Equisingular Families of Projective Curves

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Summary. In this survey, we report on progress concerning families of projective curves with fixed number and fixed (topological or analytic) types of singularities. We are, in particular, interested in numerical, universal and asymptotically proper sufficient conditions to guarantee the nonemptyness, T-smoothness and irreducibility of the variety of all projective curves with prescribed singularities in a fixed linear system. We also discuss the analogous problem for hypersurfaces of arbitrary dimension with isolated singularities, and we close with a section on open problems and conjectures.

1 Introduction

Let $\mathcal{S}$ be a topological or analytic classification of isolated plane curve singularities. Assume that Σ is a non-singular projective algebraic surface over an algebraically closed field of characteristic zero, and $D \subset \Sigma$ is an ample divisor such that $\dim |D| > 0$ and the general member of $|D|$ is irreducible and non-singular. The part of the discriminant, consisting of irreducible curves on Σ with only isolated singularities, splits into the union of *equisingular families* (*ESF*) $V^{irr}_{|D|}(S_1, \ldots, S_r)$, that is, into the varieties of irreducible curves $C \in |D|$ having exactly r isolated singular points of types $S_1, \ldots, S_r \in \mathcal{S}$, respectively. If $\Sigma = \mathbb{P}^2$, $D = dH$, H a hyperplane divisor, we simply write $V^{irr}_d(S_1, \ldots, S_r)$ for the variety of irreducible plane curves of degree d with r singularities of the prescribed types. We focus on the following geometric problems which have been of interest to algebraic geometers since the early 20th century but which are still widely open in general:

A. Existence Problem: Is $V^{irr}_{|D|}(S_1, \ldots, S_r)$ non-empty, that is, does there exist a curve $F \in |D|$ with the given collection of singularities? In particular, the question about the minimal degree of a plane curve having a given singularity is of special interest.

B. T-Smoothness Problem: If $V_{|D|}^{irr}(S_1,\dots,S_r)$ is non-empty, is it smooth and of the expected dimension (expressible via local invariants of the singularities)? More precisely, let $C\in V_{|D|}^{irr}(S_1,\dots,S_r)$ have singular points $z_1,\dots,z_r$ of types $S_1,\dots,S_r$, respectively. We say that the family $V_{|D|}^{irr}(S_1,\dots,S_r)$ is *T-smooth* at C if, for every $i=1,\dots,r$, the germ at C of the family of curves $C'\in|D|$ with a singular point of type S_i in a neighbourhood of z_i is smooth and has the expected dimension (to be explained later), and, furthermore, all these r germs intersect transversally at C (whence the name T-smooth).

C. Irreducibility Problem: Is $V_{|D|}^{irr}(S_1,\dots,S_r)$ irreducible?

D. Deformation Problem: What are the adjacency relations of ESF in the discriminant? In other words, which simultaneous deformations of the singularities of $F\in V_{|D|}^{irr}(S_1,\dots,S_r)$ can be realized by a variation of F in $|D|$? In fact, this question is closely related to Problem B. For instance, for $\Sigma=\mathbb{P}^2$, the T-smoothness of $V_d^{irr}(S_1,\dots,S_r)$ for analytic singularities $S_1,\dots,S_r$ is equivalent to the linear system $|dH|$ inducing a joint versal deformation of all singular points of any member $C\in V_d^{irr}(S_1,\dots,S_r)$. Similarly, the T-smoothness of ESF for semiquasihomogeneous topological singularities $S_1,\dots,S_r$ implies that the independence of simultaneous "lower" (w.r.t. the Newton diagram) deformations of the singularities of $C\in V_d^{irr}(S_1,\dots,S_r)$ (see Section 4).

Of course, the same questions can be posed for $V_{|D|}(S_1,\dots,S_r)$, the variety of reduced (but not necessarily irreducible) curves $C\in|D|$ with given singularities $S_1,\dots,S_r\in\mathcal{S}$.

No complete solution to the above problems is known, except for the case of plane nodal curves. However, in this overview article we demonstrate that the approach using deformation theory and cohomology vanishing (developed by the authors of this article over the last ten years) enables us to obtain reasonably proper sufficient conditions for the affirmative answers to the problems stated. More precisely, we intend to give sufficient conditions for a positive answer to the above questions which are

- *numerical*, that is, presented in the form of inequalities relating numerical invariants of the surface, the linear system, and the singularities,
- *universal*, that is, applicable to curves in any ample linear system with any number of arbitrary singularities, and
- *asymptotically proper* (see the definition below).

