Mathematics Between Research, Application, and Communication

Gert-Martin Greuel

Abstract Possibly more than any other science, mathematics of today finds itself between the conflicting demands of research, application, and communication.

A great part of modern mathematics regards itself as searching for inner mathematical structures just for their own sake, only committed to its own axioms and logical conclusions. To do so, neither assumptions nor experience nor applications are needed or desired.

On the other hand, mathematics has become one of the driving forces in scientific progress and moreover, has even become a cornerstone for industrial and economic innovation. However, public opinion stands in strange contrast to this, often displaying a large amount of mathematical ignorance.

In this article, which is addressed to a readership without any special mathematical education, I shall look at these tensions and try to reveal some of the causes that lie underneath. I will also try to explain a current scientific research topic, but mainly focus on the question whether it is possible or necessary to transmit an understanding of mathematics to the general public.

I am aware that my ideas on mathematics, which are presented here as theses in a rather dense form, certainly do need further elaboration. Nevertheless I hope that they are interesting enough to stimulate further discussions.

Wise Words

Let me introduce my conception regarding research, application, and communication by first quoting some celebrated personalities, and developing my own point of view afterwards.

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Research

In this context, I mean by research pure scientific work carried out at universities and research institutes. Trying to explain concrete mathematical research to a non-mathematician is one of the hardest tasks, if at all possible. But it is possible to explain the motivation of a mathematician to do research.

Therefore, my first thesis refers to this motivation and is a quote from Albert Einstein (physicist 1879–1955) from 1932:

> "The scientist finds his reward in what Henri Poincaré calls the joy of comprehension, and not in the possibilities of application to which any discovery of his may lead." [1]

Indeed the "joy of comprehension" is both motivation and reward at the same time and from my own experience I know that most scientists would fully agree with this statement.

Application

One could hold numerous lectures on the application of mathematics, probably forever. The involvement of mathematics in other sciences, economics, and society is so dynamic that after having demonstrated one application one could immediately continue to lecture on the resulting new applications.

The thesis concerning the application of mathematics consists of three quotes, by Galileo Galilei (mathematician, physicist, astronomer; 1564–1642), Alexander von Humboldt (natural scientist, explorer; 1769–1859), and Werner von Siemens (inventor, industrialist; 1816–1892), in chronological order:

> "Mathematics is the language in which the universe is written." [2]
> "Mathematical studies are the soul of all industrial progress." [3]
> "Without mathematics one is left in the dark." [4]

These statements seem to indicate that there is a direct relation between mathematical research and applications. In fact, recent research directions in geometry are motivated by new developments in theoretical physics, while research in numerical analysis and stochastics is often directed by challenges from various fields of application. On the other hand, the development of an axiomatic foundation of mathematics is guided by trying to formalize mathematical structures in a coherent way and not by the motivation to understand nature or to be useful in the sense of applications. Partly due to this development, it appears that the relationship between research-orientated or pure mathematics on the one hand and application-orientated or applied mathematics on the other hand is not without its strains. Some provocative statements in this article will illustrate this.

Communication

Each of us, whether a mathematician or not, is aware how difficult it is to communicate mathematics. Hans Magnus Enzensberger (German poet and essayist, born 1929) discussed the problem of communicating mathematics on a literary basis in 1999. He writes:

> "Surely it is an audacious undertaking to attempt to interprete mathematics to a culture distinguished by such profound mathematical ignorance." [5]

The exhibition *IMAGINARY—Through the Eyes of Mathematics* is one attempt to interprete and communicate mathematics to a broad audience and there exist many other examples of successful communication. Nevertheless, the problem remains and will be discussed later when I shall give some reasons why it is so difficult.

Let me now start with a more detailed analysis of the above topics.

On the Birth of Mathematical Ideas

Mathematical research has several aspects indeed, but here I am going to have a closer look at only two of them: the research topics themselves and the way in which mathematical research is performed. The latter aspect concerns the way in which mathematical ideas arise. This is an extremely creative process, which happens quite frequently in interaction with other mathematicians.

I am the Director of the Mathematisches Forschungsinstitut Oberwolfach (MFO), an internationally renowned research institute situated in the heart of the Black Forest. Those who are not familiar with mathematical research might get an impression of the process by the following description of a research center which is specially designed to foster interactions among mathematicians and to inspire creative thinking. Mathematicians simply refer to the institute after the village where it is located: Oberwolfach (Fig. 1).

The MFO has become world famous as a birthplace for mathematical ideas; some people call it a "paradise of mathematics". The following anecdotes will illustrate the degree of awareness and esteem of the institute. One time at lunch, when I asked a young American mathematician whether she had known Oberwolfach before, she answered: "To be honest, Oberwolfach is the only German word I know". And a well-known senior mathematician said with a twinkle in his eyes that the only invitations he accepts without his wife's permission, are those to an Oberwolfach workshop.

The Oberwolfach model has become so successful that many institutes all over the world have followed its example. So let me explain the main aspects of this model.

Fig. 1 General view of the Oberwolfach Institute [6]

How Do Mathematical Ideas Arise in Oberwolfach?

Mathematical research mainly studies the structure and inner relationships of mathematical objects and tries to develop more comprehensive theories about them. Many mathematical questions derive from the effort to describe nature in mathematical terms, but it often happens that the mathematical frame was created before the applications. The process of research, when successful, leads to mathematical theorems, whose proofs are typically complicated.

Historically, coincidence also plays a big role. Improving the chances for progress by coincidence is one of the main purposes of the meetings at the Oberwolfach Institute. When getting to know the background of an important result during a talk, one can suddenly have a flash of insight, perhaps leading to considerable progress in one's own research. Small group discussions, inviting the fresh thoughts and comments of colleagues, can lead to a sharing of these insights and to finding the right direction for further work. It happens quite often that two or three colleagues, during such discussions, become aware that they, though coming from different backgrounds and with different motivations, are interested in similar problems and are able to unify their ideas in order to establish a common research project.

This happens nearly daily in these workshops, so that a great number of important papers have been initiated at Oberwolfach in this manner. In contrast to the typically large conferences all over the world, the small workshops at Oberwolfach focus on active research where open questions abound [7].

The final write-up of a proof is usually best done at the home institute, but the development of a mathematical theory and, within such a theory, the promising idea for a proof, is an extremely creative process depending very much on intuition and experience and certainly benefiting from an exchange of ideas. This is why personal contact between the researchers is a crucial point at Oberwolfach. The famous

"Oberwolfach atmosphere" is completely free from distractions and enables discussions among specialists, but also between young mathematicians at the beginning of their career and famous experts.

History of Oberwolfach

In order to be able to understand the myth of Oberwolfach we have to look back to its beginnings after World War II. As early as 1946 the first small meetings were held at the old hunting lodge "Lorenzenhof". Among the participants were mathematicians like the Frenchman Henri Cartan, whose home country had been an "arch-enemy" for centuries. His family had suffered tremendously under the regime of the National Socialists so that his participation was not at all a matter of course. The first famous guests visiting the Lorenzenhof were Heinz Hopf (a world-famous topologist from Zurich, a German of Jewish descent who had moved from Germany to Switzerland in 1931) and Henri Cartan (the "grand maître" of complex analysis from Paris). It was said that "Without Hopf and Cartan Oberwolfach would have remained a summer resort for mathematicians, where in a leisurely atmosphere dignified gentlemen would polish classic theories" [8].

