The Ernst equation and ergosurfaces

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Abstract

We show that analytic solutions $\mathscr{E}$ of the Ernst equation with non-empty zero-level-set of $\Re\mathscr{E}$ lead to smooth ergosurfaces in space-time. In fact, the space-time metric is smooth near a "Ernst ergosurface" E_f if and only if $\mathscr{E}$ is smooth near E_f and does not have zeros of infinite order there.

1 Introduction

A standard procedure for constructing stationary axi-symmetric solutions of the Einstein equations proceeds by a reduction of the Einstein equations

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to a two-dimensional nonlinear equation — the Ernst equation [5] — using the asymptotically timelike Killing vector field X as the starting point of the reduction: One finds a complex valued field

$$\mathscr{E} = f + \mathrm{i}b, \tag{1.1}$$

by e.g. solving a boundary-value problem [17]. The space-time metric is then obtained by solving ODEs for the metric functions. The following difficulties arise in this construction:

1. Singularities of $\mathscr{E}$, which might — or might not — lead to singularities of the metric.
2. Struts or causality violations at the rotation axis.
3. Singularities of the equations arising at zeros of $\Re\mathscr{E}$.

The aim of this work is to address this last question. Indeed, the equations determining the components of the metric are singular at the zero-level-set[1]

$$E_f := \{f = 0\ ,\ \rho > 0\}$$

of $f := \Re\mathscr{E}$; we will refer to E_f as the *$\mathscr{E}$-ergosurface.* We show, assuming smoothness of $\mathscr{E}$ in a neighborhood of E_f, and excluding zeros of infinite order, that the singularities of the solutions of those ODEs conspire to produce a smooth space-time metric. More precisely, we have:

THEOREM 1.1 *Consider a smooth solution $f + \mathrm{i}b$ of the Ernst equations (2.5)-(2.6) below such that f has no zeros of infinite order at the $\mathscr{E}$–ergosurface E_f. Then there exists a neighborhood of E_f on which the metric (2.1) obtained by solving (2.3)-(2.4) is smooth and has Lorentzian signature.*

An immediate corollary of Theorem 1.1 is the following: Any point $\vec{x}_0$ off the axis in the Weyl coordinate chart corresponds to space-time points at which the metric is regular $\Longleftrightarrow$ the Ernst potential is a real-analytic function of the Weyl coordinates near $\vec{x}_0$ $\Longleftrightarrow$ the Ernst potential is a smooth function of the Weyl coordinates near $\vec{x}_0$ and zeros of $\Re\mathscr{E}$ have finite order.

The proof of Theorem 1.1 can be found at the end of Section 5.2.

[1]The equations are of course singular at $f = \rho = 0$ as well, but the singularity of the whole system of equations has a different nature there, because of the $\partial_\rho f/\rho$ terms in the Ernst equation (2.2), and will not be considered here. Geometrically, the set $\{\rho = f = 0\}$ has a rather different nature, corresponding to Killing horizons, with the boundary conditions there being reasonably well understood in any case [3, 10].

The condition of zeros of finite order is necessary, in the following sense: any zero of $\Re\mathscr{E}$ on a smooth space-time ergosurface is of finite order. This is proved at the end of Section 2.

It is an interesting consequence of our analysis below that a critical zero of f of order k corresponds to a smooth two-dimensional surface in space-time at which k distinct components of the ergoregion $\{f < 0\}$ "almost meet", in the sense that their closures intersect there, with the boundaries branching out. Two exact solutions with this behavior for $k = 2$ is presented in Section 5.

Section 4 below appeared in preprint form in [4]; the reader will also find in [4] some more information about second order zeros of f.

2 The field equations and ergosurfaces

We consider a vacuum gravitational field in Weyl-Lewis-Papapetrou coordinates

$$ds^2 = f^{-1}\left[h\left(d\rho^2 + d\zeta^2\right) + \rho^2 d\phi^2\right] - f\left(dt + a d\phi\right)^2 \tag{2.1}$$

with all functions depending only upon ρ and ζ. The vacuum Einstein equations for the metric functions h, f, and a are equivalent to the Ernst equation

$$(\Re\mathscr{E})\left(\mathscr{E}_{,\rho\rho} + \mathscr{E}_{,\zeta\zeta} + \frac{1}{\rho}\mathscr{E}_{,\rho}\right) = \mathscr{E}_{,\rho}^2 + \mathscr{E}_{,\zeta}^2 \tag{2.2}$$

for the complex function $\mathscr{E}(\rho,\zeta) = f(\rho,\zeta) + \mathrm{i}b(\rho,\zeta)$, where b replaces a via

$$a_{,\rho} = \rho f^{-2} b_{,\zeta}, \quad a_{,\zeta} = -\rho f^{-2} b_{,\rho} \tag{2.3}$$

and h can be calculated from

$$h_{,\rho} = \frac{\rho h}{2f^2}\left[f_{,\rho}^2 - f_{,\zeta}^2 + b_{,\rho}^2 - b_{,\zeta}^2\right], \quad h_{,\zeta} = \frac{\rho h}{f^2}\left[f_{,\rho}f_{,\zeta} + b_{,\rho}b_{,\zeta}\right]. \tag{2.4}$$

We will think of ρ and ζ as being cylindrical coordinates in $\mathbb{R}^3$ equipped with the flat metric

$$\mathring{g} = d\rho^2 + \rho^2 d\varphi^2 + d\zeta^2\ ,$$

with all the above functions being φ–independent functions on $\mathbb{R}^3$. Then (2.2) can be rewritten as

$$f\Delta f = |Df|^2 - |Db|^2\ , \tag{2.5}$$

$$f\Delta b = 2(Df, Db)\ . \tag{2.6}$$

where Δ is the flat Laplace operator of the metric $\mathring{g}$, and $(\cdot,\cdot)$ denotes the $\mathring{g}$-scalar product, similarly the norm $|\cdot|$ is the one associated with $\mathring{g}$.

Equations (2.5)-(2.6) degenerate at $\{f = 0\}$, and it is not clear that f or b will smoothly extend across $\{f = 0\}$, if at all. In Section 3 below we give examples of solutions which do *not*. On the other hand, there are large classes of solutions which are smooth across $\mathscr{E}_f$. Examples can be obtained as follows:

First, every space-time obtained from an Ernst map $\mathscr{E}'$ associated to the reduction that uses the axial Killing vector ∂_φ (see, e.g., [3, 21]) will lead to a solution $\mathscr{E}$ as considered here that extends smoothly across the *space-time ergosurfaces* (if any; recall that an ergosurface is defined to be a *timelike* hypersurface where the Killing vector X, which asymptotes a time translation in the asymptotic region, becomes null. Those ergosurfaces correspond then to $\mathscr{E}$-ergosurfaces across which f does indeed extend smoothly. We emphasise that we are interested in the construction of a space-time starting from $\mathscr{E}$, and we have no *a priori* reason to expect that an $\mathscr{E}$–ergosurface, defined as smooth zero-level set of $\Re\mathscr{E}$, will lead to a smooth space-time ergosurface.

Next, large classes of further examples are given in [9, 12, 14–17, 22][2]. Some of the solutions in those references have non-trivial zero-level sets of $\Re\mathscr{E}$, with $g_{\rho\rho} = g_{zz}$ and $g_{t\varphi}$ smooth across E_f (see in particular [12]), but smoothness of $g_{\varphi\varphi}$ is not manifest.