We should like to comment on the latter in more detail. Let $D=dD_0$ with a given divisor $D_0\subset\Sigma$ (e.g., $D_0=H$ if $\Sigma=\mathbb{P}^2$). Then the known general restrictions to the singular points of a curve $C\in V_{|D|}^{irr}(S_1,\dots,S_r)$ read as upper bounds to some total singularity invariants by a quadratic function in d. As an example, consider the bound obtained by the genus formula for irreducible curves,

$$\sum_{i=1}^{r}\delta(S_i)\;\le\;\frac{1}{2}(d^2D_0^2+dK_\Sigma D_0)+1\,,$$

where $\delta(S_i)$ is the difference between the arithmetic genus and the geometric genus of C imposed by the singularity S_i. Sufficient conditions for the "regular" properties of ESF appear in a similar form as upper bounds to total singularity invariants by a linear or quadratic function of the parameter d. In this sense we speak of *linear*, or *quadratic* sufficient conditions, the latter revealing the relevant asymptotics.

Furthermore, among the quadratic sufficient conditions we emphasize on *asymptotically proper* ones. We say that the inequality

$$\sum_{i=1}^{r} a(S_i) \le f_{\Sigma,D_0}(d) \ ,$$

with $a(S)$ a local invariant of singularities $S \in \mathcal{S}$, is an *asymptotically proper* sufficient condition for a regular property (such as non-emptiness, smoothness, irreducibility) of ESF, if

(1) for arbitrary $r, d \ge 1$ and $S_1, \ldots, S_r \in \mathcal{S}$ it provides the required property of the ESF $V^{irr}_{|dD_0|}(S_1, \ldots, S_r)$,
(2) there exists an absolute constant $A \ge 1$ such that, for each singularity type S, there is an infinite sequence of growing integers d, r and a (maybe, empty) finite collection of singularities $\mathcal{S}_d$ satisfying

$$r \cdot a(S) = A \cdot f_{\Sigma,D_0}(d) + o(f_{\Sigma,D_0}(d)), \qquad \sum_{S' \in \mathcal{S}_d} a(S') = o(f_{\Sigma,D_0}(d)) \ ,$$

such that the ESF $V^{irr}_{|dD_0|}(r \cdot S, \mathcal{S}_d)$ does not have that regular property.

If $A = 1$ in (2), we speak of an *asymptotically optimal* sufficient condition.

In less technical terms, we say that a condition is asymptotically optimal (resp. proper) if the necessary and the sufficient conditions for a regularity property coincide (resp. coincide up to multiplication of the right-hand side with a constant) if r and d go to infinity.

Methods and Results: from Severi to Harris

T-Smoothness. Already the Italian geometers [Sev68, Seg24, Seg29] noticed that it is possible to express the T-smoothness of the variety of plane curves with fixed number of *nodes* (A_1-singularities) and *cusps* (A_2-singularities) infinitesimally. The problem was historically called the *"completeness of the characteristic linear series of complete continuous systems"* (of plane curves with nodes and cusps). Best known is certainly Severi's [Sev68] result saying that each non-empty variety of plane nodal curves is T-smooth.

But for more complicated singularities (beginning with cusps) there are examples of irreducible curves where the T-smoothness fails (see below). That is, ether the ESF is non-smooth or its dimension exceeds the one expected by subtracting the number of (closed) conditions imposed by the individual singularities from the dimension $\frac{d(d+3)}{2}$ of the variety of all curves of degree d. In this context, each node imposes exactly one condition. This may be illustrated as follows: a plane curve $\{f=0\}$ has a node at the origin iff the 1-jet of f vanishes (= three closed conditions) and the 2-jet is reduced (= one open condition). Allowing the node to move (in $\mathbb{C}^2$) reduces the number of closed conditions by two. Similarly, it can be seen that each cusp imposes two conditions.

Various sufficient conditions for T-smoothness were found. The classical result is that the variety $V_d^{irr}(n\cdot A_1, k\cdot A_2)$ of irreducible plane curves of degree d with n nodes and k cusps as only singularities is T-smooth, that is, smooth of dimension $\frac{d(d+3)}{2}-n-2k$, if

$$k \;<\; 3d\,. \tag{1}$$

For arbitrary singularities, several generalizations and extensions of (1) were found [GK89, GL96, Shu87, Shu91a, Vas90]. All of them are of the form that the sum of certain invariants of the singularities is bounded from above by a linear function in d. On the other hand, the known restrictions for existence and T-smoothness (and the known series of non T-smooth ESF) suggested that an asymptotically proper sufficient condition should be quadratic in d (see below).