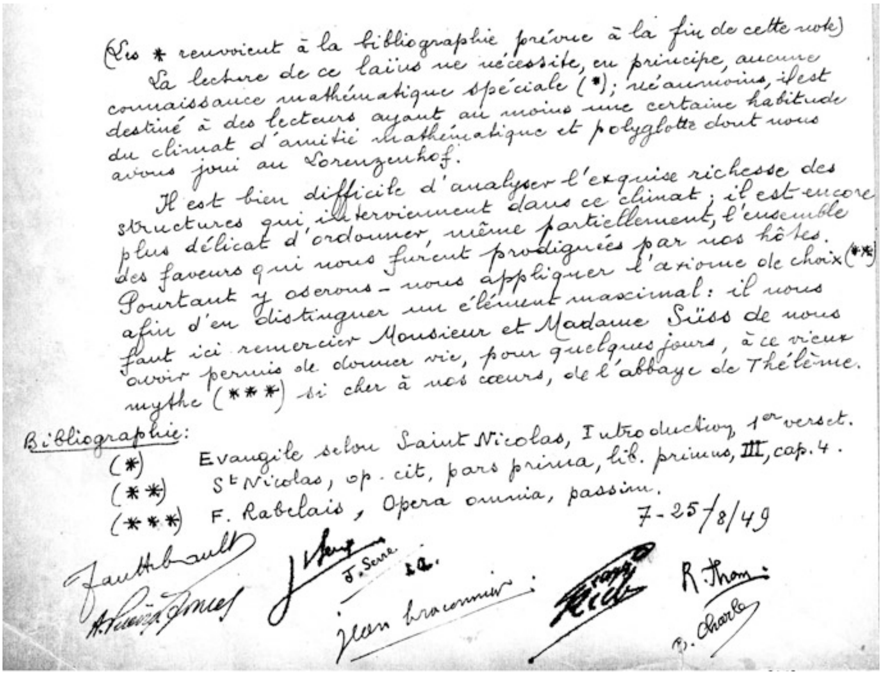
In August 1949 a group of young "wild" Frenchmen met in Oberwolfach who had taken up the cause of totally rewriting mathematics as a whole, based on the axiomatic method and aiming at a new unification. It was a truly bold venture that only young people would dare to take up. Some of their names have become famous, including Henri Cartan, Jean Dieudonné, Jean Pierre Serre, Georges Reeb, and René Thom. A photo from that time was only discovered a few years ago (Fig. 2). It shows part of the group in the autumn of 1949. Cartan himself could not come, due to the consequences of a car accident.

On the far left you can see René Thom, the later Fields Medalist and founder of "Catastrophe Theory", and in the middle Jean-Pierre Serre, also later Fields Medalist and winner of the first Abel Prize.

The Gospel According to St Nicolas and the Freedom of Research

In the first guest book they wrote down the *Evangile selon Saint Nicolas*, Fig. 3, a humorous homage on the Lorenzenhof and its famous Oberwolfach atmosphere, endorsed with mathematical hints. The name *Evangile selon Saint Nicolas* is an allusion to the works of Nicolas Bourbaki, an alias for that group of French mathematicians who wanted to rewrite mathematics entirely from scratch. During that time, almost no one in Germany had heard of Bourbaki. During the Nazi period, the so-called "Deutsche Mathematik" simply missed some important developments in mathematics, for instance in topology, the theory of distributions, and in complex and algebraic geometry. It is one of the most extraordinary achievements of the

Fig. 2 From left to right: René Thom, Jean Arbault, Jean-Pierre Serre and his wife Josiane, Jean Braconnier and Georges Reeb [9]



(Les * renvoient à la bibliographie prévue à la fin de cette note)

La lecture de ce laïus ne nécessite, en principe, aucune connaissance mathématique spéciale (*); néanmoins, il est destiné à des lecteurs ayant au moins une certaine habitude du climat d'amitié mathématique et polyglotte dont nous avons joui au Lorenzenhof.

Il est bien difficile d'analyser l'exquise richesse des structures qui interviennent dans ce climat; il est encore plus délicat d'ordonner, même partiellement, l'ensemble des faveurs qui nous furent prodiguées par nos hôtes. Pourtant y oserons-nous appliquer l'axiome de choix (**) afin d'en distinguer un élément maximal: il nous faut ici remercier Monsieur et Madame Süss de nous avoir permis de donner vie, pour quelques jours, à ce vieux mythe (***) si cher à nos cœurs, de l'abbaye de Thélème.

Bibliographie:

(*) Evangile selon Saint Nicolas, Introduction, 1er verset.

(**) St Nicolas, op. cit, pars prima, lib. primus, III, cap. 4.

(***) F. Rabelais, Opera omnia, passim.

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J. Serre

Jean Braconnier

R. Thom

Fig. 3 Evangile selon Saint Nicolas from the first guest book of Oberwolfach [11] small Oberwolfach workshops that those mathematicians who stayed in Germany and were not expelled by the Nazis were able to join the world's elite again [10].

The myth of the Abbaye de Thélème mentioned by the authors of the *Evangile selon Saint Nicolas* refers to the motto of the Abbey of Thélème, a utopic and idealized "anti-monastery" from Rabelais' *Gargantua*: the motto was "Fay ce que voudras" (Do as you please). Even today the Oberwolfach Institute is sometimes compared to an isolated monastery where mathematicians live and work together, only devoted to their science. Bourbaki has become a synonym for the modern development of mathematics being interested only in the development of its internal structures based on a few basic axioms. This restriction of the scientific objective implies a great freedom from external forces but, implicitly, also from responsibility for the consequences of its research. I think it is not a coincidence that the young Bourbaki group refers to the myth of the Abbaye de Thélème after the end of World War II.

Mathematical Research—Popularization Versus Communication

Having described some of the process of mathematical research, let me now consider the challenge of communicating mathematics and its research results.

The Popularization of Mathematics is Impossible

I would like to start with a provocative quotation by Reinhold Remmert (mathematician; born 1930) from 2007:

> We all know that it is not possible to popularize mathematics. To this day, mathematics does not have the status in the public life of our country it deserves, in view of the significance of mathematical science. Lectures exposing its audience to a Babylonian confusion and that are crowded with formulas making its audience deaf and blind, do in no way serve to the promotion of mathematics. Much less do well-intentioned speeches degrading mathematics to enumeration or even pop art. In Gauss' words mathematics is "regina et ancilla", queen and maidservant in one. The "usefulness of useless thinking" might be propagated with good publicity. Insights into real mathematical research can, in my opinion, not be given. [12] (See Fig. 4.)

Remmert's statement about the popularization of mathematics has a point. However, we have to distinguish between popularization and communication. While his statement may apply to popularization, it does, in my opinion, not apply to the communication of mathematics. Before I explain why, let me start with the difficulties that we face when trying to communicate mathematics.

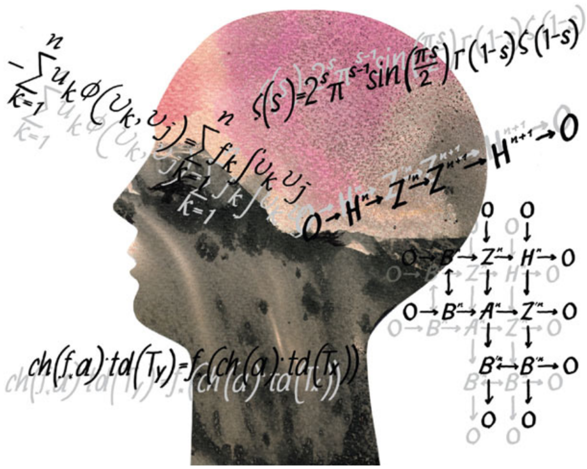


Fig. 4 "Usefulness of useless thinking" [14]

Structural Difficulties

First of all we may ask why it is not be possible to communicate mathematics. What is different in mathematics compared to other sciences? One could argue that for any research, regardless of whether it's in physics, chemistry, or biology, very specialized knowledge is required so that the popularization of research on the one hand and profoundness and correctness on the other do not go well together. Nevertheless, due to your own experience you will all have the feeling that mathematics might fall into a special category. In my opinion there are two significant structural reasons why it is so difficult to communicate or even popularize mathematics.