Quite generally, we have the following: consider a vacuum space-time $(\mathscr{M}, g)$ with two Killing vectors X, Y, and with a non-empty *space-time ergosurface* defined as

$$E_{\mathscr{M}} := \{\underbrace{g(X,X) = 0\ ,\ \ X \neq 0}_{(1)}\ ,\ \underbrace{g(X,X)g(Y,Y) - g(X,Y)^2 < 0}_{(2)}\}\ .$$

Condition (1) is the statement that X becomes null on $E_{\mathscr{M}}$, while (2) says that the planes spanned by X and Y are timelike; condition (2) distinguishes a space-time ergosurface from a horizon, where those plane are null. (For solutions in Weyl form, condition (2) translates into the requirement $\rho \neq 0$.) Now, by (2) there exists a linear combination of X and Y which is timelike near $E_{\mathscr{M}}$, and if g is sufficiently differentiable ($H^2_{\rm loc}$ in coordinates adapted to the symmetry group is more than enough), the analysis of [13] shows that there exist an atlas near $E_{\mathscr{M}}$ in which g is analytic. By chasing through the

[2]The solutions we are referring to here are not necessarily vacuum everywhere, and some of them have a function $\mathscr{E}$ which is singular somewhere in the (ρ,ζ) plane. Our analysis applies to the vacuum region, away from the rotation axis, and away from the singularities of the Ernst map $f + ib$.

construction of Weyl coordinates, this implies that f and b are real-analytic functions near E_f. In particular f will *not* have zeros of infinite order there.

3 Static solutions

In the remainder of this work only those solutions which are invariant under rotations around some fixed chosen axis are considered (when viewed as functions on subsets of $\mathbb{R}^3$), and all functions are identified with functions of two variables, ρ and ζ.

Consider a solution of (2.5)-(2.6) with $b \equiv 0$. Setting $u = \ln f$ in the region $\Omega := \{f > 0\}$, Equation (2.5) becomes

$$\Delta u = 0 \quad \text{on} \quad \Omega \,. \tag{3.1}$$

One can now obtain examples of solutions for which E_f is not empty as follows: Let $\alpha \in \mathbb{R}^*$, $\vec{x}_0 = (\rho_0, \zeta_0) \in (0,\infty)\times\mathbb{R}$; standard PDE considerations show existence of solutions of (3.1) on $\Omega := \{(0,\infty)\times\mathbb{R}\} \setminus \{\vec{x}_0\}$ such that

$$u_\alpha = \alpha \ln\Big((\rho-\rho_0)^2 + (\zeta-\zeta_0)^2\Big) + O(1) \,.$$

This leads to

$$f_\alpha := e^{u_\alpha} = \Big((\rho-\rho_0)^2 + (\zeta-\zeta_0)^2\Big)^\alpha g_\alpha \,,$$

where g_α has no zeros. We have the following:

- No such solution is smoothly extendable through the Ernst ergosurface $E_{f_\alpha} = \{\vec{x}_0\}$ except perhaps when $\alpha \in \mathbb{N}^*$.
- In that last case the solutions do not extend smoothly across E_f either, which can be seen as follows. Consider, first $\alpha = 1$, then $f = f_1$ has a zero of order two with positive-definite Hessian, but Lemma 5.2 below shows that no such solutions which are smooth across E_f exist. For general $\alpha = n \in \mathbb{N}^*$ we note that

  $$f_1 = \Big((\rho-\rho_0)^2 + (\zeta-\zeta_0)^2\Big)(g_n)^{\frac{1}{n}} \,.$$

  But smoothness of f_n would imply that of g_n, and thus of f_1, which is not the case. Thus f_n, $n \in \mathbb{N}^*$, cannot be smooth either.

Above we have considered differentiability of f_α in (ρ,ζ)–coordinates. This might *not* be equivalent to the question which is of main interest here, that of regularity of the space-time metric. In the case $b \equiv 0$ this issue is easy to handle, by noting that a can then always be made to vanish by a redefinition of t. Now, the length $g(\partial_\varphi, \partial_\varphi)$ of the Killing vector ∂_φ, generating rotations around the axis, is a smooth — hence locally bounded — function on the space-time. But $g_{\varphi\varphi} = \rho^2 f^{-1}$ by (2.1) so, in the static case, zeros of f with $\rho_0 \neq 0$ cannot correspond to ergosurfaces in space-time[3].

4 Non-critical zeros of f

We start with the following:

THEOREM 4.1 *The conclusion of Theorem 1.1 holds if one moreover assumes that $|Df|$ has no zeros at the $\mathscr{E}$–ergosurface $E_f := \{f = 0, \rho > 0\}$.*

PROOF: We need to show that the functions

$$\alpha := g_{\varphi t} = af \ , \quad \beta := \ln g_{\zeta\zeta} = \ln g_{\rho\rho} = \ln(hf^{-1}) \ ,$$

as well as

$$g_{\varphi\varphi} = \frac{\rho^2 - (af)^2}{f}$$

are smooth across $\{f = 0, \rho > 0\}$, and that $g_{\varphi t}$ does *not* vanish whenever $g_{tt} = f$ does.

We start by Taylor-expanding f and b to order two near any point (ρ_0, ζ_0) such that $f(\rho_0, \zeta_0) = 0$:

$$\begin{aligned}
f(\rho,\zeta) &= \mathring{f}_{,\rho}(\rho-\rho_0) + \mathring{f}_{,\zeta}(\zeta-\zeta_0) \\
&\quad + \frac{1}{2}\mathring{f}_{,\rho\rho}(\rho-\rho_0)^2 + \frac{1}{2}\mathring{f}_{,\zeta\zeta}(\zeta-\zeta_0)^2 + \mathring{f}_{,\rho\zeta}(\rho-\rho_0)(\zeta-\zeta_0) + \ldots, \\
b(\rho,\zeta) &= \mathring{b} + \mathring{b}_{,\rho}(\rho-\rho_0) + \mathring{b}_{,\zeta}(\zeta-\zeta_0) \\
&\quad + \frac{1}{2}\mathring{b}_{,\rho\rho}(\rho-\rho_0)^2 + \frac{1}{2}\mathring{b}_{,\zeta\zeta}(\zeta-\zeta_0)^2 + \mathring{b}_{,\rho\zeta}(\rho-\rho_0)(\zeta-\zeta_0) + \ldots,
\end{aligned}$$

where a circle over a function indicates that the value at ρ_0 and ζ_0 is taken. Inserting these expansions into (2.5)-(2.6), after tedious but elementary al-

[3]The discussion here gives a simple proof, under the supplementary condition of axisymmetry, of the Vishweshwara-Carter lemma, that there are no ergosurfaces in static space-times.

gebra one obtains either

$$\begin{array}{c} \mathring{b}_\rho = \mp \mathring{f}_\zeta , \qquad \mathring{b}_\zeta = \pm \mathring{f}_\rho , \\ \mathring{f}_{,\rho\rho} + \mathring{f}_{,\zeta\zeta} = \frac{\mathring{f}_{,\rho}}{\rho_0}, \quad \mathring{b}_{,\rho\rho} + \mathring{b}_{,\zeta\zeta} = \frac{\mathring{f}_{,\zeta}}{\rho_0}, \quad \mathring{b}_{,\rho\zeta} = \mathring{f}_{,\zeta\zeta}, \quad \mathring{f}_{,\rho\zeta} = \mathring{b}_{,\rho\rho}, \end{array} \tag{4.1}$$

or

$$\mathring{b}_\rho = \mathring{f}_\zeta = \mathring{b}_\zeta = \mathring{f}_\rho = 0 \ . \tag{4.2}$$

The second possibility is excluded by our hypothesis that $Df \neq 0$ on E_f.

