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On symplectic coverings of the projective plane

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Abstract. We prove that a resolution of singularities of any finite covering of the projective complex plane branched along a Hurwitz curve $\overline{H}$, and possibly along the line "at infinity", can be embedded as a symplectic submanifold in some projective algebraic manifold equipped with an integer Kähler symplectic form. (If $\overline{H}$ has negative nodes, then the covering is assumed to be non-singular over them.) For cyclic coverings, we can realize these embeddings in a rational complex 3-fold. Properties of the Alexander polynomial of $\overline{H}$ are investigated and applied to the calculation of the first Betti number $b_1(\overline{X}_n)$, where $\overline{X}_n$ is a resolution of singularities of an n-sheeted cyclic covering of $\mathbb{CP}^2$ branched along $\overline{H}$, and possibly along the line "at infinity". We prove that $b_1(\overline{X}_n)$ is even if $\overline{H}$ is an irreducible Hurwitz curve but, in contrast to the algebraic case, $b_1(\overline{X}_n)$ may take any non-negative value in the case when $\overline{H}$ consists of several components.

Introduction

The notion of Hurwitz curve in the projective plane $\mathbb{CP}^2$ with respect to a linear projection $\mathrm{pr}\colon \mathbb{CP}^2 \setminus \{p_\infty\} \to \mathbb{CP}^1$ (where p_∞ is the centre of the projection pr) was introduced in [18] and is a natural generalization of the notion of plane algebraic curve. (In [18], Hurwitz curves are called "semi-algebraic curves".) A precise definition of Hurwitz curves can be found, for example, in [12]. In this paper we give the following equivalent (see Lemma 1.1) definition of Hurwitz curves. Let $\mathbb{C}_i^2$ be two copies of the affine plane $\mathbb{C}^2$ with coordinates (u_i, v_i), $i = 1, 2$, $u_2 = 1/u_1$ and $v_2 = v_1/u_1$, which cover $\mathbb{CP}^2 \setminus \{p_\infty\}$ in such a way that pr is given by $(u_i, v_i) \to u_i$ in the charts $\mathbb{C}_i^2$. A set $\overline{H} \subset \mathbb{CP}^2 \setminus \{p_\infty\}$, closed in $\mathbb{CP}^2$, is called a *Hurwitz curve of degree* m if, for $i = 1, 2$, $\overline{H} \cap \mathbb{C}_i^2$ coincides with the zero set of an equation

$$F_i(u_i, v_i) := v_i^m + \sum_{j=0}^{m-1} c_{j,i}(u_i) v_i^j = 0, \tag{0.1}$$

such that

(i) $F_i(u_i, v_i)$ is a C^∞-smooth complex-valued function on $\mathbb{C}^2$,

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(ii) the function $F_i(u_i, v_i)$ has only finitely many critical values, that is, there are finitely many values of u_i, say, $u_{i,1}, \dots, u_{i,n_i}$, such that the polynomial equation

$$v_i^m + \sum_{j=0}^{m-1} c_{j,i}(u_{i,0}) v_i^j = 0 \tag{0.2}$$

has no multiple roots for $u_{i,0} \notin \{u_{i,1}, \dots, u_{i,n_i}\}$,

(iii) if $v_{i,j}$ is a multiple root of (0.2) for $u_{i,j} \in \{u_{i,1}, \dots, u_{i,n_i}\}$, then, in a neighbourhood of the point $(u_{i,j}, v_{i,j})$ (which will be called a *critical point* of $\overline{H}$), the set $\overline{H}$ coincides with the solution set of a complex-analytic equation.

We note that Hurwitz curves become symplectic surfaces in $\mathbb{CP}^2$ after a rescaling $\tilde{v}_i = \varepsilon v_i,\ 0 < \varepsilon \ll 1$ (see also the proof of Theorem 3.1).

More generally, one can consider the so-called *topological Hurwitz curves*, which have *cone singularities* (see the definition of cone singularities in [12]).

A Hurwitz (resp. topological Hurwitz) curve $\overline{H}$ is said to be *irreducible* if $\overline{H} \setminus M$ is connected for any finite set $M \subset \overline{H}$. We say that a Hurwitz curve $\overline{H}$ *consists of k irreducible components* if

$$k = \max \#\{\text{connected components of } \overline{H} \setminus M\},$$

where the maximum is taken over all finite sets $M \subset \overline{H}$.

Let H be an *affine Hurwitz curve*, that is, $H = \overline{H} \cap (\mathbb{CP}^2 \setminus L_\infty)$, where the complex line L_∞ is a fibre of pr in general position with respect to $\overline{H}$. Then the fundamental group $\pi_1 = \pi_1(\mathbb{CP}^2 \setminus (\overline{H} \cup L_\infty))$ does not depend on the choice of L_∞ and belongs to the class $\mathcal{C}$ of so-called C-groups.

By definition, a *C-group* is a group together with a finite presentation

$$G_W = \langle x_1, \dots, x_m \mid x_i = w_{i,j,k}^{-1} x_j w_{i,j,k},\ w_{i,j,k} \in W \rangle, \tag{0.3}$$

where $W = \{w_{i,j,k} \in \mathbb{F}_m \mid 1 \leqslant i, j \leqslant m,\ 1 \leqslant k \leqslant h(i,j)\}$ is a subset of the free group $\mathbb{F}_m$ (it is possible that $w_{i_1,j_1,k_1} = w_{i_2,j_2,k_2}$ for $(i_1, j_1, k_1) \neq (i_2, j_2, k_2)$) generated by free generators $x_1, \dots, x_m$, and $h\colon \{1, \dots, m\}^2 \to \mathbb{Z}$ is a function. Such a presentation is called a *C-presentation*. (C means that all relations are conjugations.)

Let $\varphi_W \colon \mathbb{F}_m \to G_W$ be the canonical epimorphism. The elements $\varphi_W(x_i) \in G$, $1 \leqslant i \leqslant m$, and their conjugates are called the *C-generators* of the C-group G. Let $f\colon G_1 \to G_2$ be a homomorphism of C-groups. It is called a *C-homomorphism* if the images of the C-generators of G_1 under f are C-generators of G_2. We shall consider C-groups up to C-isomorphisms.

A C-presentation (0.3) is called a *Hurwitz C-presentation of degree m* if the word $w_{i,i,1}$ coincides with the product $x_1 \dots x_m$ for each $i = 1, \dots, m$. A C-group G is called a *Hurwitz C-group of degree m* if it possesses a Hurwitz C-presentation of degree m. In other words, a C-group G is a Hurwitz C-group of degree m if there are C-generators $x_1, \dots, x_m$ of G such that the product $x_1 \dots x_m$ belongs to the centre of G. We note that the degree of a Hurwitz C-group G is not canonically defined and depends on the Hurwitz C-representation of G. We denote the class of all Hurwitz C-groups by $\mathcal{H}$.

Let $\overline{H}$ be a Hurwitz (resp. topological Hurwitz) curve of degree m. A Zariski–van Kampen presentation of $\pi_1 = \pi_1(\mathbb{C}^2 \setminus H)$ (where $\mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty$ and the fibre L_∞ of pr is in general position with respect to $\overline{H}$) defines the structure of a Hurwitz C-group of degree m on π_1 (see [11]). It was proved in [11] that any Hurwitz C-group G of degree m can be realized as the fundamental group $\pi_1(\mathbb{C}^2 \setminus H)$ for some Hurwitz curve $\overline{H}$ with singularities of the form $w^m - z^m = 0$, $\deg \overline{H} = 2^n m$, where n depends on the Hurwitz C-presentation of G. Since we are considering C-groups up to C-isomorphism, the class $\mathcal{H}$ coincides with the class $\{\pi_1(\mathbb{C}^2 \setminus H)\}$ of fundamental groups of the complements of affine Hurwitz (resp. topological Hurwitz) curves.

A free group $\mathbb{F}_n$ with fixed free generators is a C-group, and any C-group G possesses a well-defined canonical C-epimorphism $\nu\colon G \to \mathbb{F}_1$ that sends the C-generators of G to the C-generator of $\mathbb{F}_1$. We denote its kernel by N. We note that if all C-generators of a C-group G are conjugate to each other (such C-groups are said to be *irreducible*), then N coincides with G'.[1]

Let G be a C-group. The C-epimorphism ν induces the exact sequence of groups

$$1 \to N/N' \to G/N' \xrightarrow{\nu_*} \mathbb{F}_1 \to 1.$$

The C-generator of $\mathbb{F}_1$ acts on N/N' by conjugation $\tilde{x}^{-1} n \tilde{x}$, where $n \in N$ and $\tilde{x}$ is one of the C-generators of G. We denote this action by h and let $h_{\mathbb{C}}$ be the induced action on $(N/N') \otimes \mathbb{C}$. The characteristic polynomial $\Delta(t) = \det(h_{\mathbb{C}} - t\,\mathrm{Id})$ is called the *Alexander polynomial* of the C-group G. (If the vector space $(N/N') \otimes \mathbb{C}$ is infinite-dimensional over $\mathbb{C}$, then the Alexander polynomial $\Delta(t)$ is identically equal to zero by definition.) For a (topological) Hurwitz curve $\overline{H}$, the Alexander polynomial $\Delta(t)$ of the group $\pi_1 = \pi_1(\mathbb{C}^2 \setminus H)$ is called the *Alexander polynomial* of $\overline{H}$. We note that the Alexander polynomial $\Delta(t)$ of a (topological) Hurwitz curve $\overline{H}$ does not depend on the choice of the generic line L_∞.

Let $G = \langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle$ be a C-presentation of a C-group G and let $\mathbb{F}_m$ be the free group on the C-generators $x_1, \dots, x_m$. We denote by $\frac{\partial}{\partial x_i}$ the Fox derivative [4], that is, the endomorphism of the group ring $\mathbb{Z}[\mathbb{F}_m]$ over $\mathbb{Z}$ of the free group $\mathbb{F}_m$ to itself such that $\frac{\partial}{\partial x_i}\colon \mathbb{Z}[\mathbb{F}_m] \to \mathbb{Z}[\mathbb{F}_m]$ is a $\mathbb{Z}$-linear map defined by the following properties:

$$\begin{aligned} \frac{\partial x_j}{\partial x_i} &= \delta_{i,j}, \\ \frac{\partial uv}{\partial x_i} &= \frac{\partial u}{\partial x_i} + u \frac{\partial v}{\partial x_i} \end{aligned} \tag{0.4}$$

for any $u, v \in \mathbb{Z}[\mathbb{F}_m]$. The following fact is well known (it is proved, for example, in [19] in the case of knot groups and is generalized to the case of C-groups in [9]). The Alexander polynomial $\Delta(t)$ of a C-group G coincides (up to multiplication by a non-zero constant and a factor $\pm t^k$ invertible in $\mathbb{Z}[t, t^{-1}]$) with the greatest common divisor of the minors of order $m-1$ in the matrix

$$\nu_*\biggl(\frac{\partial r_i}{\partial x_j}\biggr) \in \mathrm{Mat}_{n \times m}(\mathbb{Z}[t, t^{-1}]),$$

[1] We use the standard notation G' for the commutator subgroup of a group G, and G'' for the commutator subgroup of G'.

where r_i $(i = 1, \dots, n)$ are the defining relations of G and $\nu_*\colon \mathbb{Z}[\mathbb{F}_m] \to \mathbb{Z}[\mathbb{F}_1] \simeq \mathbb{Z}[t, t^{-1}]$ is induced by the canonical C-epimorphism $\nu\colon \mathbb{F}_m \to \mathbb{F}_1$.

The properties of the Alexander polynomials of plane algebraic curves and their applications to the calculation of the first Betti number of a cyclic covering of the projective plane are well known (see, for example, [22], [15], [20], [6], [7], [9], [8]). One of the aims of this paper is to generalize these results to the case of Hurwitz curves and apply them to the calculation of the first Betti number of a cyclic covering of the projective plane branched along a Hurwitz curve.

The main results of this paper are the following theorems and corollaries.

Theorem 0.1. *Let $\overline{H}$ be a (topological) Hurwitz curve of degree d, and let $\Delta(t)$ be its Alexander polynomial. Then*

(i) $\Delta(t) \in \mathbb{Z}[t]$,
(ii) $\Delta(0) = \pm 1$,
(iii) *the roots of $\Delta(t)$ are dth roots of unity*,
(iv) *the action of $h_{\mathbb{C}}$ on $(N/N') \otimes \mathbb{C}$ is semisimple.*

Moreover, the Alexander polynomial $\Delta(t)$ of a Hurwitz curve $\overline{H}$ of degree d is a divisor of the polynomial $(t-1)(t^d-1)^{d-2}$ (see Theorem 5.6). If $\overline{H}$ consists of k irreducible components, then the multiplicity of the root $t = 1$ of its Alexander polynomial $\Delta(t)$ is equal to $k-1$ (see Theorem 5.9).

Theorem 0.2. *If $\overline{H}$ is an irreducible (topological) Hurwitz curve, then*

(i) *$\Delta(t)$ is a reciprocal polynomial, that is,* $\Delta(t) = t^{\deg \Delta(t)} \Delta(t^{-1})$,
(ii) $\deg \Delta(t)$ *is an even number*,
(iii) $\Delta(1) = 1$.

Corollary 0.3. *Let $\overline{H}$ be an irreducible (topological) Hurwitz curve of degree $\deg \overline{H} = p^n$, where p is a prime number. Then*

(i) $\Delta(t) \equiv 1$,
(ii) *the group π_1'/π_1'' is finite, where $\pi_1 = \pi_1(\mathbb{C}^2 \setminus H)$.*

We also note that if J is an almost complex structure on $\mathbb{CP}^2$ compatible with the Fubini–Study symplectic form and $\overline{H}$ is a J-holomorphic curve in $\mathbb{CP}^2$ of degree m, that is, the class of $[\overline{H}]$ equals $m[\mathbb{CP}^1]$ in $H_2(\mathbb{CP}^2, \mathbb{Z})$, then $\pi_1 = \pi_1(\mathbb{CP}^2 \setminus (\overline{H} \cup L_\infty))$ is a Hurwitz C-group of degree m, where L_∞ is one of the J-lines in general position with respect to $\overline{H}$. Indeed, if we choose a pencil of pseudo-holomorphic lines having L_∞ as a member, then the Zariski–van Kampen theorem yields a presentation of π_1 by a braid monodromy factorization of $\overline{H}$ with respect to this pencil. Therefore π_1 is a C-group and, as in the case of Hurwitz curves, it is easy to show (see the proof of Theorem 6.1 in [11]) that it is a Hurwitz C-group of degree m. Thus, the Alexander polynomial of a pseudo-holomorphic curve can be defined in a similar way and has the same properties as in the case of Hurwitz curves.

Let $\pi_1 = \pi_1(\mathbb{C}^2 \setminus H)$ be the fundamental group of the complement of an affine Hurwitz curve. Then the homomorphism $\nu\colon \pi_1 \to \mathbb{F}_1$ defines an infinite unramified cyclic covering $f = f_\infty\colon X'_\infty \to X' = \mathbb{C}^2 \setminus H$. We have $H_1(X'_\infty, \mathbb{Z}) = N/N'$, and the action of h on $H_1(X'_\infty, \mathbb{Z})$ coincides with that of a generator of the covering transformation group of the covering f_∞. It follows from [13] that the group $H_1(X'_\infty, \mathbb{Z})$ is finitely generated. For any $n \in \mathbb{N}$, let $\mathrm{mod}_n\colon \mathbb{F}_1 \to \mu_n = \mathbb{F}_1/\{h^n\}$ be the natural epimorphism to the cyclic group μ_n of degree n. The covering f_∞ can be factored through the cyclic covering $f_n\colon X'_n \to \mathbb{C}^2 \setminus H$ associated with the epimorphism $\mathrm{mod}_n \circ \nu$, that is, $f_\infty = g_n \circ f_n$. Since the Hurwitz curve $\overline{H}$ has only analytic singularities, the covering f_n can be extended to a smooth map $\bar{f}_n\colon \overline{X}_n \to \mathbb{CP}^2$ branched along $\overline{H}$, and possibly along L_∞ (if n is not a divisor of $\deg \overline{H}$, then $\bar{f}_n$ is branched along L_∞), where $\overline{X}_n$ is a smooth 4-dimensional manifold. The action h induces an action $\bar{h}_n$ on $\overline{X}_n$ and an action $\bar{h}_{n*}$ on $H_1(\overline{X}_n, \mathbb{Z})$.

In § 4 we show (see Theorem 4.1) that any such $\overline{X}_n$ may be embedded as a symplectic submanifold in a projective rational 3-fold whose symplectic structure is given by an integer Kähler form.

Theorem 0.4. *Let $\overline{X}_n$ be a resolution of singularities of an n-sheeted cyclic covering branched along a Hurwitz curve $\overline{H}$ and possibly along L_∞ and associated with the epimorphism* $\mathrm{mod}_n \circ \nu\colon \pi_1 \to \mathbb{Z}/n\mathbb{Z}$. *Then the first Betti number satisfies*

$$b_1(\overline{X}_n) = \dim_{\mathbb{C}} H_1(\overline{X}_n, \mathbb{C}) = r_{n,\neq 1},$$

where $r_{n,\neq 1}$ is the number of roots of the Alexander polynomial $\Delta(t)$ of the curve $\overline{H}$ which are nth roots of unity not equal to 1.

Theorems 0.1, 0.2, 0.4 and Corollary 0.3 have the following corollaries.

Corollary 0.5. *Let $\overline{X}_n$ be a resolution of singularities of an n-sheeted cyclic covering branched along a Hurwitz curve $\overline{H}$ and possibly along L_∞. If $\deg \overline{H}$ and n are coprime, then $b_1(\overline{X}_n) = 0$.*

Corollary 0.6. *Let $\overline{X}_n$ be a resolution of singularities of an n-sheeted cyclic covering branched along an irreducible Hurwitz curve $\overline{H}$ and possibly along L_∞. Then $b_1(\overline{X}_n)$ is an even number.*

We shall show that, for any $k \in \mathbb{N}$, there is an irreducible Hurwitz curve $\overline{H}_k$ such that $b_1(\overline{X}_{k,n}) = 2k$ for some n (one can take $n = 6$, see Proposition 6.5), where $\overline{X}_{k,n}$ is a resolution of singularities of an n-sheeted cyclic covering branched along $\overline{H}_k$. In addition, we show that, for any $k \in \mathbb{N}$, there is a Hurwitz curve $\overline{H}_k$ consisting of two irreducible components such that $b_1(\overline{X}_{k,6}) = k$, where $\overline{X}_{k,6}$ is a resolution of singularities of a cyclic covering of the projective plane branched along $\overline{H}_k$. We recall that $b_1(\overline{X}_n)$ is always even if $\overline{H}$ is a plane algebraic curve. Hence $\overline{H}_k$ cannot be algebraic if k is odd. Other examples of Hurwitz curves non-isotopic to algebraic curves can be found in [18].[2]

Corollary 0.7. *Let $\overline{X}_n$ be a resolution of singularities of a cyclic covering of the projective plane branched along a Hurwitz curve $\overline{H}$ consisting of k irreducible components and possibly along L_∞. If $\deg \overline{H}$ divides n, then $b_1(\overline{X}_n) = \deg \Delta(t) - k + 1$.*

[2] In [18], Moishezon proved the existence of an infinite sequence $\overline{H}_i$ of irreducible cuspidal Hurwitz curves, of degree 54 with exactly 378 cusps and 756 nodes, which have pairwise distinct braid monodromy type. In particular, they are pairwise non-isotopic, and almost all of them are not isotopic to an algebraic cuspidal curve.