The first reason is that objects in modern mathematics are abstract creations of human thought. I do not wish to enter into a discussion of whether we only discover mathematical objects, which exist independently of our thoughts, or whether these objects are abstractions of human experience. Except for very simple ideas, like natural numbers or elementary geometrical figures, mathematical objects are not perceived, even if one can argue that they are not independent of perception. Objects like e.g. groups, vector spaces, or curved spaces in arbitrary dimension cannot be experienced with our five senses. They need a formal definition, which does not rely on our senses. Gaining an understanding of mathematical objects and relations is only possible after a long time of serious theoretical consideration.

Another reason is that mathematics has developed its own language, more than any other science. This is necessarily a result from the previous point that mathematics cannot be experienced directly. Therefore, each mathematical term needs a precise formal definition. This definition includes further terms that must be defined, and so on, so that finally a cascade of terms and definitions is set up that make a simple explanation impossible. But even in ancient times the abstraction from objects of our perception has always been a decisive part of mathematics, which made it difficult to comprehend. In Euclid's words: "There is no royal road to geometry" [13].

The language of mathematics requires an extremely compact presentation, a symbolism that allows replacing pages of written text by a single symbol. The peak of mathematical precision and compact information is a mathematical formula. But mathematical formulas frighten and deter. Stephen Hawking (physicist; born 1942) wrote in 1988: "… each equation I included in the book would halve the sales" [15].

The importance of "closed" mathematical formulas or equations might change in the future, being at least partly replaced by computer programs. However, this will not make communication easier.

These structural reasons support the thesis of Remmert that the nature of mathematical research cannot be popularized. And all mathematicians engaged in research would agree, that it is nearly impossible to feasibly illustrate to a mathematically untrained person the project one is currently working on. Actually, this experience applies not only to mathematically untrained people but even to mathematicians working in a different field.

The Need to Communicate Mathematics

Nonetheless, the statement that insights into the nature of mathematical research are not possible for a non-mathematician is for me hard to accept. Because this also implies quite a lot of resignation. As much as this statement might be true when limited to genuine mathematical research, it is not true when you take into consideration the fact that mathematical research has become a cultural asset of mankind during its development over 5,000 years.

Furthermore, it is my impression that everyone has a feeling for mathematics even if it is developed to different degrees. Each of you who has been around small children would know that already from an early age they take great pleasure in counting and natural numbers and have a quantitative grasp of their surroundings. They often love to solve little calculations. Regrettably, this interest often gets lost during schooling. I would even go so far as to introduce the following:

Thesis *In an overall sense, mathematical thinking is, after speech, the most important human faculty. It was this skill especially that helped the human species in the struggle for survival and improved the competitive abilities of societies. I believe that mathematical thinking has a special place in evolution.*

By mathematical thinking I mean analytic and logical thinking in a very broad sense, which is certainly not independent of the ability to speak. Of course, the development of mathematics as a science is a cultural achievement but, in contrast to languages, it developed in a similar way in different societies. We can face the fact that the importance of mathematics for mankind has grown continuously over the centuries, regardless of the cultural and social systems. No modern science is possible without mathematics and societies with highly developed sciences are in general more competitive than others. Attaching this value to mathematics, one must conclude the following:

Thesis *Society has the fundamental right to demand an appropriate explanation of mathematics. And it is the duty of mathematicians to face this responsibility.*

However, if mathematicians want to make their science easier to understand it will be at the expense of correctness. And that's a problem for mathematicians. All their professional training is necessarily based on being exact and complete. Mathematicians simply abhor to be inexact or vague. But in order to be understood by society, they will have to be just that [16]. I admit that this remains a continual conflict for every mathematician.

How Can We Raise Public Awareness in Mathematics?

In my experience there are two approaches for raising public interest in mathematics and demonstrating its significance: First, by examples that show the applicability of mathematics, and second, by examples that demonstrate the beauty and elegance of mathematics.

The first approach is certainly the favored one and it is often the only one accepted by politicians. However, we should not underestimate the second approach: it is often much more appealing and even crucial if we wish to get children interested in mathematics.

The elegance of a mathematical proof can really be intellectually fulfilling, e.g. the proof that the square root of 2 is an irrational number, or that there are infinitely many prime numbers. Both proofs can be given in advanced school classes. More accessible and therefore even more suitable for a larger audience is the beauty of geometrical objects. An example of this kind is the mathematical exhibition IMAGINARY with its beautiful pictures. For a more detailed description of IMAGINARY and the experiences of this travelling exhibition see the chapter in this book by Andreas Matt.

Mathematical Research—Intuition and Rigor

In the following I would like to try to explain a recent problem in my own research field of algebraic geometry and singularity theory. Although the explanation will mainly be by showing pictures, I am unable to avoid formulas. Nevertheless, I will be vague and I will simplify.

Algebraic Geometry

Algebraic geometry is, generally speaking, about the description of the set of solutions of a system of polynomial equations. Here I will focus on one equation only.

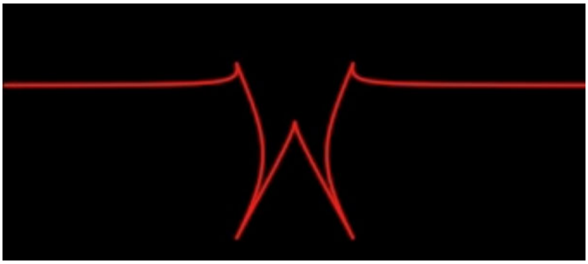


Fig. 5 $129/8x^4y - 85/8x^2y^3 + 57/32y^5 - 20x^4 - 21/4x^2y^2 + 33/8y^4 - 12x^2y + 73/8y^3 + 32x^2 = 0$

You might still remember from school the equation $y - x^2 = 0$ or $y = x^2$ for a parabola and $y^2 + x^2 - 1 = 0$ for a circle. Whereas a circle and a parabola are smooth curves, the next curve has singularities.

The equation of the curve in Fig. 5 is of degree 5, called a quintic, where the degree is the maximum sum of exponents in any of its terms. The curve has 5 peaks, called cusps, the maximum possible number for a quintic. The equation is rather complicated and we do neither realize its shape nor anything else from the equation, although it contains all the information.

Singularities

Before I get to a concrete recent research problem, let us have a look at some pictures demonstrating that singularities occur in our daily life. A parabolic mirror has an exact focus; the reflected rays meet at exactly one point, a "singularity" (usually "singular" refers to non-regular behavior, demonstrating an exception).

If the mirror is not a parabola, a focal curve, called a caustic in geometrical optics, develops and replaces the ideal focal point. This focal curve has its own singularity, a peak. Looking at the pictures, we can see the significance of a singularity: in the caustic curve it is the point where light energy has its highest intensity, the point of highest temperature.

One of the best-known caustics can be observed on sunny days in your cup of coffee, see Fig. 6. There even exist solar cookers, Fig. 7, which benefit practically from this singularity.

Curves with Many Singularities

A mathematical research problem in connection with singular curves is the following:

How many singularities can a plane curve of degree d have at most?

That an upper bound for the number of singularities should exist may be seen from the simplest example, a curve of degree $d = 1$. It is given by a linear equation,

Fig. 6 Coffee cup with caustic [17]

Fig. 7 Solar cooker [18]

which means that the curve is a line and hence has no singularities. With a bit more effort we can see that a curve of degree $d = 2$ (or $d = 3$) can have at most 1 (or 3) singularities, and these curves are realized by the union of 2 (resp. 3) lines which intersect in 1 (resp. 3) crossing points. In fact, the simplest singularities on a curve are crossing points, called nodes. The next simple ones are peaks, called cusps.

