t^5)$.

Exercise 1.4.6. Let Y be a subspace of a topological space X. Show that Y is irreducible if and only if the closure of Y in X is irreducible.

Exercise 1.4.7. (For those of you who like pathological examples. You will need some knowledge on general topological spaces.) Find a Noetherian topological space with infinite dimension. Can you find an affine variety with infinite dimension?

Exercise 1.4.8. Let $X = \{(t, t^3, t^5) \; ; \; t \in k\} \subset \mathbb{A}^3$. Show that X is an affine variety of dimension 1 and compute $I(X)$.

Exercise 1.4.9. Let $X \subset \mathbb{A}^2$ be an irreducible algebraic set. Show that either

- $X = Z(0)$, i.e. X is the whole space $\mathbb{A}^2$, or
- $X = Z(f)$ for some irreducible polynomial $f \in k[x,y]$, or
- $X = Z(x-a, y-b)$ for some $a, b \in k$, i.e. X is a single point.

Deduce that $\dim(\mathbb{A}^2) = 2$. (Hint: Show that the common zero locus of two polynomials $f, g \in k[x,y]$ without common factor is finite.)