Corollary 0.8. *Let $\overline{X}_n$ be a resolution of singularities of a cyclic covering of the projective plane of any degree n branched along an irreducible Hurwitz curve $\overline{H}$ and possibly along L_∞. If* $\deg \overline{H} = p^k$, *where p is a prime number, then* $b_1(\overline{X}_n) = 0$.

Corollary 0.9. *Let $\overline{X}_{p^k}$ be a resolution of singularities of a cyclic covering of the projective plane of degree p^k branched along any irreducible Hurwitz curve $\overline{H}$ and possibly along L_∞, where p is a prime number. Then* $b_1(\overline{X}_{p^k}) = 0$.

For any $k \in \mathbb{N}$ we shall prove the existence of a Hurwitz curve $\overline{H}_k$ which consists of $k+1$ components and is the branch curve of a 2-sheeted cyclic covering whose resolution of singularities $\overline{X}_{k,2}$ has $b_1(\overline{X}_{k,2}) = k$ (see Proposition 6.6). In particular, in our example, the Hurwitz curve $\overline{H}_1$ has $\deg \overline{H}_1 = 2^{10}$, the number of singular points of $\overline{H}_1$ is equal to 2^{16}, and all singular points are of the form $w^4 - z^4 = 0$.

Recently, Auroux and Katzarkov proved the following theorem (see [1], [2]). Let (X, ω) be a compact symplectic 4-manifold with symplectic form ω such that $[\omega] \in H^2(X, \mathbb{Z})$. Fix an ω-compatible almost-complex structure J and the corresponding Riemannian metric g. Let L be the line bundle on $\overline{X}$ with first Chern class $[\omega]$. Then for $k \gg 0$, the line bundle $L^{\otimes k}$ admits many approximately holomorphic sections, and one can choose three of them to get an approximately holomorphic generic covering $f_k \colon \overline{X} \to \mathbb{CP}^2$ of degree $N_k = k^2\omega^2$ branched over a cuspidal Hurwitz curve $\overline{H}_k$ (possibly with negative nodes).

Any such covering $f_k \colon \overline{X} \to \mathbb{CP}^2$ of degree N_k branched over a cuspidal Hurwitz curve $\overline{H}$ determines a monodromy μ, that is, an epimorphism $\mu\colon \pi_1(\mathbb{C}^2 \setminus H) \to \Sigma_{N_k}$ to the symmetric group Σ_{N_k}, with additional properties of genericity. On the other hand, any homomorphism $\mu\colon \pi_1(\mathbb{C}^2 \setminus H) \to \Sigma_N$ such that $\mu(\pi_1)$ acts transitively on the set of N elements determines an unramified covering $f\colon X \to \mathbb{C}^2 \setminus H$ of degree N. The covering f can be extended to a covering $\tilde{f}\colon \widetilde{X} \to \mathbb{CP}^2$ branched over the Hurwitz curve $\overline{H}$ and possibly over L_∞. In this paper we prove (see Corollary 3.2) that if $\widetilde{X}$ has arbitrary analytic singularities (and the covering space is non-singular over all negative nodes of $\overline{H}$ if there are any), then the resolution $\overline{X}$ of singularities of $\widetilde{X}$ can be equipped with a symplectic structure.

The proofs of Theorems 0.1, 0.2 and Corollary 0.3 are given in § 5, while § 6 is devoted to the proof of Theorem 0.4.

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§ 1. Representation of Hurwitz curves as sections of line bundles

We begin with the following lemma.

Lemma 1.1. *The definitions of Hurwitz curves in $\mathbb{CP}^2$ given in* [12] *and in the introduction are equivalent.*

Proof. We recall the definition of Hurwitz curves in [12]. Let F_1 be the relatively minimal ruled rational surface, $\mathrm{pr}\colon F_1 \to \mathbb{CP}^1$ the ruling, R a fibre of pr and E_1 the exceptional section, $E_1^2 = -1$. We identify $\mathrm{pr}\colon F_1 \to \mathbb{CP}^1$ with a linear projection $\mathrm{pr}\colon \mathbb{CP}^2 \to \mathbb{CP}^1$ with centre at a point $p \in \mathbb{CP}^2$ (p is the blow-down of E_1 to the point).

By the definition in [12], the image $\overline{H} = f(S) \subset F_1$ of a smooth map $f\colon S \to F_1 \setminus E_1$ of an oriented closed real surface S is called a *Hurwitz curve* (with respect to pr) of degree m if there is a finite subset $Z \subset \overline{H}$ such that

(i) f is an embedding of the surface $S \setminus f^{-1}(Z)$ and, for any $s \notin Z$, $\overline{H}$ and the fibre $R_{\mathrm{pr}(s)}$ of pr meet at s transversally and with positive intersection number,

(ii) every point $s \in Z$ has a neighbourhood $U \subset F_1$ such that $\overline{H} \cap U$ is a complex-analytic curve, and the complex orientation of $\overline{H} \cap U \setminus \{s\}$ coincides with the orientation transported from S by f,

(iii) the restriction of pr to $\overline{H}$ is a finite map of degree m.

To show that the definition in the introduction implies the definition in [12], we perform several monoidal transformations centred at the singular points of $\overline{H}$ (and at the singularities of the proper transforms of $\overline{H}$) to resolve all singular points of $\overline{H}$. We denote the composition of these monoidal transformations by $\sigma\colon \widetilde{\mathbb{CP}}^2 \to \mathbb{CP}^2$. Let S be the proper transform of $\overline{H}$. Then S is a smooth real surface and $f = \sigma_{|S}$ is a smooth map. To define an orientation on S, we choose an orientation at each non-critical point p of $\mathrm{pr}_{|\overline{H}}$ in such a way that the local intersection number (at p) of $\overline{H}$ and the fibre R through p is equal to $+1$. Clearly, these orientations are compatible for all non-singular points of $\overline{H}$. Near each singular point, this orientation coincides with the orientation given by the complex-analytic structure. (We recall that $\overline{H}$ is complex-analytic near the critical points of $\mathrm{pr}_{|\overline{H}}$.) Therefore this orientation can be extended to the pre-images of these critical points.

To show that the definition in [12] implies that in the introduction, we choose a fibre R_∞ of pr and put $\mathbb{C}^2 = F_1 \setminus (R_\infty \cup E_1)$. Let (u, v) be coordinates in $\mathbb{C}^2$ such that the restriction of pr is given by $(u, v) \to u$. Let $(u, v_1(u)), \dots, (u, v_m(u))$ be the coordinates of the intersection points of $\overline{H}$ and the fibre R of pr over a non-critical value u. Consider

$$F(u, v) = \prod_{i=1}^{m} (v - v_i(u)). \tag{1.1}$$

The function $F(u, v)$ is defined everywhere outside the fibres over critical values, is smooth, and can be extended to a function on the whole of $\mathbb{C}^2$ that satisfies the definition given in the introduction.

Let $\overline{H}_0$ be a Hurwitz curve of degree m given by equations (0.1). A smooth isotopy $h_t\colon \mathbb{CP}^2 \times [0, 1] \to \mathbb{CP}^2 \times [0, 1]$ is called an *H-isotopy* if, for each $t \in [0, 1]$, the image $\overline{H}_t = h_t(\overline{H}_0)$ is a Hurwitz curve given by

$$v_i^m + \sum_{j=0}^{m-1} c_{j,i}(u_i, t) v_i^j = 0, \qquad i = 1, 2,$$

where $c_{j,i}(u_i, 0) = c_{j,i}(u_i)$ for all i, j. (We note that the definition of H-isotopy in [12] also requires that the number of critical points of $\overline{H}_t$ be independent of t.) It is easy to see that if $\overline{H}_0$ and $\overline{H}_1$ are H-isotopic and the line L_∞ is generic with respect to both $\overline{H}_0$ and $\overline{H}_1$, then $\mathbb{C}^2 \setminus H_0$ and $\mathbb{C}^2 \setminus H_1$ are diffeomorphic.

We denote the centre of the projection pr by $p_\infty = \mathbb{CP}^2 \setminus (\mathbb{C}^2_1 \cup \mathbb{C}^2_2)$. In what follows we assume that the fibre of pr over $u_2 = 0$ is generic with respect to $\overline{H}_0$.

Denote it by L_∞. Clearly, there is a smooth H-isotopy h_t equal to the identity outside a small neighbourhood U of L_∞ and such that the defining function $F_2(u_2, v_2, 1)$ of $\overline{H}_1 = h_1(\overline{H}_0)$ in $\mathbb{C}_2^2$ coincides with the function $v_2^m - 1$ at the points (u_2, v_2) with $|u_2| < \varepsilon$ for some $\varepsilon > 0$. In what follows we assume that

(*) *L_∞ is given by $u_2 = 0$ and $\overline{H}$ by $v_2^m - 1 = 0$ in a neighbourhood of L_∞.*

Let $u_{1,j}$ be a critical value of the Hurwitz curve $\overline{H}_0$ of degree m given by the equation $F_1(u_1, v_1) = 0$ in $\mathbb{C}_1^2$. Then the number of distinct roots of the equation

$$F_1(u_{1,j}, v_1, 0) = 0 \tag{1.2}$$

is less than m. Let v_{1,j_0} be a root of (1.2) of multiplicity one. Then there is a smooth H-isotopy h_t equal to the identity outside a small neighbourhood $U = \{|u_1 - u_{1,j}| < \varepsilon\}$ and such that the defining function $F_1(u_1, v_1, 1)$ of $\overline{H}_1 = h_1(\overline{H}_0)$ in $\mathbb{C}_1^2$ has the following property: $v_1 = v_{1,j_0}$ is a root of the equation $F_1(u_1, v_1, 1) = 0$ for all u_1 such that $|u_1 - u_{1,j}| < \varepsilon_1$ for some positive $\varepsilon_1 < \varepsilon$. Therefore we may assume that if $(u_{1,j}, v_{1,j})$ is a critical point of $\overline{H}$, then

(**) *there is $\varepsilon > 0$ such that the defining function $F_1(u_1, v_1)$ of $\overline{H}$ is analytic at (u_1, v_1) for $|u_1 - u_{1,j}| < \varepsilon$ and $|v_1 - v_{1,j}| < \varepsilon$.*

Let $p\colon \mathcal{L}(k) \to \mathbb{CP}^2$ be the line bundle associated with the sheaf $\mathcal{O}_{\mathbb{CP}^2}(k)$. We recall its definition. The projective plane $\mathbb{CP}^2$ with homogeneous coordinates $(z_0 : z_1 : z_2)$ is covered by three charts $\mathbb{C}_i^2$, $i = 1, 2, 3$, isomorphic to $\mathbb{C}^2$, with coordinates (u_i, v_i), $u_1 = z_1/z_0$, $v_1 = z_2/z_0$, $u_2 = z_0/z_1$, $v_2 = z_2/z_1$, $u_3 = z_0/z_2$, $v_3 = z_1/z_2$. The bundle $\mathcal{L}(k)$ is covered by three charts $W_i = \mathbb{C}_i^2 \times \mathbb{C}_i^1$ with third coordinate w_i, $w_1 = w_2/u_2^k$, $w_1 = w_3/u_3^k$, $w_2 = w_3/v_3^k$, and the restriction of $p_{|W_i}$ coincides with the projection to the first coordinate.

Lemma 1.2. *The defining functions $w_i = F_i(u_i, v_i)$ of $\overline{H}$ $(i = 1, 2)$ determine a smooth section s of $\mathcal{L}(m)$ over $\mathbb{CP}^2 \setminus \{p_\infty\}$.*

Proof. In $\mathbb{C}_1^2 \cap \mathbb{C}_2^2$ we have

$$\begin{aligned} F_1(u_1, v_1) &= v_1^m + \sum_{j=0}^{m-1} c_{j,1}(u_1) v_1^j = \left(\frac{v_2}{u_2}\right)^m + \sum_{j=0}^{m-1} c_{j,1}\left(\frac{1}{u_2}\right)\left(\frac{v_2}{u_2}\right)^j \\ &= \left(\frac{1}{u_2}\right)^m \left(v_2^m + \sum_{j=0}^{m-1} c_{j,1}\left(\frac{1}{u_2}\right) u_2^{m-j} v_2^j\right). \end{aligned}$$

The functions

$$F_2(u_2, v_2) = v_2^m + \sum_{j=0}^{m-1} c_{j,2}(u_2) v_2^j$$

and

$$v_2^m + \sum_{j=0}^{m-1} c_{j,1}\left(\frac{1}{u_2}\right) u_2^{m-j} v_2^j$$

coincide since they are smooth and, for all but finitely many values $u_{2,0}$ of u_2, the polynomials

$$v_2^m + \sum_{j=0}^{m-1} c_{j,2}(u_{2,0}) v_2^j$$

and

$$v_2^m + \sum_{j=0}^{m-1} c_{j,1}\left(\frac{1}{u_{2,0}}\right) u_{2,0}^{m-j} v_2^j$$

have the same sets of roots.

Lemma 1.3. *Let $f_0\colon S^3 \to \mathbb{C}^* = \mathbb{C}\setminus\{0\}$ be a smooth function on $S^3 = \{(u,v) \in \mathbb{C}^2 \mid u\bar{u} + v\bar{v} = \varepsilon^2\}$, $0 < \varepsilon \ll 1$. Suppose that f_0 coincides with the function $v^m - 1$ in a neighbourhood $U \subset S^3$ of the circle $u = 0$. Then there is a smooth function $F\colon S^3 \times [0,1] \to \mathbb{C}^*$ such that*

(i) $F(u,v,0) = f_0(u,v)$,

(ii) $F(u,v,t) = v^m - 1$ *for* $(u,v) \in U$ *and* $t \in [0,1]$,

(iii) $F(u,v,1) = v^m - 1$.

Proof. Since S^3 is simply connected, there is a lift $\tilde{f}_0\colon S^3 \to \widetilde{\mathbb{C}}^*$ of the function f_0 such that $f_0 = e \circ \tilde{f}_0$, where $\widetilde{\mathbb{C}}^*$ is the complex plane $\mathbb{C}$ with complex coordinate x and $e\colon \widetilde{\mathbb{C}}^* \to \mathbb{C}^*$ is the universal covering given by $y = e^x$. Without loss of generality, we may assume that $\tilde{f}_0(0,\varepsilon) = \ln(1-\varepsilon^m) + \pi i$. Let $f_1\colon S^3 \to \mathbb{C}^*$ be the function $v^m - 1$, and let $\tilde{f}_1\colon S^3 \to \widetilde{\mathbb{C}}^*$ be a lift such that $\tilde{f}_1(0,\varepsilon) = \ln(1-\varepsilon^m) + \pi i$. Then we have $\tilde{f}_{0|U} \equiv \tilde{f}_{1|U}$.

Consider a function $\widetilde{F}\colon S^3 \times [0,1] \to \mathbb{C}$ given by

$$x = t\tilde{f}_1(u,v) + (1-t)\tilde{f}_0(u,v).$$

Clearly, the function $F = e \circ \widetilde{F}$ has the desired properties.

Lemma 1.4. *There is a real number ε_1, $0 < \varepsilon_1 \ll 1$, and a smooth section $\bar{s}_m$ of $\mathcal{L}(m)$ over $\mathbb{CP}^2$ such that*

(i) $\overline{H} \subset \mathbb{CP}^2 \setminus B(\varepsilon_1)$, *where $B(\varepsilon_1) = \{u_3\bar{u}_3 + v_3\bar{v}_3 \leqslant \varepsilon_1^2\}$ is a ball in $\mathbb{CP}^2$ centred at p_∞,*

(ii) *the section $\bar{s}_m$ coincides with s (see Lemma 1.2) over $\mathbb{CP}^2 \setminus B(\varepsilon_1)$,*

(iii) *$\bar{s}_m$ is complex-analytic in a neighbourhood of the line L_∞.*

Proof. By the definition of Hurwitz curve, there is a ball $B(\varepsilon_1) = \{u_3\bar{u}_3 + v_3\bar{v}_3 \leqslant \varepsilon_1^2\}$ with positive ε_1 such that $\overline{H} \subset \mathbb{CP}^2 \setminus B(\varepsilon_1)$.

The line bundle $\mathcal{L}_m$ is trivial over $B(\varepsilon_1)$. Therefore the restriction of the section s (defined in Lemma 1.2) to $\partial B(\varepsilon_1) = S^3$ defines a function $f_0\colon S^3 \to \mathbb{C}^*$. Let $F\colon S^3 \times [0,1] \to \mathbb{C}^*$ be the function whose existence is proved in Lemma 1.3. (In the notation of Lemma 1.3, we take $u = u_3$, $v = v_3$ and $\varepsilon = \varepsilon_1$.) We choose $\varepsilon_2 < \varepsilon_1$ and a smooth monotone function $r\colon [\varepsilon_2, \varepsilon_1] \to [0,1]$ such that $r(\varepsilon_1) = 0$ and $r(\varepsilon_2) = 1$. Define a map $h\colon B(\varepsilon_1, \varepsilon_2) = B(\varepsilon_1) \setminus B(\varepsilon_2) \to \partial B(\varepsilon_1)$ by

$$h(u_3, v_3) = \left(\frac{\varepsilon_1 u_3}{\sqrt{u_3\bar{u}_3 + v_3\bar{v}_3}}, \frac{\varepsilon_1 v_3}{\sqrt{u_3\bar{u}_3 + v_3\bar{v}_3}} \right).$$

Put $\widetilde{F}(u_3, v_3, t) = h^*(F)$ and

$$\overline{F}(u_3, v_3) = \left(\frac{\sqrt{u_3\bar{u}_3 + v_3\bar{v}_3}}{\varepsilon_1} \right)^m \bigl(\widetilde{F}(u_3, v_3, r(\sqrt{u_3\bar{u}_3 + v_3\bar{v}_3}\,)) + 1 \bigr) - 1.$$

If we now define the section $\tilde{s}$ by

$$\tilde{s}(p) = \begin{cases} s(p) & \text{for} \quad p \in \mathbb{CP}^2 \setminus B(\varepsilon_1), \\ \overline{F}(u_3, v_3) & \text{for} \quad p \in B(\varepsilon_1) \setminus B(\varepsilon_2), \\ v_3^m - 1 & \text{for} \quad p \in B(\varepsilon_2), \end{cases}$$

then all the conditions of Lemma 1.4 hold except possibly that $\tilde{s}$ may be non-smooth (but only continuous) at the points of $B = (\partial B(\varepsilon_1) \cup \partial B(\varepsilon_2)) \setminus U$, where U is a neighbourhood of L_∞. By standard theorems of analysis, there is a smooth section $\bar{s}_m$ which is close enough to $\tilde{s}$ and coincides with $\tilde{s}$ outside a small neighbourhood V of B such that $\overline{V} \cap (\overline{H} \cup L_\infty) = \varnothing$, where $\overline{V}$ is the closure of V.