Now let C denote a plane curve of degree d with n nodes and k cusps. It was classically known and proved around 1920 by the Italian geometer Francesco Severi (1879–1961) that such a curve must satisfy

$$k + 2n \leqq 1/2d^2 + 3/2d$$

for very large d [19]. That is, the number of nodes plus two times the number of cusps can grow at most quadratically with the degree d, when d goes to infinity. However, it remained open for quite a while, whether curves with so many crossings and cusps really do exist. It was only known that $k + 2n$ can grow linearly in d, but it

Table 1 Upper and lower bounds for the number of nodes on a surface of degree d

d	1	2	3	4	5	6	7	8	d
$n \leq$	0	1	4	16	31	65	104	174	$4/9d^3$
$n \geq$	0	1	4	16	31	65	99	168	$5/12d^3$

was unknown whether there are curves of arbitrary high degree with $k+2n$ growing quadratically with d.

This problem was finally solved affirmatively in 1989. In fact it was shown that for any k and n such that $k + 2n \leqq 1/2d^2 - 2d + 3$ there exist curves with n nodes and k cusps [20]. The proof of this result required profound theorems of modern algebraic geometry such as vanishing theorems of cohomology and the resolution of singularities but also computer computations of concrete examples (using the computer algebra system SINGULAR [21]). It is worth noting that for any given degree the strongest results are obtained by combining theoretical results with computer computations or, in other words, by combining theory and practice.

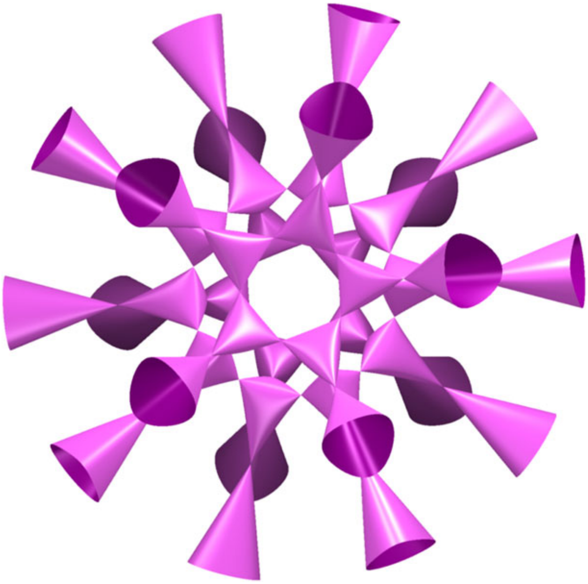
World Record Surfaces

A similar question to the one for curves can be posed for surfaces or for algebraic varieties of any dimension, namely what is the maximum number of singularities on a variety of given degree? For such varieties of higher dimension the problem is more difficult than for curves. At the moment we do not know whether for surfaces there is an asymptotic behavior for the maximum number of singularities similar to that for curves.

Instead of looking for the asymptotic behavior of the number of singularities when the degree goes to infinity, we may look for "world record surfaces", that is, for surfaces of a small degree with the maximum possible number of singular points. This is a current research topic. A complete answer is known only up to degree $d = 6$ where the theoretical upper and the known lower bounds for the number of nodes coincide. For higher d there are upper and lower bounds but the exact maximum number of nodes is unknown for $d > 6$. The Table 1 presents the known results, where the second row shows the theoretical upper bound and the third row the maximum known lower bound.

Surfaces with singularities look amazingly attractive. For example, the Barth sextic [22] is a beautiful surface of degree 6 with the symmetry of an icosahedron (and with a terrible complicated equation) Fig. 8. It holds the world record with 65 singularities and this record can never be improved for $d = 6$. From the table you can see that for degree 7 the maximum possible number is 104, but the actual known number of singularities is only 99. To fill in this gap for $d \geq 7$ is a topical research problem in algebraic geometry.

Fig. 8 Barth sextic [23]



Geometry Versus Algebra

The reader may wonder whether producing a nice picture like Fig. 8 is the essence of algebraic geometry. One may ask: what significance do such pictures have for research? It might be surprising, but it is a fact that images are nearly irrelevant in research in modern algebraic geometry, at least as far as proofs are concerned.

However, for many mathematicians like me, pictures are an important source of intuition. Geometry and algebra stimulate different parts of your brain. By means of images you will get ideas, which you then try to prove rigorously by means of algebra. Pictures are also a means of communication. I am tempted to give a digression on algebra versus geometry (or topology), an apparent conflict passing through the history of mathematics from its beginning until now. Consider the well-known quotation by Hermann Weyl (mathematician; 1885–1955) from 1939:

> "In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual mathematical domain." [24]

More than sixty years later Sir Michael Francis Atiyah (mathematician; born 1929) wrote in a similar vein:

> "Algebra is the offer made by the devil to the mathematician. The devil says: 'I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine.' [...] when you pass over into algebraic calculation, essentially you stop thinking; you stop thinking geometrically, you stop thinking about the meaning." [25]

I must suppress a further digression but instead I would like to introduce my own thesis into this conflict, including the role of computers:

Thesis *Geometry gives intuition, algebra provides rigor, and the computer forces merciless precision. Step by step, from geometry to the computer, we are gaining certainty but we are losing some of the liberty in our thinking. Rigor and precision are prerequisites for correctness, but they are of limited value if they are applied without intuition.*

Research and Application—Theory and Practice

It cannot be denied and is a simple and easily verifiable fact that mathematics is applied in our everyday life. But the reason behind this fact remains hidden, as emphasized by Bourbaki:

> "That there is an intimate connection between experimental phenomena and mathematical structures, seems to be fully confirmed in the most unexpected manner by the recent discoveries of contemporary physics. But we are completely ignorant as to the underlying reasons for this fact (supposing that one could indeed attribute a meaning to these words) and we shall perhaps always remain ignorant of them." [26]

I am afraid we do not know much more about this connection than Bourbaki in 1950. On the other hand, Bourbaki clearly believed that the formal axiomatic method is a better preparation for new interpretations of nature, at least in physics, than any method that tries to derive mathematics from experimental truths. Of course, this view creates tensions. In the following I am going to touch on the historical and current tensions between pure research and applications, between theory and practice.

Everyday Applications of Mathematics

When we talk about the application of mathematics, we have to face the fact that mathematics is essential for new and innovative developments in other sciences as well as in the economy and for industry. I do not claim that only mathematics can provide innovation, but it is no exaggeration to claim that mathematics has become a key technology behind almost all common and everyday applications, which includes the design of a car, its electronic components and all security issues, safe data transfer, error correction codes in digital music players and mobile phones, and includes the design and optimization of logistics in any large enterprise. We may say that

Mathematics is the technology of technologies.

However, since the mathematical kernel of an innovation is in most cases not visible, the relevance of mathematics is either not acknowledged by the general public or simply attributed to the advances of computers. In 2008, the German Year of Mathematics, the book "*Mathematik—Motor der Wirtschaft*" [27], (*Mathematics—Motor of the Economy*) was published, giving 19 international enterprises and the German Federal Employment Agency a platform to describe how essential mathematics has become for their success. The main point of this publication was not to demonstrate new mathematics, but to show that, in contrast to a great proportion of the general public, the representatives of economy and industry are well aware of the important role of mathematics.