Restrictions for the Existence. Various restrictions for the existence of plane curves with prescribed singularities $S_1,\dots,S_r$ have been found. First, one should mention the general classical bounds

$$\sum_{i=1}^{r}\delta(S_i) \;\le\; \frac{(d-1)(d-2)}{2}\,, \tag{2}$$

(for irreducible curves), resulting from the genus formula, respectively

$$\sum_{i=1}^{r}\mu(S_i) \;\le\; (d-1)^2,$$

resulting from the intersection of two generic polars and Bézout's theorem. If C has only nodes and cusps as singularities then the Plücker formulas give (among others) the necessary conditions

$$\begin{aligned} 2\cdot\#(\text{nodes}) + 3\cdot\#(\text{cusps}) &\le d^2-d-2\,,\\ 6\cdot\#(\text{nodes}) + 8\cdot\#(\text{cusps}) &\le 3d^2-6d\,. \end{aligned}$$

By applying the log-Miyaoka inequality, F. Sakai [Sak93] obtained the necessary condition

$$\sum_{i=1}^{r}\mu(S_i) \;<\; \frac{2\nu}{2\nu+1}\cdot\left(d^2-\frac{3}{2}d\right)\,,$$

where ν denotes the maximum of the multiplicities $\mathrm{mt}\, S_i,\ i = 1, \ldots, r$. If $S_1, \ldots, S_r$ are ADE-singularities then

$$\sum_{i=1}^{r} \mu(S_i) < \begin{cases} \frac{3}{4}d^2 - \frac{3}{2}d + 2 & \text{if } d \text{ is even}, \\ \frac{3}{4}d^2 - d + \frac{1}{4} & \text{if } d \text{ is odd}, \end{cases}$$

is necessary for the existence of a plane curve with r singularities of types $S_1, \ldots, S_r$ (cf. [Hir86, Ivi85], resp. [Sak93]). Further necessary conditions can be obtained, for instance, by applying the semicontinuity of the singularity spectrum (see [Var83a]).

Methods of Construction. The first method is to construct (somehow) a curve of the given degree which is degenerate with respect to the required curve, and then to deform it in order to obtain the prescribed singularities.

For instance, Severi [Sev68] showed that singular points of a nodal curve, irreducible or not, can be smoothed, or preserved, independently. Hence, starting with the union of generic straight lines in the projective plane and smoothing suitable intersection points, one obtains irreducible curves with any prescribed number of nodes, allowed by the genus bound (2), see Fig. 1.

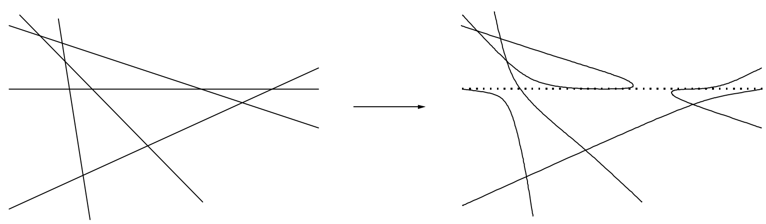

Fig. 1. Constructing irreducible nodal curves.

Attempts to extend this construction to other singularities give curves with a number of singularities bounded from above by a linear function in the degree d (see, for example, [GM88] for curves with nodes, cusps and ordinary triple points), because of the very restrictive requirement of the independence of deformations.

The second method consists of a construction especially adapted to the given degree and given collection of singularities. It may be based on a sequence of birational transformations of the plane applied to a more or less simple initial curve in order to obtain the required curve. Or it may consist of an invention of a polynomial defining the required curve. This is illustrated by constructions of singular curves of small degrees as, for instance, in [Wal95], [Wal96], or by Hirano's [Hir92] construction of cuspidal curves, which led to a series of irreducible cuspidal curves of degree $d = 2 \cdot 3^k$, $k \geq 1$ with precisely $9(9^k - 1)/8$ cusps. Note that in this case the number of conditions imposed by the cusps is $d^2/16 + O(d)$ more than the dimension of the space of curves of degree d.

Two main difficulties do not allow to apply this approach to a wide class of degrees and singularities:

- for any new degree or singularity one has to invent a new construction,
- even if one has constructed a curve with many singularities, it is hard to check that these singular points can be smoothed independently or, at least, that any intermediate number of singularities can be realized (for instance, for Hirano's examples the latter can hardly be expected).

Irreducibility. Severi [Sev68] claimed that all non-empty ESF $V_d^{irr}(n \cdot A_1)$ of irreducible plane nodal curves are not only T-smooth but also irreducible. However, as was realized later, his proof of the irreducibility was incomplete, and the problem has become known later as *"Severi's conjecture"*.