§ 2. Symplectic varieties with analytic singularities

Let Y be a projective complex manifold, $\dim_{\mathbb{C}} Y = n$, and let ω be a Kähler form on Y, $[\omega] \in H^2(Y, \mathbb{Z})$. We consider (Y, ω) as a symplectic manifold, $\dim_{\mathbb{R}} Y = 2n$. A closed subvariety X of Y is called a *symplectic variety with analytic singularities* if there are open subsets $U_0 \subset U \subset Y$ such that the closure $\overline{U}_0$ in Y is a subset of U, $X \cap U$ is a complex-analytic subset of U and $X \setminus \overline{U}_0$ is a smooth symplectic submanifold. Let $\operatorname{Sing} X$ be the set of points of X at which X is not smooth. Then $\operatorname{Sing} X$ is a projective algebraic subvariety of Y.

Lemma 2.1. *Let X be a symplectic variety with analytic singularities in a projective complex manifold Y with Kähler form ω. Let $Z \subset \operatorname{Sing} X$ be a non-singular projective subvariety of Y, $\sigma\colon \overline{Y} \to Y$ the monoidal transformation of Y with centre at Z, and $\overline{X}$ the proper transform of X. Then there is a Kähler form $\overline{\omega}$ on $\overline{Y}$ such that $\overline{X}$ is a symplectic subvariety with analytic singularities in $(\overline{Y}, \overline{\omega})$.*

Proof. The manifold $\overline{Y}$ is projective algebraic. Consider an embedding $i\colon \overline{Y} \hookrightarrow \mathbb{CP}^N$ in some projective space and denote by $\varphi = i \circ \sigma^{-1}$ a rational map from Y to $\mathbb{CP}^N$. Let $\Gamma \subset Y \times \mathbb{CP}^N$ be the closure of the graph of φ and let p_i $(i = 1, 2)$ be the projections of $Y \times \mathbb{CP}^N$ to the factors. The morphisms i and σ define the morphism $\sigma \times i\colon \overline{Y} \to \Gamma \subset Y \times \mathbb{CP}^N$. Since the composite $p_2 \circ (\sigma \times i)\colon \overline{Y} \to \overline{Y}$ is an isomorphism, so is $p_{2|\Gamma}\colon \Gamma \to \overline{Y}$. Moreover, if we identify $\overline{Y}$ with Γ by means of $p_{2|\Gamma}$, then $p_{1|\Gamma}$ coincides with σ.

Let $\Omega = \Omega_N$ be the Fubini–Study symplectic form on $\mathbb{CP}^N$:

$$\Omega_N = \frac{i}{\left(\sum_{j=0}^N \bar{z}_j z_j\right)^2} \sum_{k=0}^{N} \sum_{j\neq k} (\bar{z}_j z_j \, dz_k \wedge d\bar{z}_k - \bar{z}_j z_k \, dz_j \wedge d\bar{z}_k).$$

This is an integer Kähler form. The restriction of $\omega_\varepsilon = p_1^*(\omega) + \varepsilon p_2^*(\Omega)$ to Γ is a Kähler form on Γ for each $\varepsilon > 0$.

Choose open neighbourhoods $V_0 \subset V \subset Y$ of $\operatorname{Sing} X$ such that $V \cap X$ is an analytic subvariety and the closure $\overline{V}_0$ of V_0 in Y is a subset of V. Put $X_0 = X \setminus V_0$. This set is compact and $\sigma^{-1}_{|X_0}\colon X_0 \to \overline{X}_0 = \sigma^{-1}(X_0)$ is an isomorphism. Therefore $\overline{X}_0$ is compact.

Clearly, the restriction of ω_ε to $\Gamma \cap p_1^{-1}(V)$ is a symplectic form at each non-singular point of $\Gamma \cap p_1^{-1}(V)$ for all $\varepsilon > 0$ because $\Gamma \cap p_1^{-1}(V)$ is an analytic set in $p_1^{-1}(V)$. Since the restriction of ω to X_0 is a symplectic form at each point of X_0 and $\overline{X}_0$ is compact, we can choose ε to be small enough for the restriction of $\omega_\varepsilon = p_1^*(\omega) + \varepsilon p_2^*(\Omega)$ to $\overline{X}_0 = \Gamma \cap p_1^{-1}(X_0)$ to be a symplectic form at each point of $\overline{X}_0$. If we take $\varepsilon = \frac{m}{n}$ rational, then $n\omega_\varepsilon$ is an integer form.

§ 3. Symplectic structure of coverings of the projective plane branched along Hurwitz curves

In this section we use the notation and assumptions of § 1.

Let $\overline{H}$ be a Hurwitz curve, possibly with *negative nodes*. Hence, in a neighbourhood U of each critical point p, either $\overline{H}$ is given by an analytic equation or $\overline{H} \cap U$ consists of two smooth branches meeting transversally at p with intersection number -1 and each branch meets the fibre $\operatorname{pr}^{-1}(\operatorname{pr}(p))$ transversally at p with intersection number $+1$.

We fix a point $p \in \mathbb{CP}^2 \setminus (\overline{H} \cup L_\infty)$. Consider the fundamental group $\pi_1 = \pi_1(\mathbb{C}^2 \setminus H, p)$ of the complement of the affine Hurwitz curve $H = (\mathbb{CP}^2 \setminus L_\infty) \cap \overline{H}$. Choose a point $x \in \overline{H} \setminus \operatorname{Sing} \overline{H}$ and consider a line $L \subset \mathbb{C}^2$ meeting H transversally at x. Let $\gamma \subset L$ be a circle of small radius with centre at x. The choice of an orientation on $\mathbb{C}^2$ defines an orientation on γ. Let Γ be a loop consisting of a path l in $\mathbb{C}^2 \setminus H$ that joins p with a point $q \in \gamma$, the loop γ (with positive direction) starting and ending at q, and a return path to p along l in the opposite direction. Such a loop Γ (and the corresponding element of π_1) is called a *geometric generator* (with centre at x) of the fundamental group $\pi_1 = \pi_1(\mathbb{C}^2 \setminus H, p)$. It is well known that π_1 is generated by geometric generators.

For each critical point s_i of H, we choose a neighbourhood $U_i \subset \mathbb{C}^2$ such that either $H \cap U_i$ is given by an analytic equation (in the local coordinates on U_i) or, if s_i is a negative node, $H \cap U_i$ consists of two smooth branches meeting transversally at p. We note that if s_i is a negative node, then $\pi_1(U_i \setminus H, p_i)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and is generated by two commuting geometric generators.

Choose smooth paths γ_i that lie in $\mathbb{C}^2 \setminus H$ and connect the points p_i to p. This choice defines homomorphisms $\psi_i\colon \pi_1(U_i \setminus H, p_i) \to \pi_1$. We call $\psi_i(\pi_1(U_i \setminus H, p_i)) = G_i$ the *local fundamental group* of the singular point s_i. Local fundamental groups are uniquely defined up to conjugacy in π_1.

Consider a homomorphism $\mu\colon \pi_1 \to \Sigma_N$ from the fundamental group $\pi_1 = \pi_1(\mathbb{C}^2 \setminus H)$ of the complement of the affine Hurwitz curve $H = (\mathbb{CP}^2 \setminus L_\infty) \cap \overline{H}$ to the symmetric group Σ_N such that the image $\operatorname{Im}\mu$ acts transitively on the set of N elements.

Let s_i be a negative node of H. As mentioned above, the local fundamental group G_i is generated by two commuting geometric generators, say, $\Gamma_{i,1}$ and $\Gamma_{i,2}$. We put

$$N_{i,j} = \{1 \leqslant n \leqslant N \mid \mu(\Gamma_{i,j})(n) \neq n\}.$$

We say that μ is *good at the negative node* s_i if $N_{i,1} \cap N_{i,2} = \varnothing$. The homomorphism μ is called a *monodromy of degree* N if it is good at all negative nodes.

The homomorphism μ determines an unramified covering $f = f_\mu\colon Y \to \mathbb{C}^2 \setminus H$ of degree N. This covering can be extended to a finite ramified covering $\tilde{f}\colon \widetilde{Y} \to \mathbb{CP}^2$ branched along $\overline{H}$ and possibly along L_∞.

To describe this extension, we consider a geometric generator Γ with centre at $x \in H \setminus \operatorname{Crit} H$, where $\operatorname{Crit} H$ is the set of critical points of H. The image $\mu(\Gamma)$ in Σ_N is a product of cyclic permutations $\sigma_1, \dots, \sigma_{n_x}$. (The orders of some of the σ_i may be equal to one.) Each σ_l cyclically permutes the elements of the set $\{n_{1,l}, \dots, n_{r_l,l}\}$, $1 \leqslant n_{j,l} \leqslant N$, where r_l is the order of the permutation. Then the number of pre-images $\tilde{f}^{-1}(x)$ is equal to n_x, and each point y of $\tilde{f}^{-1}(x)$ corresponds to a cyclic permutation σ_l. Near the point y_l corresponding to σ_l, the covering $\tilde{f}$ is a cyclic covering of degree r_l branched along H and locally isomorphic to a subvariety of $\mathbb{C}^3$ given by $w^{r_l} = v - v_j(u)$, where $v - v_j(u) = 0$ is a local equation of $\overline{H}$ at the point x (see (1.1)). These local isomorphisms equip $\widetilde{Y}$ with the structure of a smooth manifold at each point y lying over $x \in H \setminus \operatorname{Crit} H$.

Let $s_i \in \operatorname{Crit} H$ be a negative node. As mentioned above, the local fundamental group G_i is generated by two geometric generators $\Gamma_{i,1}$ and $\Gamma_{i,2}$. The images $\mu(\Gamma_{i,j})$ in Σ_N are products of cyclic permutations $\sigma_{1,i,j}, \dots, \sigma_{k_{i,j},i,j}$. Let

$$\sigma_{l,i,j} = (n_{1,l,i,j}, \dots, n_{r_{l,i,j},l,i,j})$$

be a permutation of order $r_{l,i,j}$. Put $N_{i,1,2} = \{1 \leqslant n \leqslant N \mid \mu(\Gamma_{i,1})(n) = n$ and $\mu(\Gamma_{i,2})(n) = n\}$. Since μ is a monodromy, the set $\tilde{f}^{-1}(s_i)$ is in one-to-one correspondence with the union of the set $N_{i,1,2}$ and all cyclic permutations $\sigma_{l,i,j}$ $(j = 1, 2)$ of orders greater than one. Moreover, if $y \in \tilde{f}^{-1}(s_i)$ corresponds to an element of $N_{i,1,2}$, then $\tilde{f}$ is an isomorphism of a neighbourhood V of y onto its image $\tilde{f}(V)$. If $y \in \tilde{f}^{-1}(s_i)$ corresponds to a cyclic permutation $\sigma_{l,i,j}$ of order greater than one, then the restriction of $\tilde{f}$ to a neighbourhood of y is a cyclic covering of a neighbourhood of s_i of degree $r_{l,i,j}$ branched along the jth branch of the negative node. It is locally isomorphic to a subvariety of $\mathbb{C}^3$ given by $w^{r_{l,i,j}} = v - v_j(u)$, where $v - v_j(u) = 0$ is a local equation of the jth branch of $\overline{H}$ at the point s_i. These local isomorphisms equip $\widetilde{Y}$ with the structure of a smooth manifold at each point $y \in \tilde{f}^{-1}(s_i)$.

Suppose that $x = s_i \in \mathrm{Crit}_{\mathrm{analytic}} H$, that is, s_i is a critical point of H which is not a negative node. Consider a small neighbourhood U of s_i in which H is given by an analytic equation. Then the pre-image $\tilde{f}^{-1}(U)$ is a disjoint union of n_{s_i} open neighbourhoods which are in one-to-one correspondence with the orbits of the action of $\mu(G_i)$ on the set of N elements. The theorem of Grauert, Remmert and Stein (see [21] for a proof) implies that the variety $\widetilde{Y}$ can be equipped with the structure of a two-dimensional complex-analytic variety over the neighbourhood U of s_i.

By assumption, in a neighbourhood U of L_∞ (where $U = \mathbb{CP}^2 \setminus B(R)$ and $B(R) \subset \mathbb{C}^2_1$ is a ball of large radius R), the curve $\overline{H}$ coincides with the algebraic curve $\overline{C} \subset \mathbb{CP}^2$ of degree m given by the equation $v_2^m - 1 = 0$ in U. If we choose the base point p to lie in U, we can consider Zariski–van Kampen presentations of π_1 and the fundamental group $\tilde{\pi}_1 = \pi_1(\mathbb{CP}^2 \setminus (\overline{C} \cup L_\infty), p)$ having the same number of generators. It is easy to see that these presentations define an epimorphism $e\colon \tilde{\pi}_1 \to \pi_1$. The composite $\mu \circ e$ determines a ramified covering $\tilde{g}\colon \widetilde{Z} \to \mathbb{CP}^2$ branched along $\overline{C}$ and possibly along L_∞. It is easy to see that the coverings $\tilde{f}$ and $\tilde{g}$ are isomorphic over U. Therefore the variety $\tilde{f}^{-1}(U)$ can be identified with $\tilde{g}^{-1}(U)$ by an isomorphism $h\colon \tilde{f}^{-1}(U) \to \tilde{g}^{-1}(U)$. Hence $\tilde{f}^{-1}(U)$ may also be regarded as a complex-analytic variety.

Let $i\colon \widetilde{Z} \hookrightarrow \mathbb{CP}^{m_\infty}$ be an embedding such that $\tilde{g}$ is defined by the projection

$$(z'_0 : z'_1 : \ldots : z'_{m_\infty}) \to (z'_0 : z'_1 : z'_2).$$

Put

$$\begin{aligned} z_j &= h^*(z'_j), \qquad j = 3, \ldots, m_\infty, \\ w'_{j,\infty} &= \frac{z_j}{z_0}, \qquad j = 3, \ldots, m_\infty. \end{aligned} \tag{3.1}$$

Theorem 3.1. *Let $\overline{H}$ be a Hurwitz curve with negative nodes, $\mu\colon \pi_1(\mathbb{C}^2 \setminus H) \to \Sigma_N$ a monodromy, and $\tilde{f}\colon Y \to \mathbb{CP}^2$ the covering associated with μ. Then $\widetilde{Y}$ can be embedded in some projective space $\mathbb{CP}^M$ as a symplectic subvariety with analytic singularities.*

Proof. For every point $p \in \mathbb{C}^2_1$, let $V_p \subset U_p$ be the small balls $V_p = \{|u_1 - u_1(p)|^2 + |v_1 - v_1(p)|^2 < \delta_1^2\}$ and $U_p = \{|u_1 - u_1(p)|^2 + |v_1 - v_1(p)|^2 < \delta_2^2\}$ of radii δ_1 and δ_2 respectively, $0 < \delta_1 < \delta_2 \ll 1$, and let $\rho_p\colon \mathbb{CP}^2 \to \mathbb{R}$ be a smooth non-negative function such that $\rho_{p|V_p} \equiv 1$ and $\rho_{p|\mathbb{CP}^2 \setminus U_p} \equiv 0$.

To construct the desired embedding, we choose open coverings $\{U_i\}$ and $\{V_i\}$ of $\mathbb{CP}^2$ as follows.

For each point $s_i \in \mathrm{Crit}_{\mathrm{analytic}} \overline{H}$ which is not a negative mode, we choose small neighbourhoods $V'_{s_i} \subset V_{s_i} \subset U_{s_i} \subset U'_{s_i}$ such that

(c_1) the curve $U'_{s_i} \cap \overline{H}$ is analytic in U'_{s_i},

(c_2) the pre-image $\tilde{f}^{-1}(U'_{s_i})$ splits into a disjoint union of neighbourhoods of the points $y_{i,j} \in \tilde{f}^{-1}(s_i)$,

(c_3) the radius of the ball V'_{s_i} (resp. U_{s_i}) is strictly less than that of V_{s_i} (resp. U'_{s_i}).

Let $V'_\infty \subset V_\infty \subset U_\infty \subset U'_\infty$ be open neighbourhoods of L_∞ such that

(c_4) in U'_∞, the Hurwitz curve $\overline{H}$ coincides with the curve $\overline{C}$ given in U'_∞ by $v_2^m - 1 = 0$,

(c_5) $U'_\infty \cap V'_{s_i} = \varnothing$ for all the neighbourhoods V'_{s_i} of s_i chosen above,

(c_6) we have $\overline{V}'_\infty \subset V_\infty$ and $\overline{U}_\infty \subset U'_\infty$, where $\overline{V}'_\infty$ (resp. $\overline{U}_\infty$) is the closure of V'_∞ (resp. U_∞).