Hilbert's Vision

The application of mathematics in industry and the economy is certainly a part of our utilization of nature but, according to David Hilbert (mathematician; 1862–1943), mathematics, and *only* mathematics, is the foundation of nature and our culture in a fundamental sense:

> "*The tool implementing the mediation between theory and practice, between thought and observation is mathematics. Mathematics builds the connecting bridges and is constantly enhancing their capabilities. Therefore it happens that our entire contemporary culture, in so far as it rests on intellectual penetration and utilization of nature, finds its foundation in mathematics*." [28]

Based on his belief, Hilbert tried to lay the foundation of mathematics on pure axiomatic grounds, and he was convinced that it was possible to prove that they were without contradictions. The inscription on his gravestone in Göttingen expresses this vision with the words: "*We must know—we will know*". Today it is no longer possible to fully adhere to Hilbert's optimism, due to the work of Gödel on mathematical logic showing that the truth of some mathematical theories is not decidable within mathematics. But Hilbert's statement about the *utilization* of nature is truer than ever.

On the other hand, this is no reason to glorify mathematics or to consider it superior to other sciences. First of all, the utilization of nature is not possible with mathematics alone. Many other sciences contribute, though differently, in the same substantial way. Secondly, the utilization of nature cannot be considered as an absolute value, as we know today. We are all a part of nature and utilization, as necessary as it is, can destroy nature and therefore part of our life.

Pure Versus Applied Mathematics

I use the terms "pure" and "applied" mathematics although it might be better to say "science-driven" and "application-driven" mathematics. In any case, here is a very provocative and certainly arrogant quotation of Godfrey Harold Hardy (mathematician; 1877–1947) from his much quoted essay *A Mathematician's Apology*:

> "It is undeniable that a good deal of elementary mathematics [...] has considerable practical utility. These parts of mathematics are, on the whole, rather dull; they are just the parts which have least aesthetic value. The 'real' mathematicians, the mathematics of Fermat and Euler and Gauss and Abel and Riemann, are almost wholly 'useless'." [29]

Hardy distinguishes between "elementary" and "real" (in the sense of interesting and deep) mathematics. The essence of his statement has two aspects: elementary mathematics, which can be applied, is unaesthetic and dull, while "real" mathematics is useless.

I think that Hardy is wrong in both aspects. Of course there exist interesting and dull mathematics. Mathematics is interesting when new ideas and methods prove to be fruitful in either solving difficult problems or in creating new structures for a deeper understanding. Routine development of known methods almost always turns out to be rather dull, and it is true that many applications of mathematics to, say, engineering problems are routine. However, this is not the whole story. Before applying mathematics, one has to find a good mathematical model for a real world problem, and this is often not at all elementary or trivial but a very creative process. This point is completely missing in Hardy's essay. Maybe, because pure mathematicians do not consider this as mathematics at all.

His other claim must also be refuted. Very deep and interesting results of "real" mathematics have become applicable, as we now know. That is, the border between interesting and dull mathematics is not between pure and applicable mathematics, but goes through any sub-discipline of mathematics, independent of whether it is applied or pure. In fact, many great mathematicians worked in pure as well as in applied mathematics. Let me quote Felix Klein (mathematician; 1848–1925) writing about Gauss, one of the greatest mathematicians ever:

> "*The work of Gauss in the field of applied mathematics I would like to call the high point of his life's work. The true core and basis of his achievements is founded in pure mathematics, a field he dedicated himself to youth.*" [30]

Klein is seen by many as one of the last great mathematicians who combined both applied and pure mathematics in his work. Like Gauß, he started in pure mathematics and then later turned to applied mathematics. In my opinion, studying first pure and then applied mathematics has its advantages.

Applications Cannot Be Predicted—The Lost Innocence

Nowadays we know better than in Hardy's time that his statement about the uselessness of pure mathematics is wrong. The following quote by Hardy concerns his own research field, number theory:

> "I have never done anything 'useful'. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world." [31]

Shortly after Hardy's death the methods and results from number theory became the most important elements for public-key cryptography, which today is used millions of times daily for electronic data transfer in mobile phones and electronic banking. For his claim that deep and interesting mathematics is useless, Hardy calls Gauß and Riemann and also Einstein his witnesses:

> "The great modern achievements of applied mathematics have been in relativity and quantum mechanics, and these subjects are, at present at any rate, almost as 'useless' as the theory of numbers. It is the dull and elementary parts of applied mathematics, as it is the dull and elementary parts of pure mathematics, that work for good or ill. Time may change all this. No one foresaw the applications of matrices and groups and other purely mathematical theories to modern physics, and it may be that some of the 'highbrow' applied mathematics will become 'useful' in as unexpected a way." [32]

The last sentence indicates that Hardy himself was skeptical about his own statements, although he did not really believe in the possibility of "real" mathematics becoming useful. However, the development of GPS, relying on the deep work of Gauss and Riemann on curved spaces and on Einstein's work on relativity, proves the applicability of their "useless" work.

This is not about blaming Hardy because he did not foresee GPS or the use of number theory in cryptography. Nevertheless, his strong statements are somewhat surprising as he was of course aware that, for example, Kepler used the theory of conic sections, a development of Greek mathematics without intended purpose, in order to describe the planetary orbits. So, what was the reason that Hardy insisted on the uselessness of "real" mathematics?

In my opinion we can understand Hardy's strong statements, made in 1940 at the beginning of World War II, only if we know that he was a passionate pacifist. It would have been unbearable for him to see that his own mathematics could be useful for the purpose of war. He was bitterly mistaken.

We all know that nowadays the most sophisticated mathematics, pure and applied, is a decisive factor in the development of modern weapon systems. Without GPS, and hence without the mathematics of Gauss and Riemann, this would have been impossible. Even before that, the atomic bomb marked the first big disillusionment for many scientists regarding the innocence of their work. If there was ever a paradise of innocence with no possibility for mathematics to do 'good or ill' to mankind, it was lost then.

Applicability Versus Quality of Research

History shows, and the statements of Hardy prove this, that it is impossible to predict which theoretical developments in mathematics will become "useful" and will have an impact on important applications in the future. The distinction between pure and applied mathematics is more a distinction between fields than between applicability. Quite often we notice that ideas from pure research, only aiming to explore the structure of mathematical objects and their relations, become the basis for innovative ideas creating whole new branches of economic and industrial applications. Besides number theory for cryptography, I would like to mention logic for formal verification in chip design, algebraic geometry for coding theory, computer algebra for robotics, and combinatorics for optimization applied to logistics, to name just a few.

Although the list of applications of pure mathematics could be easily enlarged, it is also clear that some parts of mathematics are closer to applications than others. These are politically preferred and we can see that more and more national and international programs support only research with a strong focus on applications or even on collaboration with industry.

I think the above examples show that it would be a big mistake, if applicability were to become the main or even the only criterion for judging and supporting mathematics. In this connection I like to formulate the following:

Thesis *The value of a fundamental science like mathematics cannot be measured by its applicability but only by its quality.*

History has shown that in the long run, quality is the only criterion that matters and that only high-quality research survives. It is worthwhile to emphasize again that any kind of mathematics, either science driven or application driven, can be of high or low quality.

In view of the above and many more examples, one could argue that we would miss unexpected but important applications by restricting mathematical research to a priori applicable mathematics. This is certainly true, it is, however, not the main reason why I consider it a mistake to judge mathematics by its applicability. My main reason is that it would reduce the mathematical sciences to a useful tool, without a right to understand and to further develop the many thousands of years of cultural achievements of the utmost importance. This leads us to reflect on freedom of research.

Freedom of Research and Responsibility

Freedom of research has many facets, for example in the sense of "Fay ce que voudras" of the Abbaye de Thélème, mentioned earlier, or just emphasizing unconditional research. It implies in any case that the scientist himself defines the direction of research.