Suppose, first, that the lower signs arise in the first line of (4.1). From (2.3) we obtain

$$\alpha_{,\rho} = \frac{f_{,\rho}}{f}\alpha + \frac{\rho}{f} b_{,\zeta} \ , \tag{4.3}$$

$$\alpha_{,\zeta} = \frac{f_{,\zeta}}{f}\alpha - \frac{\rho}{f} b_{,\rho} \ , \tag{4.4}$$

so that

$$\left(\frac{\alpha - \rho}{f}\right)_{,\rho} = \underbrace{[\rho(b_{,\zeta} + f_{,\rho}) - f]}_{=:\sigma_\rho} f^{-2} \ , \tag{4.5}$$

$$\left(\frac{\alpha - \rho}{f}\right)_{,\zeta} = \underbrace{\rho(f_{,\zeta} - b_{,\rho})}_{=:\sigma_\zeta} f^{-2} \ . \tag{4.6}$$

Inserting (4.1) into the definitions of σ_ρ and σ_ζ we find

$$\sigma_\rho = \sigma_\zeta = 0 = d\sigma_\rho = d\sigma_\zeta$$

at every point (ρ_0, ζ_0) lying on the $\mathscr{E}$-ergosurface. Here, as elsewhere, $d\mu$ denotes the differential of a function μ.

Recall that Df does not vanish on $E_f = \{f = 0\}$. We can thus introduce coordinates (x, y) near each connected component of E_f so that $f = x$. Since the σ_a's are smooth we have the Taylor expansions

$$\sigma_a = \sigma_a|_{E_f} + (\partial_x \sigma_a)|_{E_f} x + r_a x^2 \ ,$$

for some remainder terms r_a which are smooth functions on space-time. But we have shown that $\sigma_a|_{E_f} = (\partial_x \sigma_a)|_{E_f} = 0$. Hence

$$\sigma_a = r_a x^2 = r_a f^2 \ ,$$

It follows that the right-hand-sides of (4.5)-(4.6) extend by continuity across E_f to smooth functions. Hence the derivatives of $(\alpha - \rho)/f$ extend by continuity to smooth functions, and by integration

$$\alpha - \rho = f\hat{\alpha} \ , \tag{4.7}$$

for some smooth function $\hat{\alpha}(\rho, \zeta)$. This proves smoothness both of $g_{t\varphi}$ and of $g_{\varphi\varphi}$. We also obtain that $g_{t\varphi} = \rho$ when $f = 0$, and since $\rho > 0$ by assumption we obtain non-vanishing of $g_{t\varphi}$ on that part of the $\mathscr{E}$–ergosurface which does not intersect the rotation axis $\{\rho = 0\}$.

In the case where the upper choice of sign in (4.1) occurs, instead of (4.5)-(4.6) we write equations for $(\alpha + \rho)/f$, and an identical argument applies.

We pass now to the analysis of $g_{\rho\rho} = g_{zz}$. From (2.4),

$$\ln(h/f)_{,\rho} = \underbrace{\frac{1}{2}\left[\rho(f_{,\rho}^2 - f_{,\zeta}^2 + b_{,\rho}^2 - b_{,\zeta}^2) - 2ff_{,\rho}\right]}_{=:\kappa_\rho} f^{-2}, \tag{4.8}$$

$$\ln(h/f)_{,\zeta} = \underbrace{\left[\rho(f_{,\rho}f_{,\zeta} + b_{,\rho}b_{,\zeta}) - ff_{,\zeta}\right]}_{=:\kappa_\zeta} f^{-2}. \tag{4.9}$$

Evaluating κ_a and its derivatives on E_f and using (4.1) one obtains again

$$\kappa_a = d\kappa_a = 0$$

on E_f. As before we conclude that $g_{\rho\rho}$ and $g_{\zeta\zeta}$ are smooth across E_f.

□

5 Higher order zeros of f

We shall say that f has a zero of order n, $n \geq 1$, at $\vec{x}_0 = (\rho_0, \zeta_0)$, if

$$f(\vec{x}_0) = \ldots = \underbrace{D \cdots D}_{n-1 \text{ factors}} f(\vec{x}_0) = 0 \ \ \underline{\text{but}} \ \ \underbrace{D \cdots D}_{n \text{ factors}} f(\vec{x}_0) \neq 0 \ .$$

It is legitimate to raise the question whether solutions of the Ernst equations (2.2) with higher order zeros on E_f exist. A simple mechanism[4] for producing such solutions is the following: consider a family of solutions depending continuously on one parameter, such that for large parameter values

[4] We are grateful to M. Ansorg and D. Petroff for pointing this out to us.

there exist two disjoint ergosurfaces, while for small parameter values the ergosurface is connected. Elementary considerations show that there exists a value of the parameter where f has a zero of higher order. Examples of such behavior have been found numerically by Ansorg [1] in families of differentially rotating disks (however, the merging of the ergosurfaces in that work occurs in the matter region, which is not covered by our analysis). In Figure 1, due to D. Petroff[5], the reader will find an example in a family of solutions with a black hole surrounded by a ring of fluid. Those solutions are globally regular, and the coalescence of ergosurfaces takes place in the vacuum region. A purely vacuum example of this kind is hinted at in [18, Fig. 2]. Finally, Figures 2-4 show a purely vacuum example within the Kramer-Neugebauer family of solutions [9], where the parameters which are being varied are the β_i's of [20]. While the value of the parameters found numerically, for which f has a zero of order two, is only approximate, the existence of a nearby value with a second order zero follows from what has been said above together with the remaining results in this paper.

Similarly three ergosurfaces merging simultaneously will lead to a zero of order precisely three, and so on.

In order to study the zeros of higher order it is convenient to consider Taylor expansions of f and b to order $n \geq k$,

$$f(\rho,\zeta) \;=\; \sum_{0\leq i+j\leq n} \frac{\mathring{f}_{i,j}}{i!j!}(\rho-\rho_0)^i(\zeta-\zeta_0)^j + r_n \;, \qquad (5.1)$$

where

$$\mathring{f}_{i,j} := \frac{\partial^{i+j} f}{\partial^i\rho\,\partial^j\zeta}(\rho_0,\zeta_0) \;.$$

Similarly we denote the Taylor coefficients of b by $\mathring{b}_{i,j}$.