Let $\rho_\infty \colon \mathbb{CP}^2 \to \mathbb{R}$ be a smooth non-negative function such that $\rho_{\infty|V_\infty} \equiv 1$ and $\rho_{\infty|\mathbb{CP}^2\setminus U_\infty} \equiv 0$. We add the neighbourhoods U_∞ and V_∞ to the sets $\{U_{s_i}\}$ and $\{V_{s_i}\}$ chosen above. For each point $p \in \mathbb{C}^2_1 \setminus \big(\big(\bigcup V_{s_i}\big) \cup V_\infty\big)$ one can find open neighbourhoods $V_p \subset U_p$ of p such that

(c_7) $U_p \cap V'_{s_i} = \varnothing$ and $U_p \cap V'_\infty = \varnothing$ for the neighbourhoods V'_{s_i} and V'_∞ chosen above,

(c_8) the pre-image $\tilde{f}^{-1}(U_p)$ splits into a disjoint union of neighbourhoods of the points $y_j \in \tilde{f}^{-1}(p)$,

(c_9) if $p \in U'_{s_i}$ (resp. $p \in U'_\infty$) for some neighbourhood U'_{s_i} (resp. U'_∞) chosen above, then $U_p \subset U'_{s_i}$ (resp. $U_p \subset U'_\infty$),

(c_{10}) if $p \notin \big(\bigcup_{s_i \in \mathrm{Crit}_{\mathrm{analytic}}\overline{H}} U'_{s_i}\big) \cup U'_\infty$, then $U_p \cap \big(\big(\bigcup_{s_i \in \mathrm{Crit}_{\mathrm{analytic}}\overline{H}} U_{s_i}\big) \cup U_\infty\big) = \varnothing$.

We add the neighbourhoods U_p and V_p to the sets $\{U_{s_i}\}$ and $\{V_{s_i}\}$ chosen above (here one of s_i equals ∞). As a result, we get open coverings $\mathcal{V} = \{V_p\}$ and $\mathcal{U} = \{U_p\}$ of $\mathbb{CP}^2$.

For $U' = U'_\infty$ we define functions $(w_{3,\infty}, \dots, w_{m_\infty,\infty})$ on $\widetilde{Y} \setminus \tilde{f}^{-1}(L_\infty)$ by $w_{j,\infty} = \tilde{f}^*(\rho_\infty) w'_j$ for $j = 3, \dots, m_\infty$, where the functions w'_j were defined in (3.1).

For each U'_{s_i}, where $s_i \in \mathrm{Crit}_{\mathrm{analytic}}\overline{H}$, there are complex-analytic functions $w'_{1,s_i}, \dots, w'_{m_{s_i},s_i}$ on $\widetilde{U}'_{s_i} = \tilde{f}^{-1}(U'_{s_i})$ such that these functions together with $\tilde{f}^*(u_1)$ and $\tilde{f}^*(v_1)$ give an analytic embedding of $\widetilde{U}'_{s_i}$ in $\mathbb{C}^{m_{s_i}+2}$. Let $w_{j,s_i} = \tilde{f}^*(\rho_{s_i}) w'_{j,s_i}$, $1 \leqslant j \leqslant m_{s_i}$ be the corresponding functions on $\widetilde{Y}$.

By the construction of the open covering, the pre-image $\tilde{f}^{-1}(V_p) = \bigsqcup \widetilde{V}_{p,j}$ (resp. $\tilde{f}^{-1}(U_p) = \bigsqcup \widetilde{U}_{p,j}$) splits into a disjoint union of $m_p = n_p$ connected neighbourhoods $\widetilde{V}_{p,j}$ (resp. $\widetilde{U}_{p,j}$), $j = 1, \dots, n_p$. If $p \in \overline{H}$, then the neighbourhood $\widetilde{U}_{p,j}$ is isomorphic to a subvariety of $\mathbb{C}^3$ given in the coordinates $(u_1, v_1, w'_{j,p})$ by

$$(w'_{j,p} - w^0_{j,p})^{r_{j,p}} = v_1 - v_{1,p}(u_1),$$

where $v_1 - v_{1,p}(u_1) = 0$ is the equation of $\overline{H}$ in U_p. We extend the functions $w'_{j,p}$ by putting $w'_{j,p|\widetilde{U}_{p,l}} \equiv 0$ for $l \neq j$ and choose constants $w^0_{j,p}$ $(j = 1, \dots, m_p)$ such that the functions $(u_1, v_1, w'_{1,p}, \dots, w'_{m_p,p})$ define a smooth embedding of $\widetilde{U}_p$ in $\mathbb{C}^{m_p+2}$. Let $w_{j,p} = \tilde{f}^*(\rho_p) w'_{j,p}$, $j = 1, \dots, m_p$, be the corresponding functions on $\widetilde{Y}$.

If $y \in \tilde{f}^{-1}(\overline{H}) \cap \widetilde{V}_{p,j} \cap \widetilde{U}_{q,j}$, where $p, q \neq \infty$ and $p, q \notin \mathrm{Crit}_{\mathrm{analytic}}\overline{H}$, then the definition of the functions $w_{j,p}$ and $w_{j,q}$ implies that $r_{j,p} = r_{j,q} = r_j$ and there is a r_jth root $\zeta_{p,q}$ of unity such that

$$\begin{aligned} w_{j,q} &= \rho_q(u_1, v_1)(\zeta_{p,q}(w_{j,p} - w^0_{j,p}) + w^0_{j,p}), \\ w_{j',q} &\equiv 0, \qquad j' \neq j, \end{aligned} \tag{3.2}$$

in a neighbourhood of y.

Similarly, if $y \in \tilde{f}^{-1}(\overline{H}) \cap \widetilde{V}_{p,j} \cap \widetilde{U}_q$ (resp. $y \in \tilde{f}^{-1}(\overline{H}) \cap \widetilde{V}_q \cap \widetilde{U}_{p,j}$), where $q = \infty$ or $q \in \mathrm{Sing}_{\mathrm{analytic}}\, \overline{H}$, then the definition of the functions $w_{j,p}$ and $w_{i,q}$ and properties (c_1), (c_4), (c_9) and (c_{10}) imply that

$$(w_{j,p} - w_{j,p}^0)^{r_j} = v_1 - F(u_1) \tag{3.3}$$

and

$$w_{i,q} = \rho_q(u_1, v_1) h_i(u_1, v_1, w_{j,p}), \qquad 1 \leqslant i \leqslant m_q \tag{3.4}$$

(resp. $w_{j,p} = \rho_p(u_1, v_1) h_j(u_1, v_1, w_{1,q}, w_{2,q}, w_{3,q}, \dots, w_{m_q,q})$, where $w_{1,q}$ and $w_{2,q}$ are constants if $q = \infty$) in a neighbourhood of y. Here F and all the h_i (resp. h_j) are analytic functions, and $v_1 - F(u_1) = 0$ is an analytic equation of some branch of $\overline{H}$.

If $p \notin \overline{H} \cup L_\infty$, then we can assume that $\tilde{f}$ defines an isomorphism of the neighbourhoods $\widetilde{U}_{p,j}$ and U_p for $j = 1, \dots, m_p = N$. We choose N distinct constants $w_{j,p}^0$ and consider the functions $w_{j,p} = \tilde{f}^*(\rho_p) w'_{j,p}$ $(j = 1, \dots, m_p)$ on $\widetilde{Y}$, where the functions $w'_{j,p}$ are defined on $\widetilde{U}_p$ by

$$w'_{j,p}(q) \equiv \begin{cases} w_{j,p}^0 & \text{if } q \in \widetilde{U}_{p,j}, \\ 0 & \text{if } q \notin \widetilde{U}_{p,j}. \end{cases}$$

Choose a finite covering $\widetilde{\mathcal{V}}_0 = \{\widetilde{V}_{p_i,j} \mid 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant m_{p_i}\} \cup \{\widetilde{V}_\infty\}$ of $\widetilde{Y}$ and put

$$M = m_\infty + \sum_{j=1}^{k} m_{p_j},$$

$w_3 = w_{3,\infty}$, ..., $w_{m_\infty} = w_{m_\infty,\infty}$. We enumerate the set of functions

$$\{w_{j,p_i} \mid 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant m_{p_i}\}$$

by the numbers $m_\infty + 1, \dots, M$.

Consider a linear projection $p\colon \mathbb{CP}^M \to \mathbb{CP}^2$ given by

$$(z_0 : z_1 : z_2 : \dots : z_M) \to (z_0 : z_1 : z_2).$$

The base locus of p is the projective space $P \simeq \mathbb{CP}^{M-3}$ given by $z_0 = z_1 = z_2 = 0$. The restriction of p to $\mathcal{L} = \mathbb{CP}^M \setminus P$ equips $\mathcal{L}$ with the structure of a vector bundle over $\mathbb{CP}^2$ whose zero section is given by $z_3 = \dots = z_M = 0$. The bundle $\mathcal{L}$ is trivial over the charts $\mathbb{C}_i^2$ with coordinates (u_i, v_i), $i = 1, 2, 3$. Its restriction to $\mathbb{C}_i^2$ is isomorphic to $\mathbb{C}_i^M \simeq \mathbb{C}_i^2 \times \mathbb{C}_i^{M-2}$. In particular, $(z_3/z_0, \dots, z_M/z_0)$ are coordinates on $\mathbb{C}_1^{M-2}$.

Over $\mathbb{C}_1^2$ we define a map $\alpha'\colon \tilde{f}^{-1}(\mathbb{C}_1^2) \to \mathbb{C}_1^M$ by

$$\alpha'(y) = (\tilde{f}^*(u_1)(y), \tilde{f}^*(v_1)(y), w_3(y), \dots, w_M(y)).$$

Each U_{p_i} is a subset of $\mathbb{C}_1^2$ for $p_i \neq \infty$, and all the functions w_{j,p_i} are identically equal to zero at the points lying over the complement of U_{p_i}. Therefore we can extend α' to a map $\alpha\colon \widetilde{Y} \to \mathcal{L}$ by declaring that α equals $i\circ h$ over the neighbourhood $\tilde{f}^{-1}(V'_\infty)$, where h was defined above and i is the linear embedding of $\mathbb{CP}^{m_\infty}$ in $\mathbb{CP}^M$ given by

$$i((z_0 : \ldots : z_{m_\infty})) = (z_0 : \ldots : z_{m_\infty} : 0 : \ldots : 0).$$

It is easy to see that α is an embedding, and $\alpha(\widetilde{V}'_{s_i})$ and $\alpha(\widetilde{V}'_\infty)$ are analytic subsets of $\mathcal{L}$ for $\widetilde{V}'_\infty = \tilde{f}^{-1}(V'_\infty)$ and all neighbourhoods $\widetilde{V}'_{s_i} = \tilde{f}^{-1}(V'_{s_i})$, $s_i \in \mathrm{Crit}_{\mathrm{analytic}}\,\overline{H}$.

Let Ω be the restriction of the Fubini–Study form Ω_M to $\mathcal{L}$. In the chart $\mathbb{C}^M_1$, it is given by

$$\Omega = \frac{i\sum_{k=1}^{M}\bigl(dw_k\wedge d\overline{w}_k + \sum_{j\neq k}(\overline{w}_j w_j\, dw_k\wedge d\overline{w}_k - \overline{w}_j w_k\, dw_j\wedge d\overline{w}_k)\bigr)}{\bigl(1+\sum_{j=1}^{M}\overline{w}_j w_j\bigr)^2}\,, \tag{3.5}$$

where $w_k = \frac{z_k}{z_0}$ and $w_1 = u_1,\ \ w_2 = v_1$.

We shall use the same symbol $\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ to denote a set of two positive numbers and the linear transformation $\bar{\varepsilon}\colon \mathbb{CP}^M \to \mathbb{CP}^M$ given by

$$(z_0 : z_1 : z_2 : z_3 : \ldots : z_M) \to (z_0 : z_1 : \varepsilon_1 z_2 : \varepsilon_2 z_3 : \ldots : \varepsilon_2 z_M).$$

Let $\omega_{\bar{\varepsilon}}$ be the restriction of Ω to $\widetilde{Y}_{\bar{\varepsilon}} = (\bar{\varepsilon}\circ\alpha)(\widetilde{Y})$. Let us show that one can find a positive constant c_1 and a positive function $c_2(t)$, $\ t \in (0, c_1]$, such that $\widetilde{Y}_{\bar{\varepsilon}}$ is a symplectic submanifold of $\mathcal{L}$ for all $\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ with $\varepsilon_1 \leqslant c_1$ and $\varepsilon_2 \leqslant c_2(\varepsilon_1)$. We note that the image of $\overline{H}$ under the map $(z_0 : z_1 : z_2) \to (z_0 : z_1 : \varepsilon_1 z_2)$ becomes symplectic if ε_1 is small enough.

For each $\bar{\varepsilon}$, the form $\omega_{\bar{\varepsilon}}$ is a symplectic form on the sets $(\bar{\varepsilon}\circ\alpha)(\widetilde{V}'_{s_i})$, where s_i is an analytic singular point of $\overline{H}$, and on the set $(\bar{\varepsilon}\circ\alpha)(\widetilde{V}'_\infty)$, because $\widetilde{Y}_{\bar{\varepsilon}}$ is an analytic subvariety of $\mathcal{L}$ at every point of these sets.

Consider a point $y \in \tilde{f}^{-1}(\overline{H}\,)$ that belongs to the ramification locus of $\tilde{f}$ and satisfies $\tilde{f}(y) \notin \widetilde{U}_p$ for $p \in \mathrm{Crit}_{\mathrm{analytic}}\,\overline{H}$ and $p = \infty$. By (3.2), renumbering the coordinates $w_3, \ldots, w_M$, we can assume that $\widetilde{Y}$ is given in a neighbourhood of $\alpha(y)$ by

$$\begin{aligned} (w_3 - w_{3,0})^r &= v_1 - F(u_1), \\ w_j &= \rho_j(u_1, v_1) h_j(w_3), \qquad j \geqslant 4, \end{aligned} \tag{3.6}$$

where $r \geqslant 2$, $\ \rho_j$ are smooth functions, $h_j = \zeta_{3,j}(w_3 - w_{3,0}) + w_{j,0}$ are analytic functions (here $\zeta_{3,j}$ is either an rth root of unity or $\zeta_{3,j} = 0$), $\ v_1 - F(u_1) = 0$ is the equation of a branch of $\overline{H}$ at the point $\tilde{f}(y)$, and $\alpha(y) = (u_{1,0}, F(u_{1,0}), w_{3,0}, \ldots, w_{M,0})$. Then the variety $\widetilde{Y}_{\bar{\varepsilon}}$ is given by

$$\begin{aligned} \varepsilon_1(w_3 - \varepsilon_2 w_{3,0})^r &= \varepsilon_2^r(v_1 - \varepsilon_1 F(u_1)), \\ w_j &= \varepsilon_2\rho_j\left(u_1, \frac{v_1}{\varepsilon_1}\right) h_j\left(\frac{w_3}{\varepsilon_2}\right), \qquad j \geqslant 4, \end{aligned} \tag{3.7}$$

in a neighbourhood of $(\bar{\varepsilon}\circ\alpha)(y) = (u_{1,0}, \varepsilon_1 F(u_{1,0}), \varepsilon_2 w_{3,0}, \ldots, \varepsilon_2 w_{M,0})$.

We put $A_1 = \frac{\partial F}{\partial u_1}(\alpha(y))$, $A_2 = \frac{\partial F}{\partial \bar{u}_1}(\alpha(y))$, $B_j = \frac{\partial h_j}{\partial w_3}(\alpha(y))$, $C_{j,1} = \frac{\partial \rho_j}{\partial u_1}(\alpha(y))$, $C_{j,2} = \frac{\partial \rho_j}{\partial \bar{u}_1}(\alpha(y))$, $D_{j,1} = \frac{\partial \rho_j}{\partial v_1}(\alpha(y))$, $D_{j,2} = \frac{\rho_j}{\partial \bar{v}_1}(\alpha(y))$, $\rho_{j,0} = \rho_j(\alpha(y))$, $h_{j,0} = h_j(w_{3,0})$, $j = 4, \dots, M$. It follows from (3.7) that at the point $(\bar{\varepsilon} \circ \alpha)(y)$ we have

$$\begin{aligned} dv_1 &= \varepsilon_1(A_1\, du_1 + A_2\, d\bar{u}_1), \\ d\bar{v}_1 &= \varepsilon_1(\bar{A}_2\, du_1 + \bar{A}_1\, d\bar{u}_1), \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} dw_j &= \rho_{j,0} B_j\, dw_3 + \varepsilon_2 h_{j,0}\left(C_{j,1}\, du_1 + C_{j,2}\, d\bar{u}_1 + D_{j,1}\frac{dv_1}{\varepsilon_1} + D_{j,2}\frac{d\bar{v}_1}{\varepsilon_1}\right), \\ d\overline{w}_j &= \rho_{j,0}\overline{B}_j\, d\overline{w}_3 + \varepsilon_2 \bar{h}_{j,0}\left(\overline{C}_{j,2}\, d\bar{u}_1 + \overline{C}_{j,1}\, du_1 + \overline{D}_{j,2}\frac{dv_1}{\varepsilon_1} + \overline{D}_{j,1}\frac{d\bar{v}_1}{\varepsilon_1}\right), \end{aligned} \tag{3.9}$$

$j = 4, \dots, M$.

Substituting (3.8) in (3.9), we get

$$\begin{aligned} dw_j &= \rho_{j,0} B_j\, dw_3 + \varepsilon_2 \nu_j, \\ d\overline{w}_j &= \rho_{j,0}\overline{B}_j\, d\overline{w}_3 + \varepsilon_2 \bar{\nu}_j, \end{aligned} \tag{3.10}$$

where the forms ν_j and $\bar{\nu}_j$ are independent of $\bar{\varepsilon}$ for $j = 4, \dots, M$.

It follows from (3.8) and (3.10) that, for each sufficiently small $\bar{\varepsilon}$, the tangent space to $\widetilde{Y}_{\bar{\varepsilon}}$ at the point $(\bar{\varepsilon} \circ \alpha)(y)$ is very close to the tangent space at the point $(\bar{\varepsilon} \circ \alpha)(y)$ of the linear algebraic variety Z given by $v_1 = \varepsilon_1 F(u_{1,0})$, $w_j - w_{j,0} = \rho_{j,0} B_j (w_3 - w_{3,0})$, $j = 4, \dots, M$. Therefore the form $\omega_{\bar{\varepsilon}}$ is symplectic at $(\bar{\varepsilon} \circ \alpha)(y)$ for all sufficiently small $\bar{\varepsilon}$. By continuity, it is symplectic in some neighbourhood of $(\bar{\varepsilon} \circ \alpha)(y)$.