In mathematics there is an even more fundamental aspect. Today's mathematics is often searching for inner mathematical structures, only committed to its own axioms and logical conclusions and thus keeping it free from any external restriction.

This was clearly intended by the creators of modern axiomatic mathematics. Georg Cantor (mathematician; 1845–1918), the originator of set theory, proclaims: “The nature of mathematics is its freedom” [33] and David Hilbert considers this freedom as a paradise: “Nobody shall expel us from the paradise created for us by Cantor” [34].

This kind of freedom was certainly felt by the group of young Bourbakists meeting in Oberwolfach in 1949 when they referred to the myth of the Abbaye de Thélème mentioned above, and many mathematicians of today feel the same way.

On the other hand, there are reasons to question this freedom as an absolute value into question, because it does also imply freedom from responsibility. However, we must emphasize that this does not excuse the individual scientist as a human being from his responsibilities. The physicist Max Born wrote in 1963:

> “Although I never participated in the application of scientific knowledge to any destructive purpose, like the construction of the A-bomb or H-bomb, I feel responsible.” [35]

It may be argued that the self-referential character of the science is, at least partially, responsible for the lack of responsibility. This is emphasized by Egbert Brieskorn (mathematician; born 1937) who not only deplores this character but even goes a step further in claiming that this attribute implies the possibility of assuming and misusing power:

> “The restriction on pure perception of nature by combining experiment and theoretical description by means of mathematical structures is the subjective condition to evolve this science as power. The development of mathematics as self-referential science enforces the possibility to seize power for science as a whole. [. . .] It belongs to the nature of the human being to prepare and to take possession of the reality. We should not feel sorrow about that, however, we should be concerned that the temptations of power is threatening to destroy our humanity.” [36]

The self-referential character appears clearly in Hilbert’s and Bourbaki’s concept of mathematical structures based on the axiomatic method. This concept was of great influence in the development of mathematics in the twentieth century. It was, however, never without objections and nowadays it is certainly not the driving force anymore. In theoretical mathematics the most influential new ideas arise from a deep interaction with physics, in particular with quantum field theory. Atiyah even calls this the “era of quantum mathematics”. [37] Applied mathematics such as numerical analysis or statistics, on the other hand, has always been too heterogeneous to be adequately covered by Bourbaki’s approach. It is often driven by challenging problems from real world applications. But I do not see that this fact makes it less vulnerable to the temptations of power, maybe even to the contrary.

Not denying this threat for any kind of mathematics, I like to point out that freedom of research is a precious gift, related to freedom of thought in an even broader sense. Mathematicians for example are educated to use their own brains, to doubt any unsubstantiated claim, and not to believe in authority. A mathematical theorem is true not because any person of high standing or of noble birth claims it, but because we can prove it ourselves. In this sense I like to claim:

Thesis *Mathematical education can contribute to freedom of thought in a broad sense.*

On the other hand, being aware of the "lost innocence" and the fact that mathematics can be "for good or ill" to mankind, freedom must be accompanied by responsibility. The responsibility for the impact of their work, though not a part of science itself and not easy to recognize, remains the task for each individual mathematician.

Thesis *Freedom of research must be guaranteed in mathematics and in other sciences. It has to be defended by scientists, but it must be accompanied by responsibility.*

IMAGINARY—Mathematical Creations and Experiences

Let me return to the communication of mathematics. As explained above, this is by no means an easy task, but mathematicians themselves have to make the effort to communicate their science. In fact, many mathematicians do so with remarkable success. The present book is a proof of these efforts.

IMAGINARY is mathematics as art: geometry presented as an attractive visual world. It started as a travelling exhibition, created by the Mathematisches Forschungsinstitut Oberwolfach, in the German Year of Mathematics 2008. Its aim is to interest people in mathematics by showing them the beauty of mathematical objects and to fill them with inspiration and stimulate their imaginations. The exhibition has been shown in more than 35 cities in different countries with several hundred thousand visitors, and its success has been overwhelming. It is interesting that only journalists have asked for applications while the other visitors have experienced the unexpected beauty and the "joy of comprehension". I would like to show two pictures from the IMAGINARY art gallery and otherwise refer the reader to the chapter in this book by Andreas Matt [38], and the web page www.imaginary-exhibition.com (see Figs. 9 and 10).

The pictures were created for an online competition in cooperation with a German science magazine, using the free IMAGINARY software *surfer* and won the first and third prizes. Details, including the equations, can be found at Spektrum der Wissenschaften, Mathematik-Kunst-Wettbewerb [38].

It seems that the success of IMAGINARY will continue. Besides the interest in the ongoing exhibitions there is a surprisingly high demand for online programs which allow individual mathematical experimentation, and there is also a demand for further background information. So far we have registered about 250,000 downloads of the IMAGINARY software and about 70,000 downloads of mathematical

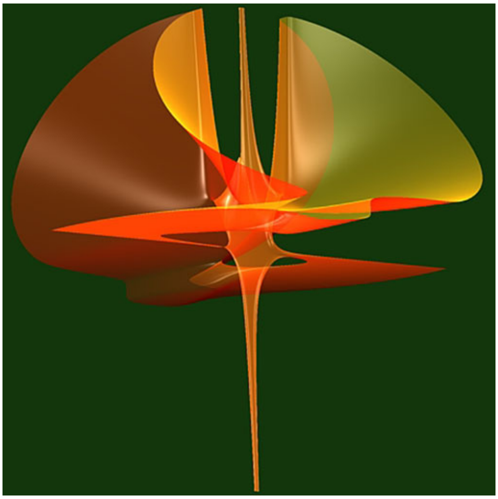


Fig. 9 Tropenwunder (tropical wonder)

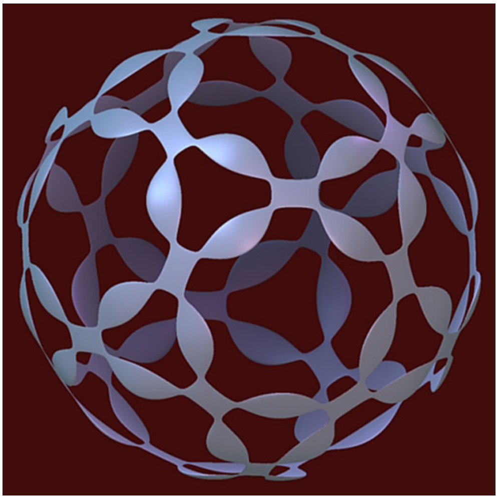


Fig. 10 Ikosidodekaeder (icosidodecahedron)

material [39]. In particular the figures demonstrate that a substantial proportion of the general public is interested in mathematics, if it is presented in an appealing manner.

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Notes and References

Several of the quotations in this article are common knowledge but, when cited, their origin is often not documented. I have made special effort to give the original source or, when this turns out to be impossible, to provide a reliable "second-hand" source. Moreover, whenever I have access to the original source of a quotation and when I find it helpful, I cite also some of the surrounding text. In order to allow a smooth reading I give in the main text an English translation when the original quotation is not in English, while the References contain the original version.

1. "What the scientist aims at is to secure a logically consistent transcript of nature. Logic is for him what the laws of proportion and perspective are to the painter, and I believe with Henri Poincaré that science is worth pursuing because it reveals the beauty of nature. And here I will say that the scientist finds his reward in what Henri Poincaré calls the joy of comprehension, and not in the possibilities of application to which any discovery of his may lead."

   Albert Einstein, in: *Epilogue, A Socratic Dialogue*, p. 211, Interlocutors: Max Planck, Albert Einstein, James Murphy. In: Max Planck, "Where is Science Going?" Norton, New York, 226 pages (1932)
2. "Philosophy is written in that great book which ever lies before our eyes—I mean the universe—but we cannot understand it if we do not first learn the language and grasp the symbols, in which it is written. This book is written in the mathematical language, and the symbols are triangles, circles and other geometrical figures, without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth."