For many years, algebraic geometers tried to solve this problem without much progress. A major step forward was made by Fulton [Ful80] and Deligne [Del81], showing that the fundamental group of the complement of each irreducible nodal curve is Abelian (which is a necessary condition for $V_d^{irr}(n \cdot A_1)$ to be irreducible). Finally, the problem was settled by Harris [Har85a]. He gave a rigorous proof for the irreducibility of the varieties $V_d^{irr}(n \cdot A_1)$, by inventing a new specialization method and by using the irreducibility of the moduli space of curves of a given genus. Since nodal curves form an open dense subset of each *Severi variety* [Alb28, Nob84a, Nob87], that is, of the variety of all irreducible plane curves of a given degree and genus, Harris' theorem extends to all Severi varieties as well. Later Ran [Ran86] and Treger [Tre88] gave different proofs. Then Ran [Ran89] generalized Harris' theorem to ESF $V_d^{irr}(O_m, n \cdot A_1)$ with one ordinary singularity O_m (of some order $m \geq 1$) and any number of nodes $n \leq (d-1)(d-2)/2 - \delta(O_m)$. Kang [Kan89, Kan89a] obtained the irreducibility of the families $V_d^{irr}(n \cdot A_1, k \cdot A_2)$ for

$$k \leq 3\,, \quad \text{or} \quad \frac{d^2 - 4d + 1}{2} \leq n \leq \frac{d^2 - 3d + 2}{2}\,.$$

The main idea of the proofs consists of using moduli spaces of curves, special degenerations of nodal curves, or degenerations of rational surfaces. In any case the proofs heavily rely on the independence of simultaneous deformations of nodes for plane curves, which (in general) does not hold for more complicated singularities, even for cusps.

Examples of Obstructed and Reducible ESF. Already, Segre [Seg29, Tan84] constructed a series of irreducible plane curves such that the corresponding germs of ESF are non-T-smooth: $V_{6m}^{irr}(6m^2 \cdot A_2)$, $m \geq 3$. Similar examples are given in [Shu94]. However, in these examples $V_d^{irr}(S_1, \ldots, S_r)$ is smooth (but of bigger dimension than the expected one). In 1987, Luengo [Lue87] provided the first examples of curves C such that the corresponding ESF is *of expected dimension, but non-smooth*, e.g. $V_9^{irr}(A_{35})$.

Concerning reducible ESF, there is mainly one classical example due to Zariski [Zar71]: $V_6^{irr}(6 \cdot A_2)$. More precisely, Zariski shows that there exist exactly two components, both being T-smooth. One component consists of cuspidal sextics C whose singularities lie on a conic. For such curves, Zariski computed the fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ to be the *non-Abelian* group $\mathbb{Z}_2 * \mathbb{Z}_3$. On the other hand, he showed that there exist curves C whose singularities do not lie all on a conic and whose complement in $\mathbb{P}^2$ has an *Abelian* fundamental group. In particular, these curves cannot be obtained by a deformation from curves in the first component.

Actually, this example belongs to a series [Shu94]: $V_{6p}^{irr}(6p^2 \cdot A_2)$, $p \geq 2$, is reducible. More precisely, for $p \geq 3$ there exist components with different dimensions. For $p = 2$ there exist two different T-smooth components, as in Zariski's example.

2 Geometry of ESF in Terms of Cohomology

In this section, we relate the ESFs $V_{|D|}(S_1, \ldots, S_r)$ to strata of a Hilbert scheme representing a deformation functor. This allows us to deduce geometric properties such as the T-smoothness or the irreducibility from the vanishing of the first cohomology group for the ideal sheaves of appropriately chosen zero-dimensional subschemes of the surface Σ.

Let T be a complex space, then by a *family of reduced (irreducible) curves on Σ over T* we mean a commutative diagram

$$\begin{array}{ccc} \mathscr{C} & \overset{j}{\hookrightarrow} & \Sigma \times T \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}} & \\ & T & \end{array}$$

where φ is a proper and flat morphism such that all fibres $\mathscr{C}_t := \varphi^{-1}(t)$, $t \in T$, are reduced (irreducible) curves on Σ, $j : \mathscr{C} \hookrightarrow \Sigma \times T$ is a closed embedding and pr denotes the natural projection.

A *family with sections* is a diagram as above, together with sections $\sigma_1, \ldots, \sigma_r : T \to \mathscr{C}$ of φ.