Consider a point $y \in \tilde{f}^{-1}(\overline{H})$ that belongs to the ramification locus of $\tilde{f}$ and satisfies $\tilde{f}(y) \in U_p$ for some $p \in \mathrm{Crit}_{\mathrm{analytic}}\, \overline{H}$ or $p = \infty$. By (3.3) and (3.4), renumbering the coordinates $w_3, \dots, w_M$, we can assume that there is an n with $3 \leqslant n \leqslant M$ such that $\widetilde{Y}$ is given in a neighbourhood of $\alpha(y)$ by

$$\begin{aligned} h_j(u_1, v_1, w_3, \dots, w_n) &= 0, & j &= 3, \dots, n, \\ \rho_j(u_1, v_1) h_j(u_1, v_1, w_3, \dots, w_n) &= w_j, & j &= n+1, \dots, M, \end{aligned} \tag{3.11}$$

where ρ_j are smooth functions and h_j are analytic functions at the point $\tilde{f}(y)$. Let $(u_{1,0}, v_{1,0}, w_{3,0}, \dots, w_{M,0})$ be the coordinates of the point $\alpha(y)$. Then the variety $\widetilde{Y}_{\bar{\varepsilon}}$ is given in a neighbourhood of $(\bar{\varepsilon} \circ \alpha)(y) = (u_{1,0}, \varepsilon_1 v_{1,0}, \varepsilon_2 w_{3,0}, \dots, \varepsilon_2 w_{M,0})$ by

$$\begin{aligned} h_j\left(u_1, \frac{v_1}{\varepsilon_1}, \frac{w_3}{\varepsilon_2}, \dots, \frac{w_n}{\varepsilon_2}\right) &= 0, & j &= 3, \dots, n, \\ \varepsilon_2 \rho_j\left(u_1, \frac{v_1}{\varepsilon_1}\right) h_j\left(u_1, \frac{v_1}{\varepsilon_1}, \frac{w_3}{\varepsilon_2}, \dots, \frac{w_n}{\varepsilon_2}\right) &= w_j, & j &= n+1, \dots, M. \end{aligned} \tag{3.12}$$

We put $A_{j,l} = \frac{\partial h_j}{\partial w_l}(\alpha(y))$ for $1 \leqslant j \leqslant M$, $1 \leqslant l \leqslant n$ (here $w_1 = u_1$ and $w_2 = v_1$), and $B_j = \rho_j(u_{1,0}, v_{1,0})$ for $n+1 \leqslant j \leqslant M$.

It follows from (3.12) that, for each fixed positive ε_1 and for all sufficiently small ε_2, the tangent space of $\widetilde{Y}_{\bar\varepsilon}$ at the point $(\bar\varepsilon\circ\alpha)(y)$ is very close to the tangent space at the point $(\bar\varepsilon\circ\alpha)(y)$ of the linear algebraic variety Z given by

$$\sum_{l=3}^{n} A_{j,l}(w_l-\varepsilon_2 w_{l,0}) = -A_{j,1}(u_1-u_{1,0}) - \frac{A_{j,2}}{\varepsilon_1}(v_1-\varepsilon_1 v_{1,0}), \qquad 3\leqslant j\leqslant n,$$
$$B_j\sum_{l=3}^{n} A_{j,l}(w_l-\varepsilon_2 w_{l,0}) = w_j-\varepsilon_2 w_{j,0}, \qquad n+1\leqslant j\leqslant M.$$

Therefore the form $\omega_{\bar\varepsilon}$ is symplectic at $(\bar\varepsilon\circ\alpha)(y)$ for fixed ε_1 and all sufficiently small ε_2. By continuity, it is symplectic in some neighbourhood of $(\bar\varepsilon\circ\alpha)(y)$.

Finally, if y does not belong to the ramification locus of $\tilde f$, then the variety $\widetilde{Y}$ is given locally at $\alpha(y)$ by $w_j=F_j(u_1,v_1)$, $j=3,\dots,M$, where $F_j(u_1,v_1)$ are smooth functions in a neighbourhood of $\tilde f(y)$. Therefore the variety $\widetilde{Y}_{\bar\varepsilon}$ is given locally at $(\bar\varepsilon\circ\alpha)(y)$ by $w_j=\varepsilon_2F_j\bigl(u_1,\frac{v_1}{\varepsilon_1}\bigr)$, $j=3,\dots,M$. It is easy to see that the form $\omega_{\bar\varepsilon}$ is symplectic at $(\bar\varepsilon\circ\alpha)(y)$ for each fixed ε_1 and all sufficiently small ε_2. Indeed, the variety $\widetilde{Y}_{\bar\varepsilon}$ is very close to the algebraic variety given by $w_j=0$, $j=3,\dots,M$, which is symplectic.

To complete the proof, it suffices to recall that $\widetilde{Y}$ is compact.

Using Lemma 2.1 and Hironaka's theorem on the resolution of singularities, we get the following corollary.

Corollary 3.2. *Let $\tilde f\colon \widetilde{Y}\to\mathbb{CP}^2$ be a finite covering branched along a Hurwitz curve $\overline{H}$ (possibly with negative modes), and possibly along L_∞, and associated with a monodromy $\mu\colon\pi_1(\mathbb{C}^2\setminus H)\to\Sigma_N$. Then there are sets $(M_1,\dots,M_k)$ and $(n_1,\dots,n_k)$ of positive integers such that a resolution $\overline{Y}$ of singularities of $\widetilde{Y}$ can be embedded as a symplectic submanifold in $\bigl(\mathbb{CP}^{M_1}\times\dots\times\mathbb{CP}^{M_k},\Omega_{n_1,\dots,n_k}\bigr)$, where $\Omega_{n_1,\dots,n_k}=n_1p_1^*(\Omega_{M_1})+\dots+n_kp_k^*(\Omega_{M_k})$ and Ω_{M_j} is the Fubini–Study symplectic form on $\mathbb{CP}^{M_j}$.*

Theorem 3.3. *In the notation of Theorem* 3.1, *let $\widetilde{Y}$ be a smooth manifold. Then the symplectic structure constructed in the proof of Theorem* 3.1 *and given by the symplectic form $\omega_{\bar\varepsilon}$ is independent of $\bar\varepsilon$ (if the coordinates of $\bar\varepsilon$ are small enough) and of the way of choosing the coverings $\mathcal{U}$ and $\mathcal{V}$ and the functions $w_{i,j}$.*

Moreover, if $i\colon\widetilde{Y}\hookrightarrow\mathbb{CP}^N$ is an algebraic embedding (in the case when $\overline{H}$ is an algebraic curve) and $\tilde f=p\circ i$, where $p\colon\mathbb{CP}^N\to\mathbb{CP}^2$ is a linear projection, then $(\widetilde{Y},\omega_{\bar\varepsilon})$ and $(\widetilde{Y},i^*(\Omega_N))$ are symplectomorphic for all sufficiently small $\bar\varepsilon$, where Ω_N is the Fubini–Study form on $\mathbb{CP}^N$ and $\omega_{\bar\varepsilon}$ is the form constructed in the proof of Theorem 3.1.

Proof. We have $\omega_{\bar\varepsilon}=(\bar\varepsilon\circ\alpha)^*(\Omega)$, where the embedding $\alpha\colon\widetilde{Y}\to\mathcal{L}$ was constructed in the proof of Theorem 3.1. Clearly, $\omega_{\bar\varepsilon}$ depends smoothly on $\bar\varepsilon$. If $0<\varepsilon_1\leqslant c_1$ and $0<\varepsilon_2\leqslant c_2(\varepsilon_1)$, then the class $[\omega_{\bar\varepsilon}]\in H^2(\widetilde{Y},\mathbb{Z})$ is dual to the class $[\tilde f^{-1}(L)]\in H_2(\widetilde{Y},\mathbb{Z})$, where L is a line in $\mathbb{CP}^2$. Therefore, by Moser's stability theorem for symplectic structures (see [16], Theorem 3.17), the forms $\omega_{\bar\varepsilon}$ define the same symplectic structure if $0<\varepsilon_1\leqslant c_1$ and $0<\varepsilon_2\leqslant c_2(\varepsilon_1)$.

The symplectic structure on $\widetilde{Y}$ defined by the forms $\omega_{\bar{\varepsilon}}$ is independent of the choice of the coverings $\{U_i\}$, $\{V_i\}$ and the functions $w_{i,j}$ that determine the embedding α. Indeed, suppose that two sets $\{w'_{i,j}\}$ and $\{w''_{i,j}\}$ of functions determine two embeddings $\alpha'\colon \widetilde{Y}\to\mathcal{L}'\subset\mathbb{CP}^{M'}$ and $\alpha''\colon\widetilde{Y}\to\mathcal{L}''\subset\mathbb{CP}^{M''}$. We put

$$\omega'_{\bar{\varepsilon}}=(\bar{\varepsilon}\circ\alpha')^*(\Omega'),\qquad \omega''_{\bar{\varepsilon}}=(\bar{\varepsilon}\circ\alpha'')^*(\Omega''),$$

where Ω' and Ω'' are the Fubini–Study symplectic forms on $\mathbb{CP}^{M'}$ and $\mathbb{CP}^{M''}$ respectively. Then we have an embedding $\alpha'\times\alpha''\colon\widetilde{Y}\to\mathcal{L}'\times_{\mathbb{CP}^2}\mathcal{L}''$. Note that the form $\Omega_t=t(p')^*(\Omega')+(1-t)(p'')^*(\Omega'')$ is a Kähler form for each $t\in[0,1]$. Since the interval $[0,1]$ is compact, we can employ the same calculation as in the proof of Theorem 3.1 to show that there is $\bar{\varepsilon}=\bar{\varepsilon}'=\bar{\varepsilon}''$ with $0<\varepsilon_1\leqslant\min(c'_1,c''_1)$ and $0<\varepsilon_2\leqslant\min(c'_2(\varepsilon_1),c''_2(\varepsilon_1))$ such that $\omega_{t,\bar{\varepsilon}}=(\bar{\varepsilon}\circ(\alpha'\times\alpha''))^*(\Omega_t)$ is a symplectic form on $\widetilde{Y}$ for all $t\in[0,1]$. On the other hand, we have $\omega_{0,\bar{\varepsilon}}=\omega'_{\bar{\varepsilon}}$ and $\omega_{1,\bar{\varepsilon}}=\omega''_{\bar{\varepsilon}}$, and the forms $\omega_{t,\bar{\varepsilon}}$ belong to the same cohomology class. Therefore, by Moser's stability theorem for symplectic structures, the forms $\omega_{t,\bar{\varepsilon}}$ determine the same symplectic structure on $\widetilde{Y}$.

Consider an algebraic embedding $i\colon\widetilde{Y}\hookrightarrow\mathbb{CP}^N$ such that $\tilde{f}=p\circ i$, where $p\colon\mathbb{CP}^N\to\mathbb{CP}^2$ is a linear projection. Let $\alpha'\colon\widetilde{Y}\to\mathcal{L}'\subset\mathbb{CP}^{M'}$ be any embedding constructed in the proof of Theorem 3.1. Put $\alpha''=i$. By the same argument as above, we easily see that $(\widetilde{Y},\omega_{\bar{\varepsilon}})$ and $(\widetilde{Y},i^*(\Omega_N))$ are symplectomorphic for all sufficiently small $\bar{\varepsilon}$ because the symplectic manifolds $(\widetilde{Y},\omega''_{\bar{\varepsilon}})$ are symplectomorphic for all positive $\bar{\varepsilon}$, where $\omega''_{\bar{\varepsilon}}=(\bar{\varepsilon}\circ\alpha'')^*(\Omega_N)$.

§ 4. Embeddings of cyclic coverings of the plane in rational projective 3-folds

In this section we use the notation and assumptions of § 1.

Let $\overline{H}$ be a Hurwitz curve of degree m. Consider the infinite cyclic covering $f=f_\infty\colon X_\infty\to X'=\mathbb{C}^2\setminus H$ corresponding to the epimorphism $\nu\colon\pi_1(\mathbb{C}^2\setminus H)\to\mathbb{F}_1$. The covering f_∞ can be factored through the cyclic covering $f_n\colon X'_n\to\mathbb{C}^2\setminus H$ associated with the epimorphism $\operatorname{mod}_n\circ\nu$, $f_\infty=g_n\circ f_n$.

In this section we will show that the covering f_n can be extended to a smooth map $\bar{f}_n\colon\overline{X}_n\to\mathbb{CP}^2$ branched along $\overline{H}$ and possibly along L_∞ (if n does not divide $\deg\overline{H}$, then $\bar{f}_n$ is branched along L_∞), where $\overline{X}_n$ is a real smooth 4-dimensional manifold.

Theorem 4.1. *Let $\overline{X}_n$ be a resolution of singularities of a cyclic covering of $\mathbb{CP}^2$ of degree n branched along a Hurwitz curve $\overline{H}$ and possibly along L_∞. Then $\overline{X}_n$ can be embedded in some rational projective 3-fold* (*equipped with an integer Kähler symplectic structure*) *as a symplectic submanifold.*

Proof. Since $\mathbb{C}^2\setminus H_1$ and $\mathbb{C}^2\setminus H_2$ are diffeomorphic for H-isotopic Hurwitz curves $\overline{H}_1$ and $\overline{H}_2$, we can assume that $\overline{H}$ satisfies conditions (*) and (**).

By Lemma 2.1 and Hironaka's theorem on the resolution of singularities, it suffices to show that there is an extension $\tilde{f}_n\colon\widetilde{X}_n\to\mathbb{CP}^2$ of $f'_n\colon X'_n\to X'$ such that the variety $\widetilde{X}_n$ can be embedded in some rational projective 3-fold (equipped with an integer Kähler symplectic structure) as a symplectic subvariety with analytic singularities.

To show this, we denote by d the smallest non-negative integer such that $m+d$ is divisible by n. Put $m+d=kn$ and consider the line bundle $p\colon \mathcal{L}(k)\to\mathbb{CP}^2$ (see § 1) associated with the sheaf $\mathcal{O}_{\mathbb{CP}^2}(k)$. By Lemma 1.4, the Hurwitz curve $\overline{H}$ coincides with the zero locus of a smooth section $\bar{s}_m$ of $\mathcal{L}(m)$ over $\mathbb{CP}^2$ such that $\bar{s}_m$ is analytic in a neighbourhood U of L_∞ and in neighbourhoods of all the critical points of $\overline{H}$.

Let $\bar{s}_d$ be the section of $\mathcal{L}(d)$ defined by $w_2=u_2^d$ over $\mathbb{C}_2^2$. The product

$$\bar{s}_{m+d}=\bar{s}_m\bar{s}_d \tag{4.1}$$

is a section of $\mathcal{L}(m+d)$, where $\bar{s}_m$ is a section of $\mathcal{L}(m)$ satisfying all the conditions of Lemma 1.4.

We define $\alpha\colon \widetilde{X}_n\hookrightarrow\mathcal{L}(k)$ by the equation

$$w_i^n=\bar{s}_{m+d}(u_i,v_i) \tag{4.2}$$

and put $\tilde{f}_n=p_{|X_n}$, where $p\colon\mathcal{L}(k)\to\mathbb{CP}^2$ is the morphism defining the structure of the line bundle on $\mathcal{L}(k)$. In particular, $\widetilde{X}_n$ is given by the equation

$$w_1^n=F_1(u_1,v_1)$$

in $\mathbb{C}_1^3$ and by

$$w_2^n=u_2^dF_2(u_2,v_2)$$

in $\mathbb{C}_2^3$.

Clearly, the covering $\tilde{f}_n$ is an unramified n-sheeted cyclic covering over $\mathbb{CP}^2\setminus(\overline{H}\cup L_\infty)$. All the singular points of the variety $\widetilde{X}_n$ lie over singular points of $\overline{H}$ and possibly over L_∞. Moreover, by the construction of $\bar{s}_{m+d}$, the set $\operatorname{Sing}\widetilde{X}_n$ is complex-analytic in some neighbourhood $U\subset\mathcal{L}(k)$.

The line bundle $\mathcal{L}(k)$ is a quasi-projective variety and can be compactified to a projective 3-dimensional rational manifold $\overline{\mathcal{L}}(k)$ by adding a section "at infinity". The variety $\overline{\mathcal{L}}(k)$ has many different embeddings in projective spaces since its Picard group is $\operatorname{Pic}(\overline{\mathcal{L}}(k))\simeq\mathbb{Z}\oplus\mathbb{Z}$. We choose one of these embeddings, for example, the following one.

In the neighbourhood $\mathbb{C}_1^3$ with coordinates (u_1,v_1,w_1), consider monomials $u_1^{a_1}v_1^{a_2}w_1^{a_3}$, $0\leqslant a_1+a_2+ka_3\leqslant k+1$, the number of which is equal to $\frac{(k+2)(k+3)}{2}+3$. Put $N=\frac{(k+2)(k+3)}{2}+2$ and consider the rational map $h\colon\overline{\mathcal{L}}(k)\to\mathbb{CP}^N$ given in $\mathbb{C}_1^3$ by $z_{\bar{a}}=u_1^{a_1}v_1^{a_2}w_1^{a_3}$, where $\bar{a}=(a_1,a_2,a_3)$ are triples of integers and $z_{\bar{a}}$ are homogeneous coordinates in $\mathbb{CP}^N$. It is easy to check that h is an embedding.

We consider the Fubini–Study form Ω_N on $\mathbb{CP}^N$ and denote its pullback by $\Omega=h^*(\Omega_N)$. As in the proof of Theorem 3.1, we shall use the same symbol $\bar{\varepsilon}=(\varepsilon_1,\varepsilon_2)$ to denote a set of two positive numbers and the automorphism of $\overline{\mathcal{L}}(k)$ given in $\mathbb{C}_1^3$ by $(u_1,v_1,w_1)\to(u_1,\varepsilon_1v_1,\varepsilon_2w_1)$.

Calculations (to be omitted) similar to those in the proof of Theorem 3.1 show that one can find a positive constant c_1 and a positive function $c_2(t)$ such that $\widetilde{X}_{\bar{\varepsilon}}=(\bar{\varepsilon}\circ\alpha)(X_n)$ is a symplectic subvariety of $\mathcal{L}$ with analytic singularities for all $\bar{\varepsilon}=(\varepsilon_1,\varepsilon_2)$ with $\varepsilon_1\leqslant c_1$ and $\varepsilon_2\leqslant c_2(\varepsilon_1)$.

§ 5. Alexander polynomials of C-groups

Let $\overline{H}$ be a Hurwitz (resp. topological Hurwitz) curve of degree m. Since any Zariski–van Kampen presentation of $\pi_1(\mathbb{C}^2 \setminus H)$ is a C-presentation of a Hurwitz C-group of degree m, Theorems 0.1 and 0.2 are corollaries of Theorems 5.1 and 5.2 below.