Suppose that f has a zero of order k at $\vec{x}_0$. Note that the value $b(\rho_0,\zeta_0) = \Im\mathscr{E}(\rho_0,\zeta_0)$ is irrelevant both for the equations and for the metric, so without loss of generality we may assume that $\mathscr{E}(\rho_0,\zeta_0)=0$. From now on we assume that this is the case. Let $\mathscr{E}_k$ be the homogeneous polynomial in $\rho-\rho_0$ and $\zeta-\zeta_0$, of order k, obtained by keeping in the Taylor expansion only the terms of first non-vanishing order, similarly f_k. Thus, f_k is a homogeneous polynomial in $\rho-\rho_0$ and $\zeta-\zeta_0$, of order k:

$$f_k(\rho,\zeta) \;=\; \sum_{i+j=k} \frac{\mathring{f}_{i,j}}{i!j!}(\rho-\rho_0)^i(\zeta-\zeta_0)^j \;. \qquad (5.2)$$

[5] We are very grateful to D. Petroff for providing this figure; a detailed analysis of configurations of this type can be found in [2].

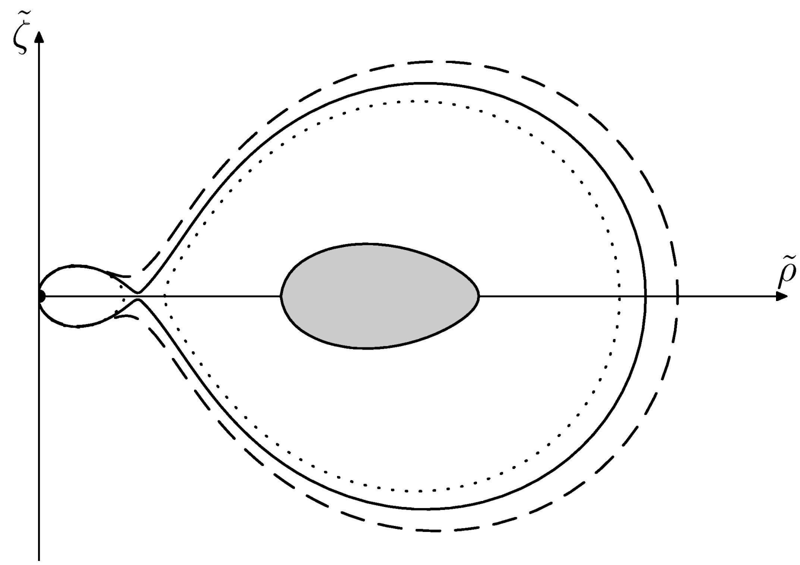

Figure 1: A coordinate representation of ergosurfaces for three configurations consisting of a black hole (coordinate origin) surrounded by a fluid ring (shaded area). The values of the parameters, in the notation of [2], are $\varrho_{\rm i}/\varrho_{\rm o} = 0.55$, $r_{\rm c}/\varrho_{\rm o} = 0.015$, $J_{\rm c}/\varrho_{\rm o}^2 = 0.05$, with $V_0 = -1.45$ (dashed line), $V_0 = -1.396$ (solid line) and $V_0 = -1.35$ (dotted line). The shape of the ring (here corresponding to the ergosurface indicated by a solid line) is represented by the shaded area and differs only minimally between the three configurations. The coordinates $\tilde{\varrho}$ and $\tilde{\zeta}$ are related to ϱ and ζ by a conformal transformation.

The polynomial f_k can be written in a convenient form, (5.4) below, as follows: suppose, for the moment, that $\rho - \rho_0$ is strictly positive, by homogeneity we can then write

$$f_k(\rho,\zeta) = (\rho-\rho_0)^k P_k(w)\;, \qquad \text{where } w := \frac{\zeta-\zeta_0}{\rho-\rho_0}\;, \tag{5.3}$$

for some non-trivial polynomial P_k of order smaller than or equal to k. Let $\alpha_i \in \mathbb{C}$, $i = 1,\ldots,n$, be distinct zeros of P_k, with multiplicities m_i, thus $P_k(w) = C\,\Pi_{i=1}^n (w-\alpha_i)^{m_i}$, for some constant $C \in \mathbb{C}^*$. Hence

$$f_k(\rho,\zeta) = C(\rho-\rho_0)^{m_0}\,\Pi_{i=1}^n\Big(\zeta-\zeta_0-\alpha_i(\rho-\rho_0)\Big)^{m_i}\;, \tag{5.4}$$

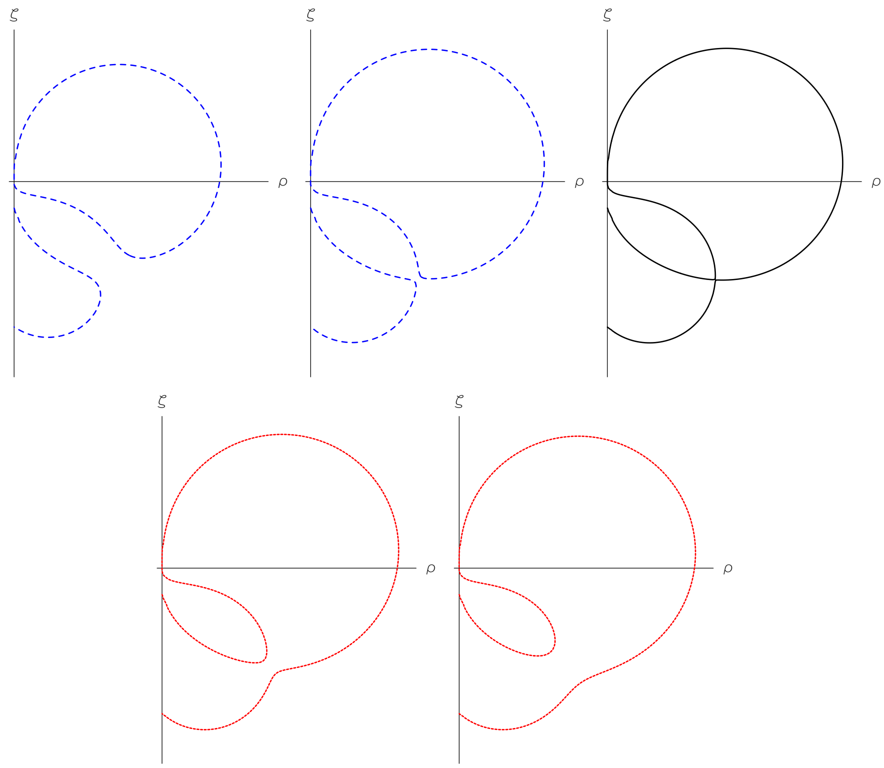

Figure 2: Coalescing ergosurfaces in the "double-Kerr" family of metrics in the (ϱ, ζ) half-plane, with one black hole extreme. We use a parameterization as in [20]. In all five cases the event horizon of the degenerate black hole lies at the origin, thus $\alpha_1 = 0$, while $\alpha_2 = -1/6$, $\alpha_3 = -1$, and with the β_a's, $a = 1, 2$ of the form $\beta_a = -b_a(1 + \mathrm{i})$, where: 1) $b_1 = 0.6$, $b_2 = 1.5$; 2) $b_1 = 0.62$, $b_2 = 1.66$; 3) $b_1 = 0.6218704381$, $b_2 = 1.668809562$; 4) $b_1 = 0.62$, $b_2 = 1.68$; 5) $b_1 = 0.6$, $b_2 = 1.7$. Those solutions have both singular struts at the axis and singular rings away from the region where the coalescing of ergosurfaces occurs, but those singularities are irrelevant for the proof that there are no local obstructions to zeros of higher order.