   Galileo Galilei, in: *Opere Complete di Galileo Galilei*, Firenze, 1842, ff, vol. IV, p. 171, as quoted by Edwin Arthur Burtt: *The Metaphysical Foundations of Modern Science*, p. 75, Dover Reprint, New York, 352 pages (2003)
3. "Ich komme aus dem Lande, das Land der Mathematiker geblieben ist, [...] auch der mathematischen Studien, welche die Seele aller industriellen Fortschritte sind."

   Attributed to Alexander von Humboldt, in: Roland Z. Bulirsch, *Weltfahrt als Dichtung*, p. 14, in: "Dokumentation zur Verleihung des Literaturpreises der Konrad-Adenauer-Stiftung e.V. an Daniel Kehlmann", 52 pages (2006)
4. " 'Ohne Mathematik tappt man doch immer im Dunkeln', das schrieb vor mehr als 150 Jahren Werner von Siemens an seinen Bruder Wilhem."

   Quote taken from: Peter Löscher, *Siemens AG*, p. 99, in: "MATHEMATIK – Motor der Wirtschaft", Eds. G.-M. Greuel, R. Remmert, G. Rupprecht; Springer Verlag, 125 pages (2008)
5. "Es gehört eine gewisse Kühnheit dazu, in einer Kultur, die sich durch ein profundes mathematisches Nichtwissen auszeichnet, derartige Übersetzungsversuche zu unternehmen." Hans Magnus Enzensberger, *Zugbrücke außer Betrieb - Drawbrigde Up*, p. 44, A K Peters LTD, Natick, MA, 48 pages (1999)
6. Archives of the Mathematisches Forschungsinstitut Oberwolfach

7. The text of this section is, slightly modified, taken from: *Editorial of the Oberwolfach Reports*, published by the Mathematisches Forschungsinstitut Oberwolfach in cooperation with the European Mathematical Society
8. "Der erste große Name im Gästebuch ist Heinz Hopf (Zürich), ein Topologe von Weltruf. Im November 1946 war Henri Cartan, dessen Familie während der deutschen Besatzung großes Leid erdulden mußte, zu Besuch. Ohne Hopf und Cartan wäre Oberwolfach damals vielleicht eine "Sommerfrische für Mathematiker" geblieben, wo würdige Herren in beschaulicher Ruhe klassische Theorien polierten. Gott sei Dank kam es anders."

   Reinhold Remmert, *Mathematik in Oberwolfach – Erinnerungen an die ersten Jahre*, p. 1. Grußwort zur Einweihung der Bibliothekserweiterung am 5. Mai 2007, published by the Mathematisches Forschungsinstitut Oberwolfach, 27 pages (2008). See also Annual Report 2007 of the Mathematisches Forschungsinstitut Oberwolfach, http://www.mfo.de/scientific-programme/publications/annual-publications
9. Archives of the Mathematisches Forschungsinstitut Oberwolfach
10. For a historical appreciation of the "Deutsch-Französische Arbeitsgemeinschaft", a meeting of young German and French mathematicians in Oberwolfach, see: Maria Remenyi, *Oberwolfach im August 1949: Deutsch-Franzosische Sommerfrische*, Math. Semesterber., Mathematische Bildergalerie, Springer (2011)
11. "Das Lesen dieser Zeilen erfordert im Prinzip keinerlei spezielle (∗) mathematische Kenntnisse, dennoch sind sie für Leser bestimmt, die zumindest ein gewisses Gefühl entwickelt haben für die mathematisch und vielsprachig freundschaftliche Atmosphäre, an der wir uns auf dem Lorenzenhof erfreut haben. Es ist äußerst schwierig, die auserlesene Vielfalt der Strukturen, die diese Atmosphäre zustande bringt, zu analysieren; es ist zudem noch viel schwieriger, die Gunstbeweise, die uns durch unsere Gastgeber zu Teil wurden, in ihrer Gesamtheit auch nur zum Teil einzuordnen. Dennoch wagen wir es hier, das Auswahlaxiom (∗∗) anzuwenden, um ein maximales Element auszuzeichnen: unseren Dank an Herrn und Frau Süss, die es uns ermöglichten, für einige Tage diesem alten Mythos (∗ ∗ ∗) der Abbaye de Thélème Leben zu verleihen, der uns so sehr am Herzen liegt.

   Literaturangaben:

   (∗) Sankt Nikolaus Evangelium, Einleitung, 1. Vers
   (∗∗) Sankt Nikolaus, op. cit. pars prima, lib. primus, III, Kapitel 4
   (∗ ∗ ∗) F. Rabelais, Opera omnia, passim.

   Jean Arbault, Jean-Pierre Serre, René Thom, A. Pereira Gomez, Josiane Serre, Georges Reeb, Bernard Charles, Jean Braconnier."

   Guestbook of the Mathematisches Forschungsinstitut Oberwolfach, No 1, p. 2. German translation see [8], p. 7 and [10]. Online at http://oda.mfo.de/view/viewer.jsf
12. "Wir alle wissen, daß Mathematik nicht popularisierbar ist. Sie hat bis heute im öffentlichen Leben unseres Landes nicht die Stellung, die ihr Kraft der

Tragweite ihrer Inhalte zukommt. Vorträge, wo die Hörer vom babylonischen Sprachgewirr und Formelgestrüpp taub und blind werden, eignen sich nicht für Werbung. Noch weniger helfen gut gemeinte Reden, wo Mathematik zu einer Rechenkunst oder gar Pop-Kultur erniedrigt wird. Mathematik ist nach Gauß ‘regina et ancilla’, Königin und Magd in einem. Die ‘Nützlichkeit nutzlosen Denkens’ kann man vielleicht öffentlichkeitswirksam propagieren, Einblicke in das Wesen mathematischer Forschung lassen sich nach meiner Erfahrung nicht geben.” [8], loc. cit., p. 20

13. According to Proclus, a neo platonist (412–485 A.D.), Euclid replied to King Ptolemy, who asked whether he could not learn geometry more easily than by studying the *Elements*: “There is no royal road to geometry.”

    Quote taken from http://www.1902encyclopedia.com/E/EUC/euclid-mathematician.html
14. Poster of the Mathematisches Forschungsinstitut Oberwolfach. Design by Boy Müller
15. “Someone told me that each equation I included in the book would halve the sales. I therefore resolved not to have any equations at all. In the end, however, I did put in one equation, Einstein’s famous equation, $E = mc^2$. I hope that this will not scare off half of my potential readers.”

    Stephen W. Hawking, *A Brief History of Time*, “Acknowledgments”, Bantam Dell Publishing Group, 224 pages (1988)
16. Compare this with the dialogue from the preface of Ian Stewart’s *The Problems of Mathematics* (Oxford Univ. Press, 1987) where a mathematician is chatting with a fictional layman “Seamus Android”:

    – *Mathematician*: It’s one of the most important discoveries of the last decade!
    – *Android*: Can you *explain* it in words ordinary mortals can understand?
    – *Mathematician*: Look, buster, if ordinary mortals could understand it, you wouldn’t need mathematicians to do the job for you, right? You can’t get a feeling for what’s going on without understanding the technical details. How can I talk about manifolds without mentioning that the theorems only work if the manifolds are finite-dimensional paracompact Hausdorff with empty boundary?
    – *Android*: Lie a bit.
    – *Mathematician*: Oh, but I couldn’t do that!
    – *Android*: Why not? Everybody *else* does.
    – *Mathematician* (tempted, but struggling against a lifetime’s conditioning): But I *must* tell the truth.
    – *Android*: Sure. But you might be prepared to bend it a little, if it helps people understand what you’re doing.
    – *Mathematician* (sceptical, but excited at his own daring): Well, I suppose I could give it a *try*.