Theorem 5.1. *Let $G \in \mathcal{H}$ be a Hurwitz C-group of degree m and let $\Delta(t)$ be its Alexander polynomial. Then*
(i) $\Delta(t) \in \mathbb{Z}[t]$,
(ii) $\Delta(0) = \pm 1$,
(iii) *the roots of $\Delta(t)$ are mth roots of unity*,
(iv) *the rank of the free part of N'/N'' is equal to* $\deg \Delta(t)$,
(v) *the action of $h_{\mathbb{C}}$ on $N/N' \otimes \mathbb{C}$ is semisimple.*

Proof. Consider the exact sequence of groups

$$1 \to N \to G \xrightarrow{\nu} \mathbb{F}_1 \to 1.$$

This exact sequence induces an automorphism $\tilde{h} \in \operatorname{Aut} N$ (the action of the C-generator $x \in \mathbb{F}_1$ on N). It is given by $\tilde{h}(n) = \tilde{x}^{-1} n \tilde{x}$ for $n \in N$, where $\tilde{x}$ is one of the C-generators of G. Clearly, $\tilde{h}$ is uniquely defined up to an inner automorphism of the group N. Therefore $\tilde{h}$ determines an automorphism $h \in \operatorname{Aut} N/N'$.

In [13], it was proved that N is finitely presented for any Hurwitz C-group G. Therefore N/N' is a finitely generated abelian group. Let $N/N' = T \oplus F$ be a decomposition into the direct sum of the torsion subgroup T and a free abelian group F. Note that T is a finite group and F is finitely generated. The automorphism h of N/N' induces an automorphism of T and, therefore, an automorphism $\bar{h}$ of $F \simeq (N/N')/T$. If one chooses a free basis of the $\mathbb{Z}$-module F over $\mathbb{Z}$, then this automorphism is given by a matrix H with integer coefficients. Since the automorphism $h_{\mathbb{C}}$ of $N/N' \otimes \mathbb{C}$ is given by the same matrix H, we have $\Delta(t) = \det(H - t \operatorname{Id}) \in \mathbb{Z}[t]$. Since $\bar{h} \in \operatorname{Aut} F$, we get $\det \widetilde{H} = \pm 1$ and, therefore, $\Delta(0) = \pm 1$.

We claim that $\tilde{h}^m$ is an inner automorphism of N. Indeed, since G is a Hurwitz C-group of degree m, it is generated by C-generators $x_1, \dots, x_m$ such that the product $x_1 \dots x_m$ belongs to the centre of G. Therefore the element $\tilde{x}^m = \tilde{n} \cdot x_1 \dots x_m$ (where $\tilde{n} \in N$) induces an inner automorphism on N. Thus the induced automorphisms h^m of N/N' and $h_{\mathbb{C}}^m$ of $N/N' \otimes \mathbb{C}$ are trivial, that is, $h_{\mathbb{C}}^m = \operatorname{Id}$.

Since $h_{\mathbb{C}}^m = \operatorname{Id}$, all roots of the polynomial $\Delta(t) = \det(h_{\mathbb{C}} - t \operatorname{Id})$ are mth roots of unity, and the action of $h_{\mathbb{C}}$ on $N/N' \otimes \mathbb{C}$ is semisimple.

Theorem 5.2. *Let G be an irreducible Hurwitz C-group. Then*
(i) $\Delta(1) = 1$,
(ii) $\deg \Delta(t)$ *is an even number*,
(iii) $\Delta(t)$ *is a reciprocal polynomial.*

Proof. It follows from Lemma 6 in [9] that $\Delta(1) = \pm 1$. Let us show that $t = -1$ is not a root of $\Delta(t)$. Indeed, if $t = -1$ is a root of $\Delta(t)$, then $\Delta(t) = (t+1)P(t)$, where $P(t)$ is a polynomial with integer coefficients. Therefore $2P(1) = \pm 1$. But this is impossible since $P(1)$ is an integer.

By Theorem 5.1, all the roots of $\Delta(t) \in \mathbb{Z}[t]$ are roots of unity. They are non-real since $t = \pm 1$ are not roots of $\Delta(t)$. Thus $\deg \Delta(t)$ is even, and (ii) is proved.

It is well known that if λ is a primitive kth root of unity with $k > 2$ and a polynomial $P(t) \in \mathbb{Z}[t]$ vanishes at λ, then all the primitive kth roots of unity are roots of $P(t)$. In particular, λ^{-1} is also a root of $P(t)$, and (iii) is proved.

Since $\Delta(t) = \det(h_{\mathbb{C}} - t\,\mathrm{Id})$ and $\deg \Delta(t)$ is even, we have $\Delta(t) = t^{\deg \Delta(t)} + \dots$. Let $\Delta(t) = \prod_i \Phi_{k_i}(t)$ be a factorization as a product of k_ith cyclotomic polynomials. Since $\Delta(1) = \pm 1$, assertion (i) follows from the following well-known lemma.

Lemma 5.3. *Let $\Phi_k(t)$ be the kth cyclotomic polynomial, $k > 1$. Then*

$$\Phi_k(1) = \begin{cases} p & \text{if } k = p^n \text{ for some prime } p, \\ 1 & \text{if } k \neq p^n \text{ for any prime } p. \end{cases}$$

Proof. Using induction on $k > 1$, we deduce the lemma from the equalities

$$\Phi_k(t) = \frac{\sum_{i=0}^{k-1} t^i}{\prod_{\substack{d \mid k \\ d<k}} \Phi_d(t)}, \qquad \Phi_k(1) = \frac{k}{\prod_{\substack{d \mid k \\ d<k}} \Phi_d(1)}, \qquad \Phi_k(0) = \frac{1}{\prod_{\substack{d \mid k \\ d<k}} \Phi_d(0)}.$$

Corollary 5.4. *Let G be an irreducible Hurwitz C-group of degree $m = p^n$, where p is a prime number. Then*

(i) $\Delta(t) \equiv 1$,

(ii) *the group G'/G'' is a finite abelian group.*

Proof. By Theorem 5.1, all the roots of $\Delta(t)$ are mth roots of unity. Let λ be one of the roots. Assume that λ is a primitive p^kth root of unity, $1 \leqslant k \leqslant n$. Then λ is a root of the p^kth cyclotomic polynomial

$$\Phi_{p^k}(t) = \sum_{i=0}^{p-1} t^{ip^{k-1}},$$

and there is a polynomial $f(t) \in \mathbb{Z}[t]$ such that $\Delta(t) = \Phi_{p^k}(t) f(t)$. By Theorem 5.2, $\Phi_{p^k}(1) f(1) = \pm 1$. On the other hand, we have $f(1) \in \mathbb{Z}$ and $\Phi_{p^k}(1) = p$. Therefore $\Delta(t)$ has no roots. Thus $\deg \Delta(t) = 0$, and the group G'/G'' has no free part, that is, it is a finite abelian group.

Corollary 0.3 follows from Corollary 5.4.

Lemma 5.5. *Let G_1, G_2 be C-groups and let $\Delta_1(t)$, $\Delta_2(t)$ be their Alexander polynomials. Assume that there is a C-epimorphism $f\colon G_1 \to G_2$. Then $\Delta_2(t)$ is a divisor of $\Delta_1(t)$.*

Proof. Let N_i be the kernel of the canonical C-epimorphism $\nu_i\colon G_i \to \mathbb{F}_1$, $i = 1, 2$. It is easy to see that the homomorphism g in the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N_1 & \longrightarrow & G_1 & \xrightarrow{\nu_1} & \mathbb{F}_1 & \longrightarrow & 1 \\ & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \simeq} & & \\ 1 & \longrightarrow & N_2 & \longrightarrow & G_2 & \xrightarrow{\nu_2} & \mathbb{F}_1 & \longrightarrow & 1 \end{array}$$

is an epimorphism. This diagram induces the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N_1/N_1' & \longrightarrow & G_1/N_1' & \xrightarrow{\nu_{1*}} & \mathbb{F}_1 & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle g_*} & & \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle \simeq} & & \\ 1 & \longrightarrow & N_2/N_2' & \longrightarrow & G_2/N_2' & \xrightarrow{\nu_{2*}} & \mathbb{F}_1 & \longrightarrow & 1 \end{array} \tag{5.1}$$

where g_* is also an isomorphism.

It follows from diagram (5.1) that $\Delta_2(t)$ is a divisor of $\Delta_1(t)$ since $h_2(g_*(n)) = g_*(h_1(n))$ for any $n \in N_1/N_1'$.

Theorem 5.6. *Let G be a Hurwitz C-group of degree m. Then its Alexander polynomial $\Delta(t)$ divides the polynomial $(t-1)(t^m-1)^{m-2}$.*

Proof. Consider the Hurwitz C-group

$$\widetilde{G}_m = \langle x_1, \dots, x_m \mid [x_i, x_1 \dots x_m] = 1,\ i = 1, \dots, m\rangle. \tag{5.2}$$

For any Hurwitz C-group G of degree m, there is a natural C-epimorphism $f\colon \widetilde{G}_m \to G$ sending the C-generators x_i of $\widetilde{G}_m$ to C-generators x_i of G whose product $x_1 \dots x_m$ belongs to the centre of G. Therefore, to prove the theorem, it suffices to show that the Alexander polynomial of $\widetilde{G}_m$ is equal to $(-1)^{m-1}(t-1) \times (t^m-1)^{m-2}$.

We denote the kernel of $\nu\colon \widetilde{G}_m \to \mathbb{F}_1$ by $\widetilde{N}_m$ and put $y = x_1 \dots x_m$. Without loss of generality, we may assume that $\tilde{h}(n) = x_m n x_m^{-1}$ for $n \in \widetilde{N}_m$. In [13], the Reidemeister–Schreier method was applied to show that $\widetilde{N}_m$ is generated by the elements

$$a_{k,j} = x_m^k x_j x_m^{-(k+1)}, \tag{5.3}$$

where $j = 2, \dots, m-1$, $k \in \mathbb{Z}$, and the elements

$$a_{k,m} = x_m^k y x_m^{-(k+m)}, \tag{5.4}$$

where $k \in \mathbb{Z}$. Then the action of $\tilde{h}$ is given by $\tilde{h}(a_{k,j}) = a_{k+1,j}$.

The relations $yx_j = x_j y$ $(j = 2, \dots, m)$ give rise to the relations (see [13])

$$a_{k,m} = a_{k+1,m} \tag{5.5}$$

for $k \in \mathbb{Z}$ and

$$a_{k,m} a_{k+m,j} a_{k+1,m}^{-1} = a_{k,j} \tag{5.6}$$

for $j = 2, \dots, m-1$ and $k \in \mathbb{Z}$. Therefore $\widetilde{N}_m$ is a free group generated by $a_{0,m}$ and $a_{k,j}$, where $k = 1, \dots, m$ and $j = 2, \dots, m-1$.

We have $\tilde{h}(a_{0,m}) = a_{0,m}$, $\tilde{h}(a_{k,j}) = a_{k+1,j}$ for $k = 1, \dots, m-1$, $j = 2, \dots, m-1$ and $\tilde{h}(a_{m,j}) = a_{0,m}^{-1} a_{1,j} a_{0,m}$ for $j = 2, \dots, m-1$. Let $\bar{a}_{k,j}$ be the image of $a_{k,j}$ in $\widetilde{N}_m/\widetilde{N}_m'$. Then the action of h on $\widetilde{N}_m/\widetilde{N}_m'$ is given by

$$\begin{aligned} h(\bar{a}_{0,m}) &= \bar{a}_{0,m}, \\ h(\bar{a}_{k,j}) &= \bar{a}_{k+1,j}, \qquad k = 1, \dots, m-1, \quad j = 2, \dots, m-1, \\ h(\bar{a}_{m,j}) &= \bar{a}_{1,j}, \qquad j = 2, \dots, m-1. \end{aligned}$$

Simple computations show that the characteristic polynomial of h is equal to $(-1)^{m-1}(t-1)(t^m-1)^{m-2}$.

It is easy to check ([9], Lemma 4) that G/G' is a finitely generated free abelian group for any C-group G. Moreover, the canonical epimorphism $\mathrm{ab}\colon G \to G/G'$ is a C-homomorphism if we choose the $\mathrm{ab}(x_i)$ as C-generators of G/G', where $\{x_i\}$ is the set of C-generators of G. We say that a C-group G *consists of n irreducible components* if $G/G' \simeq \mathbb{Z}^n$. The notion of the number of irreducible components of a Hurwitz C-group is clarified by the following simple lemma. (See [10] for the case of an arbitrary C-group.)

Lemma 5.7. *A (topological) Hurwitz curve $\overline{H}$ consists of n irreducible components if and only if its fundamental group $\pi_1 = \pi_1(\mathbb{C}^2 \setminus H)$ consists of n irreducible components.*

Trivial computations show that the Alexander polynomial $\Delta(t)$ of the abelian C-group $\mathbb{Z}^n$ is equal to $(-1)^{n-1}(t-1)^{n-1}$. Therefore Lemma 5.5 implies the following lemma.

Lemma 5.8. *The Alexander polynomial $\Delta(t)$ of a Hurwitz C-group G consisting of n irreducible components is divisible by $(t-1)^{n-1}$.*

Theorem 5.9. *Let G be a Hurwitz C-group consisting of n irreducible components and let $\Delta(t)$ be its Alexander polynomial. Then*

$$\Delta(t) = (t-1)^{n-1}P(t),$$

where the polynomial $P(t) \in \mathbb{Z}[t]$ satisfies $P(1) \neq 0$.

Proof. Let m be the degree of the Hurwitz C-group G. To obtain a C-presentation of G, it suffices to add several C-relations to the presentation (5.2).

Since G consists of n irreducible components, the set $\{1,\dots,m\}$ splits into the disjoint union of n subsets $J_1,\dots,J_n$ such that $j_1, j_2 \in J_k$ if and only if x_{j_1} and x_{j_2} are conjugate in G. Let $J_k = \{j_{1,k} < \dots < j_{m_k,k}\}$. Without loss of generality, we may assume that the added C-relations include the relations

$$x_{j_{i,k}} w_{j_{i,k},j_{i+1,k},1} = w_{j_{i,k},j_{i+1,k},1} x_{j_{i+1,k}} \tag{5.7}$$

for $i = 1,\dots,m_k-1$, $k = 1,\dots,n$ and some words $w_{j_{i,k},j_{i+1,k},1}$. Let $\nu(w_{j_{i,k},j_{i+1,k},1}) = x^{t_{j_{i,k}}}$, where x is the C-generator of $\mathbb{F}_1$.

Consider the diagram (5.1) with $G_1 = \widetilde{G}_m$ and $G_2 = G$. In the notation of the proof of Theorem 5.6, the elements $\bar{a}_{0,m}$ and $\sum_{k=1}^m \bar{a}_{k,j}$ $(j = 2,\dots,m-1)$ generate the subgroup

$$(N_1/N_1')_1 = \{y \in N_1/N_1' \mid h(y) = y\}$$

of N_1/N_1' and the subspace $(N_1/N_1' \otimes \mathbb{C})_1$ generated by the eigenvectors of $h_{\mathbb{C}}$ with eigenvalue 1. The image $g_*((N_1/N_1')_1) \otimes \mathbb{C}$ is the eigenspace of $N_2/N_2' \otimes \mathbb{C}$ corresponding to the eigenvalue 1.

Applying the Reidemeister–Schreier method, we see that the group N_2 is also generated by the element $a_{0,m}$ (defined in (5.4)) and the elements $a_{k,j}$,

$k = 1, \dots, m, \quad j = 2, \dots, m-1$, defined in (5.3) (more precisely, by their images under g_*). Moreover, each of the relations (5.7) (after substituting $x_1 = y(x_2 \dots x_m)^{-1}$) gives rise to the relations

$$a_{r,j_{i,k}} \overline{w}_{r+1,j_{i,k},j_{i+1,k}} = \overline{w}_{r,j_{i,k},j_{i+1,k}} a_{r+t_{j_{i,k}},j_{i+1,k}} \tag{5.8}$$

if $1 < j_{i,k} < j_{i+1,k} < m$,

$$a_{0,m} \bigg(\prod_{s=2}^{m-1} a^{-1}_{r+m-s,m+1-s} \bigg) \overline{w}_{r+1,1,j_{i+1,k}} = \overline{w}_{r,1,j_{i+1,k}} a_{r+t_{1,j_{i+1,k}}} \tag{5.9}$$

if $1 = j_{i,k} < j_{i+1,k} < m$,

$$a_{r,j_{i,k}} \overline{w}_{r+1,j_{i,k},m} = \overline{w}_{r,j_{i,k},m} \tag{5.10}$$

if $1 < j_{i,k} < j_{i+1,k} = m$, and

$$a_{0,m} \bigg(\prod_{s=2}^{m-1} a^{-1}_{r+m-s,m+1-s} \bigg) \overline{w}_{r+1,1,m} = \overline{w}_{r,1,m} \tag{5.11}$$

if $1 = j_{i,k} < j_{i+1,k} = m$, where $r \in \mathbb{Z}$ and each of the words

$$\overline{w}_{r,j_{i,k},j_{i+1,k}} = x^r_m w_{j_{i,k},j_{i+1,k},1} x_m^{-(r+t_{j_{i,k}})}$$

is written in the generators $a_{l,s}$.

As in [13], one can show that the words $\overline{w}_{r,j_{i,k},j_{i+1,k}}$ and $\overline{w}_{r+m,j_{i,k},j_{i+1,k}}$ are conjugate in G_2 by $a_{0,m}$. Therefore, taking the sum over r, we deduce the following relations in N_2/N_2' from relations (5.8)–(5.11):

$$\sum_{r=1}^{m} \bar{a}_{r,j_{i,k}} = \sum_{r=1}^{m} \bar{a}_{r,j_{i+1,k}} \tag{5.12}$$

if $1 < j_{i,k} < j_{i+1,k} < m$,

$$m\bar{a}_{0,m} - \sum_{s=2}^{m-1} \sum_{r=1}^{m} \bar{a}_{r,s} = \sum_{r=1}^{m} \bar{a}_{r,j_{i+1,k}} \tag{5.13}$$

if $1 = j_{i,k} < j_{i+1,k} < m$,

$$\sum_{r=1}^{m} \bar{a}_{r,j_{i,k}} = 0 \tag{5.14}$$

if $1 < j_{i,k} < j_{i+1,k} = m$, and

$$m\bar{a}_{0,m} - \sum_{s=2}^{m-1} \sum_{r=1}^{m} \bar{a}_{r,s} = 0 \tag{5.15}$$

if $1 = j_{i,k} < j_{i+1,k} = m$.