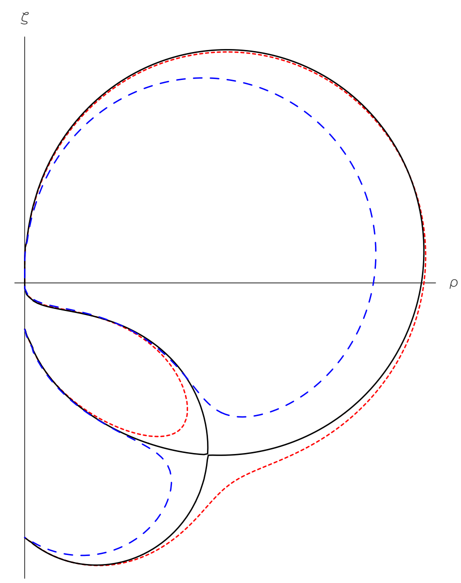

Figure 3: The first, third and fifth ergosurfaces of Figure 2 superimposed.

where $m_0 = k - m_1 - \cdots m_n$. It should be clear that (5.4) remains true for all ρ, and not only for $\rho > \rho_0$ as assumed so far.

We will need the following:

PROPOSITION 5.1 *Assume that f and b are smooth near $\vec{x}_0$. Then the function b, normalised so that $b(\vec{x}_0) = 0$, has a zero at $\vec{x}_0$ of* precisely *the same order as f.*

PROOF: Let $k \in \mathbb{N}$ denote the order of the zero. For $k = 1$ the result has already been established in Section 4, so we assume $k \geq 2$. We then have $f = O(|\vec{x} - \vec{x}_0|^k)$, $|Df| = O(|\vec{x} - \vec{x}_0|^{k-1})$, $\Delta f = O(|\vec{x} - \vec{x}_0|^{k-2})$, and (2.5) shows that

$$|Db|^2 = O(|\vec{x} - \vec{x}_0|^{2k-2}) .$$

Integrating radially around $\vec{x}_0$ gives $b = O(|\vec{x} - \vec{x}_0|^k)$, hence the order of the zero of b is larger than or equal to k.

To show the reverse inequality, suppose that $b = O(|\vec{x} - \vec{x}_0|^{k+1})$. Inserting the Taylor expansion of f into (1.1) one finds that f_k solves the equation

$$f_k \left(\frac{\partial^2 f_k}{\partial \rho^2} + \frac{\partial^2 f_k}{\partial \zeta^2} \right) = \left(\frac{\partial f_k}{\partial \rho} \right)^2 + \left(\frac{\partial f_k}{\partial \zeta} \right)^2 . \qquad (5.5)$$

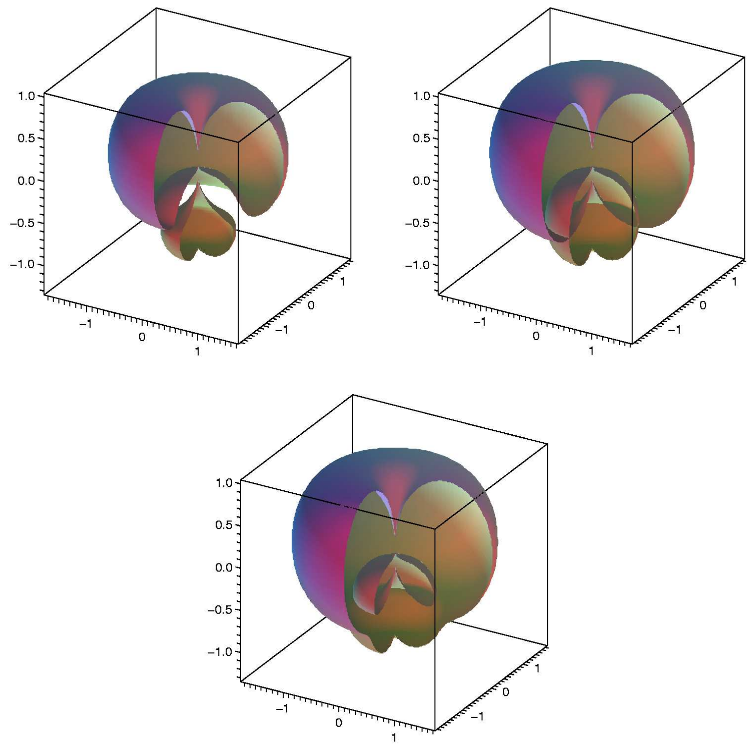

Figure 4: Coordinate representation in $\mathbb{R}^3$ of the first, third and fifth ergosurfaces from Figure 2. The cusps where the ergosurfaces meet the symmetry axis have a geometric character and also arise in the Kerr solution [19].

On the set $\Omega_k := \{f_k > 0\}$ define $u_k = \ln f_k$. Without loss of generality, changing f to $-f$ if necessary, we can assume that Ω_k is non-empty, with $\vec{x}_0$ lying in the closure of Ω_k. On Ω_k, Equation (5.5) is simply the statement that u_k is harmonic in the variables (ρ, ζ):

$$\Delta_2 u_k := \frac{\partial^2 u_k}{\partial \rho^2} + \frac{\partial^2 u_k}{\partial \zeta^2} = 0 \ . \tag{5.6}$$

From (5.4) we have, assuming $C \neq 0$,

$$u_k = \ln C + m_0 \ln(\rho - \rho_0) + \sum_{i=1}^{n} m_i \ln\Big(\zeta - \zeta_0 - \alpha_i(\rho - \rho_0)\Big) \ .$$

Inserting into (5.6) one obtains

$$\Delta_2 u_k \quad = \quad -m_0 \frac{1}{(\rho-\rho_o)^2} - \sum_{i=1}^{n} m_i(1+\alpha_i^2) \frac{1}{\Big(\zeta-\zeta_0-\alpha_i(\rho-\rho_0)\Big)^2} = 0 \ .$$

Recalling that the α_i's are distinct, this is only possible if

$$m_0 = 0 = m_i(1+\alpha_i^2) \quad \forall\ i \ .$$

Reordering the m_i's if necessary, as u_k is real-valued we have proved that

$$u_k = \ln C + m_1 \ln \Big((\zeta-\zeta_0)^2+(\rho-\rho_0)^2\Big) \quad \Longleftrightarrow \quad f_k = C\Big((\zeta-\zeta_0)^2+(\rho-\rho_0)^2\Big)^{m_1} .$$

Subsequently,

$$f = C\Big((\zeta-\zeta_0)^2+(\rho-\rho_0)^2\Big)^{m_1} + O(|\vec{x}-\vec{x}_o|^{k+1}) \ . \tag{5.7}$$

As the order of $\vec{x}_0$ is even, this proves Proposition 5.1 for all k odd.

To continue, we note the following

LEMMA 5.2 *Under the hypotheses of Proposition 5.1, let $\vec{x}_0$ be a zero of order two. Then the quadratic form defined by the Hessian $DDf(\vec{x}_0)$ of f has signature $(+-)$ or $(-+)$. This implies that second order zeros of f are isolated.*

REMARK 5.3 For further use we note that the derivation of (5.8)-(5.9) only uses the truncated equations (5.13)-(5.14) below. Furthermore, the calculations here — and therefore their conclusions — remain valid when a supplementary error term $o(|\vec{x}-\vec{x}_0|^3)$ is allowed at the right-hand-side of (2.5).