    Quote taken from: [5], loc. cit., p. 45–47
17. Picture by Christian Ucke, http://users.physik.tu-muenchen.de/cucke
18. Picture from http://www.atlascuisinesolaire.com

19. Francesco Severi, *Vorlesungen über algebraische Geometrie*, Teubner, 408 pages (1921)
20. Gert-Martin Greuel, Christoph Lossen, Eugenii Shustin, *Plane curves of minimal degree with prescribed singularities*, Invent. Math. 133, 539–580 (1998)
21. Gert-Martin Greuel, Gerhard Pfister, Hans Schoenemann, *SINGULAR—A Computer Algebra System for Polynomial Computations*, free software, http://www.singular.uni-kl.de (1990–to date)
22. Wolf Barth, *Two Projective Surfaces with Many Nodes Admitting the Symmetries of the Icosahedron*, J. Alg. Geom. 5, 173–186 (1996)
23. Picture produced with the ray tracer *surfer*, free software, http://www.imaginary-exhibition.com/surfer.php
24. "In this purely algebraic way based on the adjunction argument we master the orthogonal and the symplectic invariants. This procedure has even stood the test in certain special cases where the statement of full reducibility breaks down.

    In these days the angel of topology and the devil of abstract algebra fight for the soul of each individual mathematical domain. [. . .].

    I feel bound to add a personal confession. In my youth I was almost exclusively active in the field of analysis; the differential equations and expansions of mathematical physics were the mathematical things with which I was on the most intimate footing. I have never succeeded in completely assimilating the abstract algebraic way of reasoning, and constantly feel the necessity of translating each step into a more concrete analytic form. But for that reason I am perhaps fitter to act as intermediary between old and new than the younger generation which is swayed by the abstract axiomatic approach, both in topology and algebra."

    Hermann Weyl, *Invariants*, pp. 500–501, Duke Mathematical Journal 5, 489–502 (1939)
25. "One way to put the dichotomy in a more philosophical or literary framework is to say that algebra is to the geometer what you might call the 'Faustian offer'. As you know, Faust in Goethe's story was offered whatever he wanted (in his case the love of a beautiful woman), by the devil, in return for selling his soul. Algebra is the offer made by the devil to the mathematician. The devil says: 'I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine.' (Nowadays you can think of it as a computer!) Of course we like to have things both ways; we would probably cheat on the devil, pretend we are selling our soul, and not give it away. Nevertheless, the danger to our soul is there, because when you pass over into algebraic calculation, essentially you stop thinking; you stop thinking geometrically, you stop thinking about the meaning."

    Sir Michal Atiyah, *Special Article—Mathematics in the 20th Century*, p. 7, Bull. London Math. Soc. 34, 1–15 (2002)
26. Nicholas Bourbaki, *The Architecture of Mathematics*, p. 231, Amer. Math. Monthly 57, No. 4, 221–232 (1950)
27. Gert-Martin Greuel, Reinhold Remmert, Gerhard Rupprecht, Eds., *MATHEMATIK – Motor der Wirtschaft*, see [4]

28. "Das Instrument, welches die Vermittlung bewirkt zwischen Theorie und Praxis, zwischen Denken und Beobachten, ist die Mathematik; sie baut die verbindende Brücke und gestaltet sie immer tragfähiger. Daher kommt es, dass unsere ganze gegenwärtige Kultur, soweit sie auf der geistigen Durchdringung und Dienstbarmachung der Natur beruht, ihre Grundlage in der Mathematik findet."

    David Hilbert, *Naturerkennen und Logik*, Versammlung Deutscher Naturforscher und Ärzte in Königsberg, 1930. Quote taken from: http://quantumfuture.net/gn/zeichen/hilbert.html, linking to an mp3 version of the original speech by Hilbert. For the English translation see: http://math.ucsd.edu/~williams/motiv/hilbert.html
29. Godfrey H. Hardy, *A Mathematician's Apology*, pp. 32–33, Cambridge University Press, 52 pages (1940)
30. "Die besprochenen Arbeiten von Gauß auf dem Gebiet der angewandten Mathematik möchte ich als Krönung seines Lebenswerkes bezeichnen. Der eigentliche Kern und das Fundament seiner Leistungen aber liegt auf dem Gebiet der *reinen Mathematik*, der er sich in seinen Jugendjahren widmete."

    Felix Klein, *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, p. 24, Reprint, Springer Verlag, 208 pages (1970)
31. See [29], loc. cit. p. 49
32. See [29], loc. cit. p. 39
33. "Dagegen scheint mir aber jede überflüssige Einengung des mathematischen Forschungstriebes eine viel größere Gefahr mit sich zu bringen und eine um so größere, als dafür aus dem Wesen der Wissenschaft keinerlei Rechtfertigung gezogen werden kann; denn das *Wesen* der *Mathematik* liegt gerade in Ihrer *Freiheit*."

    Georg Cantor, *Gesammelte Abhandlungen*, p. 182, Ed. Ernst Zermelo, Springer, 486 pages (1932)
34. "Aus dem Paradies, das Cantor uns geschaffen hat, soll uns niemand vertreiben können." David Hilbert, *Über das Unendliche*, p. 170, Math. Ann. 95, 161–190 (1926)
35. "Obwohl ich an der Anwendung naturwissenschaftlicher Kenntnis für zerstörerische Zwecke, wie die Herstellung der A-Bombe oder der H-Bombe, nicht teilgenommen habe, fühle ich mich verantwortlich."

    Max Born, *Erinnerungen und Gedanken eines Physikers*, in: Max und Hedwig Born, Der Luxus des Gewissens: Erlebnisse und Einsichten im Atomzeitalter, p. 73, Nymphenburger Verlagshandlung, 200 pages (1969)
36. "Die Beschränkung auf reine Naturerkenntnis durch die Verbindung von Experiment und theoretischer Beschreibung mit Hilfe mathematischer Strukturen ist die subjektive Bedingung der Möglichkeit der Entfaltung dieser Wissenschaft als Macht. Die Entwicklung der Mathematik als selbstreferentielle Wissenschaft verstärkt die Machtförmigkeit der Wissenschaft insgesamt. [...] Daß der Mesch sich der Wirklichkeit bemächtigt, daß er sie sich zurechtmacht, gehört zu seinem Wesen. Darüber soll man nicht traurig sei, wohl aber darüber, daß die Verführung der Macht unsere Menschlichkeit zu zerstören droht."

Egbert Brieskorn, *Gibt es eine Wiedergeburt der Qualität in der Mathematik?*, p. 257–258, in: Wissenschaft zwischen Qualitas und Quantitas, Ed. Erwin Neuenschwander, Birkhäuser Verlag, 444 pages (2003)

37. "I have said the 21st century might be the era of quantum mathematics or, if you like, of infinite-dimensional mathematics. What could this mean? Quantum mathematics could mean, if we get that far, 'understanding properly the analysis, geometry, topology, algebra of various non-linear function spaces', and by 'understanding properly' I mean understanding it in such a way as to get quite rigorous proofs of all the beautiful things the physicists have been speculating about."

    Sir Michal Atiyah, *Special Article—Mathematics in the 20th Century*, p. 14, Bull. London Math. Soc. 34, 1–15 (2002)
38. The picture Tropenwunder was created by Hiltrud Heinrich and Ikosidodekaeder was created by Martin Heider. See http://www.spektrum.de/blatt/d_sdwv_extra_artikel&id=947549&_z=798888&_z=798888
39. See the IMAGINARY website: www.imaginary-exhibition.com