It is easy to see that relations (5.12)–(5.15) are linearly independent over $\mathbb{Z}$ for any decomposition $\{1, \dots, m\} = \bigsqcup_{k=1}^{n} J_k$, their number is equal to $m - n$, and the elements $\bar{a}_{0,m}$ and $\sum_{r=1}^{m} \bar{a}_{r,s}$ $(s = 2, \dots, m-1)$ generate the group $(N_1/N_1')_1$. Therefore the rank of the kernel of the restriction of g_* to $(N_1/N_1')_1$ is not less than $m - n$. Thus,

$$\dim(N_2/N_2' \otimes \mathbb{C})_1 \leqslant \dim(N_1/N_1' \otimes \mathbb{C})_1 - (m-n) = n - 1.$$

On the other hand, by Lemma 5.8, the Alexander polynomial $\Delta(t)$ of a C-group G consisting of n irreducible components is divisible by $(t-1)^{n-1}$.

Corollary 5.10. *Let G be a Hurwitz C-group consisting of n irreducible components and $\Delta(t)$ its Alexander polynomial. Then*

$$\Delta(0) = (-1)^{\deg \Delta(t)-(n-1)}.$$

Proof. The Alexander polynomial $\Delta(t)$ is given by

$$\Delta(t) = \det(h_{\mathbb{C}} - t\,\mathrm{Id}) = (-t)^{\deg \Delta(t)} + \sum_{i=0}^{\deg \Delta(t)-1} c_i t^i.$$

By Theorem 5.9, we have $\Delta(t) = (t-1)^{n-1}P(t)$, where the polynomial $P(t) \in \mathbb{Z}[t]$ is such that $P(1) \neq 0$. Therefore the polynomial

$$P(t) = (-1)^{\deg \Delta(t)} \prod_i \Phi_{n_i}(t)$$

is a product (up to a sign) of cyclotomic polynomials $\Phi_{n_i}(t)$ with $n_i > 1$. By Lemma 5.3, $\Phi_{n_i}(0) = 1$ for all $n_i > 1$. Therefore $\Delta(0) = (-1)^{n-1}(-1)^{\Delta(t)} = (-1)^{\deg \Delta(t)-(n-1)}$.

Lemma 5.11. *Let $j\colon \{1,\dots,n\} \to \{1,\dots,m\}$ be an injective map and let G be a C-group generated by C-generators $x_1,\dots,x_n$. Suppose that $w = \bar{x}_1 \dots \bar{x}_m$ is a quasipositive word in $x_1,\dots,x_n$ (that is, $\bar{x}_k$ is conjugate to some $x_{i_k} \in \{x_1,\dots,x_n\}$) such that $\bar{x}_{j(i)} = x_i$ for $i = 1,\dots,n$. If w belongs to the centre of G, then G is a Hurwitz C-group of degree m.*

Proof. Let $G = \langle x_1,\dots,x_n \mid R\rangle$ be a C-presentation of G. We put $J = \{1 \leqslant j \leqslant m \mid j = j(i),\ i = 1,\dots,n\}$. By assumption, we have $\bar{x}_{j(i)} = x_i$ and

$$\bar{x}_j = w_j^{-1} x_{i_j} w_j \tag{5.16}$$

for $j \notin J$, where w_j is a word in $x_1,\dots,x_n$ and their inverses. Note that relations (5.16) are C-relations. Therefore, if we adjoin generators $\bar{x}_j$ ($j \notin J$) to the set $\{x_1,\dots,x_n\}$ of generators and the relations (5.16) to R, then we obtain a C-presentation of the same group G. To complete the proof, it suffices to renumber the resulting set of generators.

Proposition 5.12. *Let G_1, G_2 be Hurwitz C-groups of degrees m_1, m_2 respectively, and let $\Delta_i(t)$, $i = 1,2$, be their Alexander polynomials. Then there is a Hurwitz C-group of degree $2m_1m_2$ with Alexander polynomial $\Delta(t) = \Delta_1(t)\Delta_2(t)$.*

Proof. Let $G_i = \langle x_{1,i},\dots,x_{m_i,i} \mid R_i\rangle$ be a Hurwitz C-presentation of the group G_i of degree m_i.

Consider the amalgamated free product $\widetilde{G} = G_1 *_{\{x_{m_1,1}=x_{m_2,2}\}} G_2$. This is a C-group given by the presentation

$$\widetilde{G} = \langle x_{1,i},\dots,x_{m_i,i},\ i = 1,2 \mid R_1 \cup R_2,\ x_{m_1,1} = x_{m_2,2}\rangle.$$

Put $y_i = x_{1,i} \dots x_{m_i,i}$ for $i = 1, 2$ and denote the kernel of $\nu\colon G_i \to \mathbb{F}_1$ (resp. $\nu\colon \widetilde{G} \to \mathbb{F}_1$) by N_i (resp. $\widetilde{N}$).

Applying the Reidemeister–Schreier method as in the proof of Theorem 5.6, one can show that the group N_i (resp. $\widetilde{N}$) is generated by the elements $a_{k,j,i} = x^k_{m_i,i} x_{j,i} x^{-(k+1)}_{m_i,i}$, where $j = 2, \dots, m_i - 1$, $k \in \mathbb{Z}$, and the elements $a_{k,m_i,i} = x^k_{m_i,i} y_i x^{-(k+m_i)}_{m_i,i}$, $k \in \mathbb{Z}$ (resp. by the union of these elements since $x_{m_1,1} = x_{m_2,2}$ in $\widetilde{G}$). The set of defining relations of N_i (resp. $\widetilde{N}$) is obtained from the set $\overline{R}_i = \{x^n_{m_i,i} r_{l,i} x^{-n}_{m_i,i} \mid r_{l,i} \in R_i,\ n \in \mathbb{Z}\}$ (resp. from $\overline{R} = \overline{R}_1 \cup \overline{R}_2$) by rewriting the words in the alphabet $\{a_{k,j,i}\}$ (resp. in the union of these alphabets). Therefore $\widetilde{N} = N_1 * N_2$ is the free product of the groups N_1 and N_2.

It follows from the proof of Theorem 5.6 that the elements $a_{k,j,i}$ and $a_{k+lm_i,j,i}$ are conjugate in N_i for all $l \in \mathbb{Z}$.

Let $\tilde{h}_i$ be the automorphism of N_i given by conjugation by $x_{m_i,i}$. Then the automorphism $\tilde{h}$ of $\widetilde{N}$ is given by conjugation by $x_{m_1,1} = x_{m_2,2}$ and is equal to $\tilde{h}_1 * \tilde{h}_2$. Therefore the Alexander polynomial $\tilde{\Delta}(t)$ of the group $\widetilde{G}$ is equal to $\Delta_1(t)\Delta_2(t)$.

Consider the group

$$G = \langle x_{1,i}, \dots, x_{m_i,i},\ i = 1, 2 \mid R_1 \cup R_2,\ x_{m_1,1} = x_{m_2,2}, \\ [x_{j,i}, y^{m_i}_{\bar{i}}] = 1,\ j = 1, \dots, m_i - 1,\ i = 1, 2\rangle,$$

where $\bar{i} = \{1, 2\} \setminus \{i\}$. (We recall that $x_{1,\bar{i}}, \dots, x_{m_{\bar{i}},\bar{i}}$ commute with $y_{\bar{i}}$.) Let N be the kernel of $\nu\colon G \to \mathbb{F}_1$.

To obtain a presentation of N from that of $\widetilde{N}$ described above, one must add the relations induced by the relations

$$[x_{j,i}, y^{m_i}_{\bar{i}}] = 1, \qquad j = 1, \dots, m_i - 1, \quad i = 1, 2.$$

It is easy to see that the additional relations have the form

$$a_{k,j,i} a^{m_i}_{0,m_{\bar{i}}} = a^{m_i}_{0,m_{\bar{i}}} a_{k+m_1 m_2,j,i} \tag{5.17}$$

since $a_{k,m_{\bar{i}}} = a_{k+1,m_{\bar{i}}}$ in $N_{\bar{i}}$ for all k. The additional relations (5.17) imply that $a_{k,j,i}$ and $a_{k+m_1m_2,j,i}$ are conjugate. But these elements were conjugate in $\widetilde{N}$. Therefore $\widetilde{N}/\widetilde{N}' \simeq N/N'$, and the groups $\widetilde{G}$ and G have the same Alexander polynomial. To complete the proof of Proposition 5.12, we notice that

$$y^{m_2}_1 y^{m_1}_2 = (x_{1,1} \dots x_{m_1,1})^{m_2} (x_{1,2} \dots x_{m_2,2})^{m_1}$$

belongs to the centre of G. Therefore, by Lemma 5.11, G is a Hurwitz C-group of degree $2m_1m_2$.

Let G_1 and G_2 be Hurwitz C-groups given by Hurwitz C-presentations $G_i = \langle x_{1,i}, \dots, x_{m_i,i} \mid R_i\rangle$, $i = 1, 2$, and let G be the Hurwitz C-group constructed in the proof of Proposition 5.12. Then G is called the *Hurwitz product* of G_1 and G_2 and is denoted by $G_1 \diamond G_2$. Of course, the Hurwitz product of G_1 and G_2 depends on the Hurwitz C-presentations of G_1 and G_2. However, Proposition 5.12 shows that the Alexander polynomial of $G_1 \diamond G_2$ is independent of the Hurwitz C-representations of the factors.

Lemma 5.13. *Let $C_{n,m}$ be the plane affine algebraic curve given by $w^n - z^m = 0$, where n and m are any positive integers. Then the fundamental group $G_{n,m} = \pi_1(\mathbb{C}^2 \setminus C_{n,m})$ of the complement of $C_{n,m}$ is a Hurwitz C-group.*

Remark 5.14. Lemma 5.13 does not follow from the previous assertion on the fundamental group of the complement of an affine Hurwitz curve. Indeed, that assertion requires the line at infinity to be in general position with respect to the curve. Here the line at infinity is in special position. If we consider the local fundamental group $G = \pi_1(B_\varepsilon \setminus C)$, where C is an irreducible analytic singularity in a small ball B_ε, then G always has a natural structure of an irreducible C-group. It has a non-trivial centre if and only if the singularity of C is given by $x^p = y^q$, where p and q are coprime (see [3]).

Proof of Lemma 5.13. The braid monodromy of the singularity $w^n = z^m$ with respect to the projection $(z,w) \to z$ is equal to

$$b_{n,m} = (\sigma_1 \dots \sigma_{n-1})^m \in \mathrm{Br}_n,$$

where $\sigma_1, \dots, \sigma_{n-1}$ are the standard generators of the braid group Br_n. This means that the generators satisfy

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}, \qquad i = 1, \dots, n-2,\\ [\sigma_i, \sigma_j] &= 1, \qquad |i-j| \geqslant 2. \end{aligned}$$

The group Br_n acts on the free group $\mathbb{F}_n$ generated by $x_1, \dots, x_n$. This action is given by $\sigma_j(x_i) = x_i$ if $j \neq i, i+1$ and $\sigma_j(x_{j+1}) = x_j$, $\sigma_j(x_j) = x_j x_{j+1} x_j^{-1}$. Let $B_{n,m}$ be the cyclic subgroup of Br_n generated by $b_{n,m}$. Then the group

$$G_{n,m} = \langle x_1, \dots, x_n \mid x_i = b(x_i),\ i = 1, \dots, n,\ b \in B_{n,m} \rangle$$

is a C-group (see [11]). By Lemma 5.11, it is a Hurwitz C-group because $b_{n,m}^n = (\Delta_n^2)^m \in B_{n,m}$, where Δ_n is the Garside element of the braid group Br_n, and hence the element $(x_1 \dots x_n)^m$ belongs to the centre of $G_{n,m}$ (since $\Delta_n^2(x_i) = (x_1 \dots x_n) x_i (x_1 \dots x_n)^{-1}$ and $\Delta_n^2(x_1 \dots x_n) = x_1 \dots x_n$).

Proposition 5.15 [14]. *If m and n are coprime, then the Alexander polynomial of $G_{n,m}$ is given by*

$$\Delta_{n,m}(t) = \frac{(t-1)(t^{nm}-1)}{(t^n-1)(t^m-1)}.$$

Proposition 5.16. *The Alexander polynomial of the group $G_{2,2m}$ is equal to*

$$\Delta(t) = (1-t)\sum_{i=0}^{m-1} t^{2i}.$$

Proof. It follows from the proof of Lemma 5.13 that

$$G_{2,2m} = \langle x_1, x_2 \mid [x_1, (x_1x_2)^m] = [x_2, (x_1x_2)^m] = 1 \rangle.$$

We claim that the relations

$$[x_2, (x_1x_2)^m] = 1, \qquad i = 1, 2, \tag{5.18}$$

are equivalent to the single relation $(x_1x_2)^m = (x_2x_1)^m$. Indeed, the relations (5.18) imply that

$$(x_1x_2)^m = x_1^{-1}(x_1x_2)^m x_1 = (x_2x_1)^m,$$

and the relation $(x_1x_2)^m = (x_2x_1)^m$ implies that

$$x_2(x_1x_2)^m = (x_2x_1)^m x_2 = (x_1x_2)^m x_2$$

and

$$x_1(x_1x_2)^m = x_1(x_2x_1)^m = (x_1x_2)^m x_1.$$

Therefore,

$$G_{2,2m} = \langle x_1, x_2 \mid (x_1x_2)^m(x_2x_1)^{-m} = 1\rangle.$$

We put $r = (x_1x_2)^m(x_2x_1)^{-m}$. Applying the free differential calculus of Fox [4], we easily see that

$$\nu_*\left(\frac{\partial r}{\partial x_1}\right) = 1 + t^2 + \dots + t^{2(m-1)} - t^{2m-1} - t^{2m-3} - \dots - t^1,$$
$$\nu_*\left(\frac{\partial r}{\partial x_2}\right) = t + \dots + t^{2m-1} - t^{2(m-1)} - t^{2(m-2)} - \dots - t^2 - 1.$$

Therefore,

$$\Delta(t) = (1-t)\sum_{i=0}^{m-1} t^{2i}.$$

We consider the group

$$G(2) = \langle x_1, \dots, x_4 \mid x_2^2x_1x_2^{-2} = x_4,\ x_3 = x_2,\ x_4^2x_2x_4^{-2} = x_2, \\ [x_i, x_1 \dots x_4] = 1 \text{ for } i = 1, \dots, 4\rangle. \tag{5.19}$$

Proposition 5.17. *The Alexander polynomial $\Delta(t)$ of $G(2)$ is equal to $t^2 - 1$.*

Proof. Let $N(2)$ be the kernel of the canonical C-homomorphism $\nu\colon G(2) \to \mathbb{F}_1$. We put $m = 4$ and $y = x_1 \dots x_4$. In the notation of the proof of Theorem 5.6, it follows from the relations $[x_i, y] = 1$ for $i = 1, \dots, 4$ that the group $N(2)$ is generated by the elements $a_{k,j} = x_4^k x_j x_4^{-(k+1)}$, $k = 1, \dots, 4$, $j = 2, 3$, and the element $a_{0,4} = yx_4^{-4}$. In our case, relations (5.5) and (5.6) take the form

$$a_{k,4} = a_{0,4}, \qquad a_{k+4,j} = a_{0,4}^{-1}a_{k,j}a_{0,4} \tag{5.20}$$

for all k. The relation $x_3 = x_2$ gives rise to the relations

$$a_{k,3} = a_{k,2} \tag{5.21}$$

for all k, and the relation $x_4^2x_2 = x_2x_4^2$ gives rise to the relations

$$a_{k+2,2} = a_{k,2} \tag{5.22}$$

for all k. We may write the relation $x_2^2x_1 = x_4x_2^2$ as $x_2^2y = x_4x_2^4x_4$ (since $x_2 = x_3$). This gives rise to the relations

$$a_{k,2}a_{k+1,2}a_{k+2,4} = a_{k+1,2}a_{k+2,2}a_{k+3,2}a_{k+4,2} \tag{5.23}$$

for all k.

It follows from (5.20)–(5.23) that $N(2)$ is generated by $a_{1,2}$, $a_{2,2}$ and $a_{0,4}$, which satisfy the relations

$$a_{0,4} = (a_{1,2}a_{2,2})^{-1}(a_{2,2}a_{1,2})^2 = (a_{2,2}a_{1,2})^{-1}(a_{1,2}a_{2,2})^2$$

and

$$[a_{1,2}, a_{0,4}] = [a_{2,2}, a_{0,4}] = 1.$$

Therefore the group $N(2)/N(2)'$ is a free abelian group generated by the images $\bar{a}_{1,2}$ and $\bar{a}_{2,2}$ of the elements $a_{1,2}$ and $a_{2,2}$.

As in the proof of Theorem 5.6, the action of $\tilde{h}$ on $N(2)$ is given by $\tilde{h}(a_{1,2}) = a_{2,2}$, $\tilde{h}(a_{2,2}) = a_{3,2} = a_{1,2}$. The induced action of h on $N(2)/N(2)'$ is given by $h(\bar{a}_{1,2}) = \bar{a}_{2,2}$, $h(\bar{a}_{2,2}) = \bar{a}_{1,2}$. The characteristic polynomial of this action is equal to $(t-1)(t+1)$.

Corollary 5.18. *For any $k \in \mathbb{N}$ there is a Hurwitz C-group G consisting of two irreducible components such that the Alexander polynomial $\Delta(t) = (t-1)P(t)$ of G satisfies $|P(1)| = k$.*

Proof. If $k > 2$, then Proposition 5.16 shows that $P(1) = -k$, where $P(t) = -\sum_{i=0}^{k-1} t^{2i}$ is a factor of the Alexander polynomial $\Delta(t) = (1-t)\sum_{i=0}^{k-1} t^{2i}$ of the group $G_{2,2k}$. If $k = 2$, then Proposition 5.17 shows that the group $G(2)$ has the desired property since its Alexander polynomial is $\Delta(t) = (t-1)(t+1)$. If $k = 1$, then one can take the abelian Hurwitz C-group $G = \mathbb{Z}^2$.

Proposition 5.19. *For every $k \in \mathbb{N}$ there exists*

(i) *an irreducible Hurwitz C-group whose Alexander polynomial $\Delta(t)$ has* $\deg \Delta(t) = 2k$,

(ii) *a Hurwitz C-group consisting of two irreducible components such that the Alexander polynomial $\Delta(t) = (t-1)P(t)$ satisfies* $\deg P(t) = k$.

Proof. By Propositions 5.12 and 5.15, the Alexander polynomial $\Delta(t)$ of the Hurwitz product $G_{2,3}^{\diamond k}$ is equal to $(t^2 - t + 1)^k$.

To prove (ii), it suffices to take the C-groups $G(2) \diamond G_{2,3}^{\diamond n}$ if $k = 2n+1$ is odd and $\mathbb{Z}^2 \diamond G_{2,3}^{\diamond n}$ if $k = 2n$ is even.