PROOF: The result is obtained by a calculation, the simplest way proceeds as in the proof of Theorem 5.6 below. Alternatively, one can use MAPLE or MATHEMATICA, the interested reader can download the worksheets from `http://th.if.uj.edu.pl/~szybka/CMS`; that last calculation has been done as follows: Consider the polynomials W_a, $a=1,2$, obtained by inserting the Taylor expansion of f and b, with $\mathring{f} = D\mathring{f} = 0$, into equations obtained by multiplying (2.5) and (2.6) with ρ. The requirement that those polynomials vanish up-to-and-including order two imposes the following alternative sets of conditions:

$$\text{I.} \qquad \mathring{b}_{2,0} = \mathring{b}_{1,1} = \mathring{b}_{0,2} = \mathring{f}_{1,1} = 0 \ , \quad \mathring{f}_{2,0} = \mathring{f}_{0,2} \in \mathbb{R} \ , \tag{5.8}$$

$$\text{II.} \quad \mathring{f}_{2,0} = -\mathring{f}_{0,2} = -\mathring{b}_{1,1} \in \mathbb{R} \ , \quad \mathring{b}_{0,2} = -\mathring{f}_{1,1} = -\mathring{b}_{2,0} \in \mathbb{R} \ , \tag{5.9}$$

as well as a set which is related to II. above by exchanging b with $-b$. The first set leads to $\mathring{f}_{1,1} = 0$ when requiring that the polynomials W_a just defined vanish to one order higher, so that the first set cannot occur for zeros of second order. One then checks that the set II. leads to Lorentzian signature of DDf, unless vanishing. □

Clearly the Hessian of f given by (5.7) does not have indefinite signature when $m_1 = 1$, proving Proposition 5.1 for zeros of order two.

It remains to consider $m_1 \in \mathbb{N}$ satisfying $m_1 \geq 2$. Replacing f by $-f$ if necessary, it follows from (5.7) that f is strictly positive in a neighborhood of $\vec{x}_0$, so that we can define

$$g := f^{1/m_1} .$$

Usual arguments (cf., e.g., [11]), show that g is smooth and has a zero of order two at $\vec{x}_0$. From (2.5) one has

$$g\Delta g - |Dg|^2 = \frac{1}{m_1} g^2 \underbrace{\frac{|Db|^2}{f^2}}_{O(1)} = O(|\vec{x} - \vec{x}_0|^4) . \tag{5.10}$$

Taylor expanding g up to order $o(|\vec{x} - \vec{x}_0|^4)$ and inserting into (5.10) gives $C = 0$ (see Remark 5.3), proving Proposition 5.1. □

5.1 Simple zeros

A zero of f of order k will be said to be *simple* if all the α_i's in (5.4) are real and have multiplicities one, with $m_0 \in \{0, 1\}$. We will show below that zeros of finite order of solutions of Ernst equations are simple. Somewhat to our surprise, for such zeros Theorem 4.1 generalises:

THEOREM 5.4 *The conclusions of Theorem 1.1 are valid under the supplementary condition that f has only simple zeros at the $\mathscr{E}$–ergosurface $E_f := \{f = 0, \rho > 0\}$.*

PROOF: As pointed out by Malgrange [11, end of Section 3], simplicity implies that near $\vec{x}_0$ there exist smooth functions ϕ_a, $a = 1, \ldots, k$, with $\phi_a(\vec{x}_0) = 0$ and with nowhere-vanishing gradient, together with a strictly positive smooth function g such that we can write

$$f = \phi_1 \cdots \phi_k g . \tag{5.11}$$

(Supposing that $m_0 = 0$, the ϕ_a's have the Taylor expansion $\phi_a = \zeta - \zeta_0 - \alpha_a(\rho - \rho_0) + O(|\vec{x} - \vec{x}_0|^2)$; if $m_0 = 1$, then one has $\phi_1 = \rho - \rho_0 + O(|\vec{x} - \vec{x}_0|^2)$, with the remaining Taylor expansions of the same form as before. For $k = 2$ this is a special case of Morse's theorem [6, Theorem 6.9, p. 65].)

Equation (5.11) shows that E_f is, near $\vec{x}_0$, the union of the smooth submanifolds $\{\phi_a = 0\}$. On each of those Df is non-vanishing, except at the origin. Passing to a small neighborhood of $\vec{x}_0$ if necessary, we can assume that each of the sets $\{\phi_a = 0, Df \neq 0\}$ has precisely two components.

Consider a connected component of $\{\phi_1 = 0\}$, by Section 4 Equation (4.1) holds there. Suppose that the lower sign arises on this component, then the same lower sign has to arise on the remaining component of $\{\phi_1 = 0\}$, because the inversion $\vec{x} - \vec{x}_0 \to -\vec{x} + \vec{x}_0$ maps each component to the accompanying one up to quadratic terms, and because Db has, in the leading order of its Taylor development, the same parity as Df by Proposition 5.1.

We consider the function σ_ρ as in (4.5), an identical argument applies to σ_ζ and to κ_ρ, κ_ζ. Using a coordinate system (y^1, y^2) with $\phi_1 = y^1$ we have a Taylor expansion

$$\sigma_\rho(y^1, y^2) = \sigma_{\rho,0}(y^2) + \sigma_{\rho,1}(y^2)y^1 + \sigma_{\rho,2}(y^1, y^2)(y^2)^2 \, . \tag{5.12}$$

Note that f has a simple zero away from the origin on the axis $\{y^1 = 0\}$, so by the results in Section 4 the functions $\sigma_{\rho,0}$ and $\sigma_{\rho,1}$ vanish there. By continuity they also vanish at the origin, thus σ_ρ factorises as

$$\sigma_\rho = \sigma'_\rho \phi_1^2$$

for a smooth function $\sigma'_\rho := \sigma_{\rho,2}$.

We introduce a new coordinate system (z^1, z^2) in which $z^1 = \phi_2$. We Taylor expand σ'_ρ as in (5.12), with the y^i's there replaced by z^i's, etc. The equations

$$\begin{aligned} 0 &= \sigma_\rho|_{z^1=0,z^2\neq 0} = \sigma'_\rho|_{z^1=0,z^2\neq 0} \underbrace{\phi_1^2|_{z^1=0,z^2\neq 0}}_{\neq 0} \, , \\ 0 &= d\sigma_\rho|_{z^1=0,z^2\neq 0} = d\sigma'_\rho|_{z^1=0,z^2\neq 0} \underbrace{\phi_1^2|_{z^1=0,z^2\neq 0}}_{\neq 0} \\ & \quad +2 \underbrace{\sigma'_\rho|_{z^1=0,z^2\neq 0}}_{=0} (\phi_1 d\phi_1)|_{z^1=0,z^2\neq 0} \, , \end{aligned}$$

show that the function σ'_ρ vanishes, together with its first derivatives, away from the origin on the axis $\{z^1 = 0\}$. We conclude as before that σ'_ρ factorises as $\sigma'_\rho = \sigma''_\rho \phi_2^2$ for a smooth function σ''_ρ, hence σ_ρ factorises as

$$\sigma_\rho = \sigma''_\rho \phi_1^2 \phi_2^2 \ .$$

Continuing in this way, in a finite number of steps one obtains

$$\sigma_\rho = \hat{\sigma}_\rho \phi_1^2 \cdots \phi_k^2 \ ,$$

and the result easily follows. □

5.2 Zeros of finite order are simple

Consider a zero of f of order $k < \infty$, with $\rho_0 > 0$, then the leading order Taylor polynomials f_k and b_k solve the truncated equations

$$f_k \left(\frac{\partial^2 f_k}{\partial \rho^2} + \frac{\partial^2 f_k}{\partial \zeta^2} \right) = \left(\frac{\partial f_k}{\partial \rho} \right)^2 + \left(\frac{\partial f_k}{\partial \zeta} \right)^2 - \left(\frac{\partial b_k}{\partial \rho} \right)^2 - \left(\frac{\partial b_k}{\partial \zeta} \right)^2 \ , \quad (5.13)$$

$$f_k \left(\frac{\partial^2 b_k}{\partial \rho^2} + \frac{\partial^2 b_k}{\partial \zeta^2} \right) = \frac{\partial f_k}{\partial \rho} \frac{\partial b_k}{\partial \rho} + \frac{\partial f_k}{\partial \zeta} \frac{\partial b_k}{\partial \zeta} \ . \quad (5.14)$$

Let

$$f_k + \mathrm{i} b_k \equiv \mathscr{E}_k = \alpha (z - z_0)^k \ , \quad (5.15)$$

where $\alpha \in \mathbb{C}$, with $z = \rho + \mathrm{i}\zeta$. It is straightforward, using the Cauchy-Riemann equations, to check that functions of this form satisfy (5.13)-(5.14), for all $k \in \mathbb{N}$. (In fact, both the left- and right-hand-sides then vanish identically.) Those solutions have been found by inspection of the solutions found by MAPLE for $k = 2$ and by SINGULAR [7,8] for $k = 3$ and 4. In fact, both the SINGULAR–generated solutions, as well as our remaining computer experiments using SINGULAR, played a decisive role in our solution of the problem at hand.

Let us show that:

LEMMA 5.5 *Zeros of f_k given by (5.15) are simple.*

PROOF: Indeed, the equation $f_k = 0$ is equivalent to

$$\alpha (z - z_0)^k = \mathrm{i}\beta \ ,$$

for some $\beta \in \mathbb{R}$. This is easily solved; we write $\alpha = |\alpha| e^{\mathrm{i}\theta}$, and set

$$\alpha_\ell = \tan \left(\frac{(2\ell + 1)\pi - 2\theta}{2k} \right) \ , \quad \ell = 1, \ldots, k \ .$$

Assuming $\alpha_\ell \neq \pm\infty$ for all ℓ, we obtain k *distinct* real lines $z_0 + \mathbb{R}(1 + \mathrm{i}\alpha_\ell)$ on which $\Re\mathscr{E}_k$ vanishes, and simplicity follows. The remaining cases are analysed similarly, and are left to the reader. □

Another non-trivial, "polarised", family of solutions is provided by $b_k = 0$, $f_k = C\Big((\rho-\rho_0)^2 + (\zeta-\zeta_0)^2\Big)^m$, $m \in \mathbb{N}$. As mentioned in Section 3, there exist associated *static* solutions of the Ernst equations. However, as already pointed out (compare Remark 5.3), neither those, nor any other solutions with this f_k, b_k, are smooth across E_f.

Setting $z = \rho - \rho_0 + \mathrm{i}(\zeta - \zeta_0)$, the equations satisfied by $\mathscr{E}_k = f_k + \mathrm{i}b_k$ take the form

$$(\mathscr{E}_k + \overline{\mathscr{E}}_k)\frac{\partial^2 \mathscr{E}_k}{\partial z \partial \bar z} = 2\frac{\partial \mathscr{E}_k}{\partial z}\frac{\partial \mathscr{E}_k}{\partial \bar z}\ . \tag{5.16}$$

Since $\mathscr{E}_k$ is a polyhomogeneous polynomial in x and y, it can be written as

$$\mathscr{E}_k = \sum_{m=0}^{k} \beta_m z^m \bar z^{k-m}\ .$$

Inserting this into (5.16) we obtain

$$\sum_{1\le m+j\le 2k-1} m\beta_m\{(2j-k-m)\beta_j + (k-m)\bar\beta_{k-j}\} z^{m+j-1}\bar z^{2k-m-j-1} = 0\ .$$

Hence, for $1 \le \ell \le 2k-1$,

$$\sum_{m+j=\ell}\{(k-m)\bar\beta_{k-j} - (k+m-2j)\beta_j\}m\beta_m = 0\ . \tag{5.17}$$

Since $\ell = 0$ is trivial, we obtain $2k-2$ equations for $k+1$ numbers β_m, which should be rather restrictive, especially for $k \ge 3$. Nevertheless, as already pointed out, there exist non-trivial solutions. It is instructive to find them directly by inspection of (5.17). First, there is the obvious solution $\beta_m = 0$ for $m \ge 1$, which corresponds to an anti-holomorphic $\mathscr{E}_k = \beta_0 \bar z^k$. Next, one checks that a collection with $\beta_k \neq 0$ but $\beta_m = 0$ for $m < k$ provides a solution, which corresponds to a holomorphic $\mathscr{E}_k = \beta_k z^k$. Finally, when $k = 2n$, one checks that $\beta_n \in \mathbb{R}$, but $\beta_m = 0$ for $m \neq k/2$, is a solution, which corresponds to a real $\mathscr{E}_{2n} = \beta_n z^n \bar z^n = \beta_n(x^2+y^2)^n$.

The computer algebra program SINGULAR can be used to show that the above exhaust the list of solutions for k less than or equal to eight[6]. This turns out to be true for all $k < \infty$:

[6]The SINGULAR input file is available on URL `http://th.if.uj.edu.pl/~szybka/CGMS`

THEOREM 5.6 *These are all solutions: thus the homogeneous polynomial $\mathscr{E}_k$ is either holomorphic, or anti-holomorphic, or real and radial.*

PROOF: The case $k = 1$ is a straightforward calculation, so we assume $k > 1$.

If $\mathscr{E}_k$ is a solution, then so is its complex conjugate; this implies that if an ordered collection $\{\beta_m\}_{0\le m\le k}$ satisfies (5.17), then so does $\{\bar\beta_{k-m}\}_{0\le m\le k}$. Inserting this into (5.17) one obtains, again for $1 \le \ell \le 2k-1$,

$$\sum_{m+j=\ell} \{(k-m)\beta_j - (k+m-2j)\bar\beta_{k-j}\} m\bar\beta_{k-m} = 0 \ . \tag{5.18}$$

Consider (5.17) with $\ell = 1$; since $1 \le m \le \ell$ this enforces $m = 1$, $j = 0$, giving

$$\{(k-1)\bar\beta_k - (k+1)\beta_0\}\beta_1 = 0 \ . \tag{5.19}$$

Similarly (5.18) with $\ell = 1$ gives

$$\{(k-1)\beta_0 - (k+1)\bar\beta_k\}\bar\beta_{k-1} = 0 \ . \tag{5.20}$$

Suppose, first, that $\beta_0 \neq 0$. We will use induction arguments to establish the implication (5.21) below. So assume, for contradiction, that $\beta_1 \neq 0$. Then $\bar\beta_k = (k+1)\beta_0/(k-1) \neq 0$ from (5.19), inserting into (5.20) we obtain $\beta_{k-1} = 0$. But then (5.18) with $\ell = 2$ gives

$$0 = \{(k-2)\beta_0 - (k+2)\bar\beta_k\}\bar\beta_{k-2} = \underbrace{\frac{(k-2)(k-1)-(k+2)(k+1)}{k-1}\beta_0}_{\neq 0}\bar\beta_{k-2} \ ,$$

hence $\beta_{k-2} = 0$. Equation (5.18) with $\ell = 3$ similarly gives now $\beta_{k-3} = 0$. Continuing in this way one concludes in a finite number of steps that $\beta_1 = 0$, a contradiction. It follows that $\beta_0 \neq 0$ enforces $\beta_1 = 0$.