Question 5.20. *Let $P(t) \in \mathbb{Z}[t]$ be a polynomial all of whose roots are roots of unity. Suppose that $t = 1$ is a root of $P(t)$ of multiplicity k, and $P(0) = (-1)^{\deg P(t) - k}$. Assume also that $P(1) = 1$ if $k = 0$. Does there exist a Hurwitz C-group G with Alexander polynomial $\Delta(t) = P(t)$?*

§ 6. The first Betti number of cyclic coverings of the plane

Consider the infinite cyclic covering $f = f_\infty \colon X_\infty \to X' = \mathbb{C}^2 \setminus H$ corresponding to the epimorphism $\nu \colon \pi_1(\mathbb{C}^2 \setminus H) \to \mathbb{F}_1$. Let $h \in \operatorname{Deck}(X_\infty/X') \simeq \mathbb{F}_1$ be a covering transformation corresponding to the C-generator $x \in \mathbb{F}_1$. We say that h is the *monodromy of the Hurwitz curve* H. We regard the space X' as the quotient space $X' = X_\infty/\mathbb{F}_1$. In this situation, Milnor [17] considered an exact sequence of chain complexes

$$0 \to C.(X_\infty) \xrightarrow{h-\mathrm{id}} C.(X_\infty) \xrightarrow{\varphi_*} C.(X') \to 0,$$

which gives a homology exact sequence

$$\cdots \to H_1(X_\infty) \xrightarrow{h-\mathrm{id}} H_1(X_\infty) \xrightarrow{f_*} H_1(X') \to H_0(X_\infty) \to 0. \tag{6.1}$$

(We often write h instead of h_* if no misunderstanding can occur.)

If $G_n \subset \mathbb{F}_1$ is the infinite cyclic group generated by h^n, then $X'_n = X_\infty/G_n$ and $X' = X'_n/\mu_n$, where $\mu_n = \mathbb{F}_1/G_n$ is cyclic of order n. Let h_n be the automorphism of X'_n induced by the monodromy h. Then h_n is a generator of the covering transformation group $\operatorname{Deck}(X'_n/X') = \mu_n$ acting on X'_n. We apply the sequence

$$\cdots \to H_1(X_\infty) \xrightarrow{h^n-\mathrm{id}} H_1(X_\infty) \xrightarrow{g_{n,*}} H_1(X'_n) \to H_0(X_\infty) \to 0 \tag{6.2}$$

constructed in the same way as (6.1) to the infinite cyclic covering $g_n = g_{\infty,n} \colon X_\infty \to X'_n$ to analyse the group $H_1(X'_n)$.

Let $H_1(X_\infty, \mathbb{C})_n$ be the subspace of $H_1(X_\infty, \mathbb{C})$ corresponding to those eigenvalues λ of h_* that are nth roots of unity, and let $H_1(X_\infty, \mathbb{C})_{n,\neq 1}$ be the subspace corresponding to those eigenvalues $\lambda \neq 1$ that are nth roots of unity. By Theorem 5.1, we have $\dim H_1(X_\infty, \mathbb{C})_n = r_n$ and $\dim H_1(X_\infty, \mathbb{C})_{n,\neq 1} = r_{n,\neq 1}$, where r_n (resp. $r_{n,\neq 1}$) is the number of roots of the Alexander polynomial $\Delta(t)$ of the Hurwitz curve $\overline{H}$ which are nth roots of unity (resp. nth roots of unity other than 1). Note that, by Lemma 5.7 and Theorem 5.9,

$$r_n - r_{n,\neq 1} = r_1 = \#\{\text{irreducible components of } \overline{H}\,\} - 1.$$

Proposition 6.1. *We have*

(i) $\dim H_1(X'_n, \mathbb{C}) = r_n + 1$,

(ii) $\dim H_1(X'_n, \mathbb{C})_1 = r_1 + 1 = \#\{$*irreducible components of* $\overline{H}\,\}$.

Proof. This follows from the exact sequence (6.2).

Let $\overline{H}$ be a Hurwitz curve consisting of k irreducible components $\overline{H}_1, \dots, \overline{H}_k$. Choose a line $L \subset \mathbb{C}^2$ belonging to the pencil of lines (with respect to which $\overline{H}$ is defined) and transversally intersecting the curve H. Let γ_i be a circle of small radius in L centred at one of the intersection points $H_i \cap L$. It is easy to see that the corresponding cycles $\gamma_1, \dots, \gamma_k$ form a basis in $H_1(\mathbb{C}^2 \setminus H, \mathbb{Z})$ and are independent of the choice of the line L. Let $\bar{\gamma}_i$ be a cycle in $H_1(X'_n, \mathbb{Z})$ corresponding to the simple path $f_n^{-1}(\gamma_i)$, $i = 1, \dots, k$.

Lemma 6.2. *The cycles* $\bar{\gamma}_i$ $(i = 1, \dots, k)$ *are linearly independent in* $H_1(X'_n, \mathbb{Z})$ *and form a basis in* $H_1(X'_n)_1$.

Proof. Clearly, all the $\bar{\gamma}_i$ are invariant under the action of h_n. The proof now follows from Proposition 6.1, (ii) and the remark that we have $(f_n)_*(\bar{\gamma}_i) = n\gamma_i$ under the homomorphism $(f_n)_*\colon H_1(X'_n, \mathbb{Z}) \to H_1(\mathbb{C}^2 \setminus H, \mathbb{Z})$.

In the notation of the proof of Theorem 4.1, the covering f_n can be extended to a map $\tilde{f}_n\colon \widetilde{X}_n \to \mathbb{CP}^2$ branched along $\overline{H}$ and possibly along $\bar{L}_\infty$. Here $\widetilde{X}_n$ is a closed four-dimensional variety. Over each singular point of $\overline{H}$, $\widetilde{X}_n$ is locally isomorphic to the complex-analytic singularity given by $w_1^n = \widetilde{F}_1(u_1, v_1)$, where

$$\widetilde{F}_1(u_1, v_1) = \prod_j (v_1 - v_{1,j}(u_1))$$

and the product is taken over those branches of $\overline{H}$ whose closure contains the singular point of $\overline{H}$. Over a neighbourhood of each intersection point of $\overline{H}$ and L_∞, $\widetilde{X}_n$ is locally isomorphic to the singularity given by $w_2^n = \big(v_2 - e^{\frac{2\pi i}{m}}\big)u^d$, where d is the smallest non-negative integer such that $m + d$ is divisible by n. If $\tilde{f}_n^{-1}(L_\infty) \subset \operatorname{Sing} \widetilde{X}_n$, then $\widetilde{X}_n$ can be normalized (as in the algebraic case), and we obtain a covering $\tilde{f}_{n,\mathrm{norm}}\colon \widetilde{X}_{n,\mathrm{norm}} \to \mathbb{CP}^2$, where $\widetilde{X}_{n,\mathrm{norm}}$ is a singular analytic variety at its finitely many singular points. One can resolve them and obtain a smooth manifold $\overline{X}_n$. Let $\sigma\colon \overline{X}_n \to \widetilde{X}_{n,\mathrm{norm}}$ be the resolution of singularities. We put $E = \sigma^{-1}(\operatorname{Sing} \widetilde{X}_{n,\mathrm{norm}})$, $\bar{f}_n = \tilde{f}_{n,\mathrm{norm}} \circ \sigma$, $R_i = \tilde{f}_{n,\mathrm{norm}}^{-1}(\overline{H}_i)$ for $i = 1, \dots, k$, and $R_\infty = \tilde{f}_{n,\mathrm{norm}}^{-1}(L_\infty)$. Note that the restriction of $\tilde{f}_{n,\mathrm{norm}}$ to each R_i $(i = 1, \dots, k)$ is one-to-one, and the restriction of $\tilde{f}_{n,\mathrm{norm}}$ to R_∞ is an n_0-sheeted cyclic covering, where $n_0 = \mathrm{GCD}(n, d)$ and the ramification index n_∞ of $\tilde{f}_{n,\mathrm{norm}}$ along R_∞ is equal to $\frac{n}{n_0}$. As in the algebraic case, it is easy to show that R_∞ is irreducible. We denote the proper transform of R_i by $\overline{R}_i = \sigma^{-1}(R_i)$ for $i = 1, \dots, k, \infty$.

We have the embeddings $i_1\colon X'_n \hookrightarrow X_n = \overline{X}_n \setminus E$ and $i_2\colon X_n \hookrightarrow \overline{X}_n$.

Lemma 6.3. *The induced homomorphism* $i_{1*}\colon H_1(X'_n) \to H_1(X_n)$ *is an epimorphism with* $\ker i_{1*} = H_1(X'_n)_1$.

Proof. We have

$$X'_n = X_n \Big\backslash \left(\bigcup_{i=1}^{k} \overline{R}_i\right) \cup \overline{R}_\infty$$

and each $\overline{R}_i$ $(i = 1, \dots, k, \infty)$ is a submanifold of codimension 2 in X_n. Therefore every one-dimensional cycle $\gamma \subset X_n$ can be moved outside $\big(\bigcup_{i=1}^{k} \overline{R}_i\big) \cup \overline{R}_\infty$. Thus $i_{1,*}$ is an epimorphism.

Let a complex line $L \subset \mathbb{CP}^2$ meet L_∞ transversally at $q \in L_\infty \setminus \overline{H}$ and let γ_∞ be a small simple loop around L_∞ lying in L. Then $f_n^{-1}(\gamma_\infty)$ splits into the disjoint union of n_0 simple loops $\bar{\gamma}_{\infty,i}$, $i = 1, \dots, n_0$. Since R_∞ is irreducible, any two loops $\bar{\gamma}_{\infty,i}$ and $\bar{\gamma}_{\infty,j}$ belong to the same homology class in $H_1(X'_n)$. (We denote it by $\bar{\gamma}_\infty$.) Therefore $n_0\bar{\gamma}_\infty \in H_1(X'_n)_1$. The lemma follows from the remark that $\bar{\gamma}_1, \dots, \bar{\gamma}_k, \bar{\gamma}_\infty$ generate $\ker i_{1*}$ and $\bar{\gamma}_1, \dots, \bar{\gamma}_k$ generate $H_1(X'_n)_1$.

Lemma 6.4. *The homomorphism $i_{2*}\colon H_1(X_n,\mathbb{C}) \to H_1(\overline{X}_n,\mathbb{C})$ is an isomorphism.*

Proof. We have $X_n = \overline{X}_n \setminus E$. Let $T \subset \overline{X}_n$ be a closed regular neighbourhood of E and ∂T its boundary. We put $T' = T \setminus E$ and $T^0 = T \setminus \partial T$. It is known (see, for example, the proof of Proposition 3.4 in [5]) that the embedding $i\colon \partial T \hookrightarrow T$ induces an isomorphism $i_*\colon H_1(\partial T,\mathbb{C}) \to H_1(T,\mathbb{C})$ and there is a deformation retraction $T' \searrow \partial T$. Hence there is a deformation retraction $X_n \searrow X_n^0$, where $X_n^0 = X_n \setminus T^0$. The lemma now follows from the Mayer–Vietoris sequence

$$H_2(\overline{X}_n) \to H_1(\partial T) \to H_1(T) \oplus H_1(X_n^0) \to H_1(\overline{X}_n) \to 0.$$

The proof of Theorem 0.4 follows from Lemmas 6.3, 6.4 and Proposition 6.1.

Proposition 6.5. *For any $k \in \mathbb{N}$, there exists*

(i) *an irreducible Hurwitz curve $\overline{H}_k$ such that the first Betti number $b_1(\overline{X}_{k,6})$ is equal to $2k$, where $\overline{X}_{k,6}$ is a resolution of singularities of the cyclic covering of $\mathbb{CP}^2$ of degree six branched along $\overline{H}_k$,*

(ii) *a Hurwitz curve $\overline{H}_k$ consisting of two irreducible components such that the first Betti number $b_1(\overline{X}_{k,6})$ is equal to k, where $\overline{X}_{k,6}$ is a resolution of singularities of the cyclic covering of degree six branched along $\overline{H}$.*

Proof. In the proof of Proposition 5.19, it was show that the Alexander polynomial $\Delta(t)$ of the Hurwitz product $G_{2,3}(k) = G_{2,3}^{\diamond k}$ is equal to $(t^2 - t + 1)^k$, and the Alexander polynomials $\Delta(t)$ of the Hurwitz products $G_{2,3}(2,n) = G(2) \diamond G_{2,3}^{\diamond n}$ and $G_{2,3}(\mathrm{ab},n) = \mathbb{Z}^2 \diamond G_{2,3}^{\diamond n}$ are equal to $(t-1)\times(t+1)(t^2-t+1)^n$ and $(1-t)(t^2-t+1)^n$ respectively.

The groups $G_{2,3}(k)$, $G_{2,3}(2,n)$ and $G_{2,3}(\mathrm{ab},n)$ are Hurwitz C-groups. Moreover, one can assume that the degrees of these Hurwitz C-groups are divisible by 6. (One can apply Lemma 5.11 to y^6, where y is the product of the C-generators in a Hurwitz C-presentation of the corresponding group.) Therefore, by Theorem 6.2 of [11], each of these groups may be realized as the fundamental group $\pi_1(\mathbb{C}^2 \setminus H)$ for some Hurwitz curve of degree divisible by 6. The curve $\overline{H}$ is irreducible in the case of $G_{2,3}(k)$ and consists of two irreducible components in the other two cases. Proposition 6.5 now follows from Theorem 0.4.

Proposition 6.6. *For every $k \in \mathbb{N}$ there is a Hurwitz curve $\overline{H}_k$ which consists of $k+1$ irreducible components, has singularities of the form $w^{2^{3k-1}} - z^{2^{3k-1}} = 0$ and is the branch curve of a two-sheeted cyclic covering $\bar{f}_2\colon \overline{X}_{k,2} \to \mathbb{CP}^2$ with $b_1(\overline{X}_{k,2}) = k$.*

In particular, the Hurwitz curve $\overline{H}_1$ has $\deg \overline{H}_1 = 2^{10}$, the number of singular points of $\overline{H}_1$ is equal to 2^{16}, and all its singular points are of the form $w^4 - z^4 = 0$.

Proof. The Hurwitz product $G(2,k) = G(2)^{\diamond k}$ is a Hurwitz C-group of degree $m = 2^{3k-1}$. By Propositions 5.12 and 5.17, its Alexander polynomial is

$$\Delta_k(t) = (t-1)^k(t+1)^k.$$

By Theorem 6.2 of [11], each of the $G(2,k)$ can be realized as the fundamental group $\pi_1(\mathbb{C}^2 \setminus H_k)$ for some Hurwitz curve $\overline{H}_k$ of even degree with singularities of the form $w^m - z^m = 0$. Since the multiplicity of the root $t = 1$ of the Alexander polynomial $\Delta_k(t)$ equals k, it follows from Lemma 5.7 and Theorem 5.9 that the curve $\overline{H}_k$ consists of $k+1$ irreducible components.

By Theorem 0.4, the first Betti number $b_1(\overline{X}_{k,2})$ is equal to k since the multiplicity of the root $t = -1$ of the Alexander polynomial $\Delta_k(t)$ is equal to k.

To prove the existence of a Hurwitz curve $\overline{H}_1$ with the desired properties, we must find an integer m and a braid monodromy factorization (see [12])

$$\Delta_m^2 = b_1 \cdot \ldots \cdot b_n$$

of a Hurwitz curve $\overline{H}$ such that the group given by the presentation

$$\langle x_m, \ldots, x_m \mid x_i = b_j(x_i) \text{ for } i = 1, \ldots, m,\ j = 1, \ldots, n \rangle$$

is C-isomorphic to $G(2)$.

To find such a presentation, we briefly recall the proof of Theorem 6.2 of [11] and apply it to calculate the invariants of a Hurwitz curve $\overline{H}_1$ for which $\pi_1(\mathbb{C} \setminus H_1) \simeq G(2)$.

The group $G(2)$ is given by the presentation (5.19). Consider the group

$$G_{4,4} \simeq \langle x_1, \ldots, x_4 \mid x_i = \Delta_4^2(x_i) \text{ for } i = 1, \ldots, 4 \rangle, \tag{6.3}$$

where Δ_4 is the Garside element of Br_4. The braid Δ_4^2 is the braid monodromy of the singularity given by $w^4 - z^4 = 0$, and $s_0 = \Delta_4^2$ (a factorization with a single factor) is the braid monodromy factorization of four lines in $\mathbb{CP}^2$ passing through a given point.

To obtain the presentation (5.19), we must adjoin the relations

$$x_2^2 x_1 x_2^{-2} = x_4, \qquad x_4^2 x_2 x_4^{-2} = x_2, \qquad x_3 = x_2 \tag{6.4}$$

to the presentation (6.3). According to the notation and notions of [11], the first step in this process is to double (see Theorem 3.2 of [11]) the braid monodromy factorization $s_0 = \Delta_4^2$ several times, which enables one to move apart the generators $x_1, \ldots, x_4$ and to replace each relation in (6.4) by the relations $x_i = x_{i+4}$ and relations of the form $x_i = b(x_i)$, where b is a braid conjugate to a standard generator of the braid group. In our case, it suffices to double twice. We obtain the braid monodromy factorization $s_1 = d^2(s_0)$ of Δ_{16}^2 (the doubling $d^2(s_0)$ is defined in [11] by formula (25)), where each factor is either conjugate to Δ_4^2 or conjugate to a standard generator of Br_{16}, and the group

$$\langle x_1, \ldots, x_{16} \mid x_i = b(x_i),\ i = 1, \ldots, 16, \text{ and } b \text{ is a factor of } s_1 \rangle$$

is C-isomorphic to $G_{4,4}$. Precisely 4^2 factors of s_1 are conjugate to Δ_4^2.

Then, to adjoin the relations (6.4) to (6.3), it suffices to apply Lemma 3.4 of [11] three times. This yields a braid monodromy factorization s_2 of $\Delta_{2^{10}}^2$, each factor of which is either conjugate to Δ_4^2 or conjugate to a standard generator of $\mathrm{Br}_{2^{10}}$. Precisely 4^8 factors of s_2 are conjugate to Δ_4^2. The factorization s_2 is a braid monodromy factorization of a Hurwitz curve $\overline{H}_1$. We see that $\deg \overline{H}_1 = 2^{10}$, $\overline{H}_1$ has 4^8 singular points of the form $w^4 - z^4 = 0$, and $\pi_1(\mathbb{C}^2 \setminus H) \cong G(2)$ by the construction of s_2.

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