Assume now, again for contradiction, that $\beta_0 \neq 0$ and $\beta_1 = 0$ but $\beta_2 \neq 0$. Equation (5.17) with $\ell = 2$ gives

$$\{(k-2)\bar\beta_k - (k+2)\beta_0\}\beta_2 = 0 \ .$$

If $k = 2$ we obtain immediately a contradiction; otherwise $\bar\beta_k = (k+2)\beta_0/(k-2) \neq 0$, inserting into (5.20) we find $\beta_{k-1} = 0$. But then (5.18) with $\ell = 2$ gives

$$0 = \{(k-2)\beta_0 - (k+2)\bar\beta_k\}\bar\beta_{k-2} = \underbrace{\frac{(k-2)^2-(k+2)^2}{k-1}\bar\beta_k}_{\neq 0}\bar\beta_{k-2} \ ,$$

hence $\beta_{k-2} = 0$. Continuing in this way one concludes in a finite number of steps that $\beta_2 = 0$, a contradiction. This shows that $\beta_0 \neq 0$ and $\beta_1 = 0$ but $\beta_2 \neq 0$ is incompatible with the equations.

It should be clear to the reader how to iterate this argument to obtain the implication

$$\beta_0 \neq 0 \text{ implies } \beta_m = 0 \text{ for } m = 1, \ldots, k \ . \tag{5.21}$$

Using symmetry under complex conjugation, the hypothesis $\beta_k \neq 0$ leads to $\beta_m = 0$ for $m = 0, \ldots, k-1$.

It remains to analyse what happens when $\beta_0 = \beta_k = 0$, which we assume from now on. Suppose, for contradiction, that $\beta_1 \neq 0$. Recalling that $k > 1$, (5.17)-(5.18) with $\ell = 2$ give

$$(\bar{\beta}_{k-1} - \beta_1)\beta_1 = 0 = (\beta_1 - \bar{\beta}_{k-1})\bar{\beta}_{k-1} \ .$$

If $k = 2$ we obtain $\beta_1 \in \mathbb{R}$, and we are done.

Otherwise $\beta_{k-1} = \bar{\beta}_1 \neq 0$ and (5.18) with $\ell = 3$ gives

$$\{(k-1)\beta_2 - (k+1)\bar{\beta}_{k-2}\}\beta_1 = 0 \ .$$

When $k = 3$ this gives a contradiction, and the result is established for this value of k.

For $k \geq 4$ the proof will be finished by more induction arguments, as follows: Suppose, to start with, that $\beta_k = 0$ and that there exist $k_0, k_1 \in \mathbb{N}$, $1 \leq k_1 \leq k_0 \leq k/2$, such that $\beta_m = 0$ for $0 \leq m \leq k_0 - 1$ and for $k - k_1 < m \leq k$ but $\beta_{k_0} \neq 0$. (The case $k_1 > k_0$ can be reduced to this one by replacing $\mathscr{E}_k$ with its complex conjugate.) With these hypotheses (5.17) can be rewritten as

$$\sum_{k_0 \leq m \leq \min(k-k_1, \ell-k_1)} \{(k-m)\bar{\beta}_{k-(\ell-m)} - (k+3m-2\ell)\beta_{\ell-m}\} m\beta_m = 0 \ . \tag{5.22}$$

Equation (5.22) with $\ell = k_0 + k_1 \leq k$ gives:

$$(k-k_0)\bar{\beta}_{k-k_1} = (k+k_0-2k_1)\beta_{k_1}$$

which equals zero unless $k_1 = k_0$. It follows that we can without loss of generality assume that $k_1 = k_0$ and

$$\beta_{k-k_0} = \bar{\beta}_{k_0} \ .$$

We can now rewrite (5.18) as

$$\sum_{k_0 \le m \le \min(k-k_0, \ell-k_0)} \{(k-m)\beta_{\ell-m} - (k+3m-2\ell)\bar{\beta}_{k-(\ell-m)}\} m\bar{\beta}_{k-m} = 0 \ . \tag{5.23}$$

Suppose that $k = 2k_0$; then (5.23) leads immediately to the restriction $\beta_{k_0} \in \mathbb{R}$, giving a real radial solution, as desired. Otherwise, choosing $\ell = 2k_0 + 1$ in (5.23) one obtains

$$(k - k_0)k_0\beta_{k_0+1} = [(k-k_0)k_0 + 2]\bar{\beta}_{k-k_0-1} \ .$$

Equation (5.22) with $\ell = 2k_0 + 1$ gives

$$(k - k_0)k_0\bar{\beta}_{k-k_0-1} = [(k-k_0)k_0 + 2]\beta_{k_0+1} \ .$$

It follows that

$$\beta_{k_0+1} = \beta_{k-k_0-1} = 0 \ .$$

Our aim now is to show (5.24) below, by a last induction. So, suppose there exists $k_2 \in \mathbb{N}$ satisfying $k_0 < k_2 < k - k_0$ such that $\beta_m = 0$ for $k_0 < m < k_2$ and for $k - k_2 < m < k - k_0$; we have shown that this is true with $k_2 = k_0 + 2$. Equation (5.23) with $\ell = k_0 + k_2$ gives

$$(k - k_0)k_0\beta_{k_2} = [(k-k_0)k_0 + 2(k_2 - k_0)^2]\bar{\beta}_{k-k_2} \ .$$

But from (5.22) again with $\ell = k_0 + k_2$ one obtains

$$(k - k_0)k_0\bar{\beta}_{k-k_2} = [(k-k_0)k_0 + 2(k_2 - k_0)^2]\beta_{k_2} \ .$$

This allows us to conclude that

$$\beta_m = 0 \ \text{ except if } \ m = k_0 \ \text{ or if } \ m = k - k_0 \ , \quad \text{with} \ \ \beta_{k-k_0} = \bar{\beta}_{k_0} \ . \tag{5.24}$$

Equation (5.23) with $\ell = k$ gives now $\beta_{k_0} = 0$ (recall that we have assumed $k \neq 2k_0$), a contradiction, and the theorem is proved. □

We can now pass to the

Proof of Theorem 1.1: Theorem 5.6 gives the list of all possible $\mathscr{E}_k$'s. The real ones do not lead to smooth f's by Proposition 5.1. The holomorphic ones lead to simple zeros by Lemma 5.5; the same is true for the anti-holomorphic ones, because the condition of simplicity is preserved by complex-conjugation of $\mathscr{E}$. The result follows now from Theorem 5.4. □

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