To a family of reduced plane curves and a point $t_0 \in T$ we can associate, in a functorial way, the deformation $(\mathscr{C}, z_1) \amalg \ldots \amalg (\mathscr{C}, z_r) \to (T, t_0)$ of the multigerm $(C, \mathrm{Sing}\ C) = \coprod_i (C, z_i)$ over the germ (T, t_0), where $C = \mathscr{C}_{t_0}$ is the fibre over t_0. Having a family with sections $\sigma_1, \ldots, \sigma_r$, $\sigma_i(t_0) = z_i$, we obtain in the same way a deformation of $\coprod_i (C, z_i)$ over (T, t_0) with sections.

A family $\mathscr{C} \hookrightarrow \Sigma \times T \to T$ of reduced curves (with sections) is called *equianalytic* (along the sections) if, for each $t \in T$, the induced deformation of the multigerm $(\mathscr{C}_t, \mathrm{Sing}\ \mathscr{C}_t)$ is isomorphic (isomorphic as deformation with section) to the trivial deformation (along the trivial sections). It is called *equisingular* (along the sections) if, for each $t \in T$, the induced deformation of the multigerm $(\mathscr{C}_t, \mathrm{Sing}\, \mathscr{C}_t)$ is isomorphic (isomorphic as deformation with section) to an equisingular deformation along the trivial sections. In other words, a family with sections is equianalytic (resp. equisingular) if the analytic (resp. topological) type of the fibre $\mathscr{C}_t$ does not change along the sections (cf. [Wah, GLS06]). Here, the *analytic type* of a reduced plane curve singularity (C, z) is given by the isomorphism class of its analytic local ring $\mathcal{O}_{C,z}$, while the *(embedded) topological type* is given by the Puiseux pairs of its branches and their mutual intersection multiplicities or, alternatively, by the system of multiplicity sequences (see [BK86]).

The *Hilbert functor* $\mathcal{H}ilb_{\Sigma}$ on the category of complex spaces defined by

$$\mathcal{H}ilb_{\Sigma}(T) := \{\mathscr{C} \hookrightarrow \Sigma \times T \to T,\ \text{family of reduced curves over } T\}$$

is known to be representable by a complex space Hilb_{Σ} (see [Gro61] for algebraic varieties and [Dou] for arbitrary complex spaces). Moreover, the universal family of reduced curves on Σ "breaks up" into strata with constant Hilbert polynomials, more precisely, $\mathrm{Hilb}_{\Sigma} = \coprod_{h \in \mathbb{C}[z]} \mathrm{Hilb}_{\Sigma}^h$ where the Hilb_{Σ}^h are (unions of) connected components of Hilb_{Σ} whose points correspond to curves on Σ with the fixed Hilbert polynomial h.

Let $V_h(S_1, \ldots, S_r) \subset \mathrm{Hilb}_{\Sigma}^h$ denote the locally closed subspace (*equisingular stratum*) of reduced curves with Hilbert polynomial h having precisely r singularities of types $S_1, \ldots, S_r$ ([GL96]). Further, let $V_h^{irr}(S_1, \ldots, S_r) \subset V_h(S_1, \ldots, S_r)$ denote the open subspace parametrizing irreducible curves.

This notion of (not necessarily reduced) equisingular strata in the Hilbert scheme Hilb_{Σ} is closely related to the ESFs $V_{|D|}(S_1, \ldots, S_r)$ considered before: if $U \subset |C| = \mathbb{P}\big(H^0(\mathcal{O}_{\Sigma}(C))\big)$ denotes the open subspace corresponding to reduced curves, then there exists a unique morphism $U \hookrightarrow \mathrm{Hilb}_{\Sigma}^h$ which on the tangent level corresponds to $H^0\big(\mathcal{O}_{\Sigma}(C)\big)/H^0\big(\mathcal{O}_{\Sigma}\big) \hookrightarrow H^0\big(\mathcal{O}_C(C)\big)$. Via this morphism, we may consider U as a locally closed subscheme of Hilb_{Σ}^h. In particular, for a regular surface (that is, a surface satisfying $H^1(\mathcal{O}_{\Sigma}) = 0$) the above injection is an isomorphism and U is an open subscheme of Hilb_{Σ}^h (see [GL01, GLS06] for details).

In the following, we give a geometric interpretation of the zeroth and first cohomology of the ideal sheaves of certain zero-dimensional schemes. We write $C \in V_h(S_1, \ldots, S_r)$ to denote either the point in $V_h(S_1, \ldots, S_r)$ or the curve corresponding to the point, that is, the corresponding fibre of the universal family $\mathscr{U}_h$ over the Hilbert scheme. Consider the map

$$\Phi_h : V_h(S_1, \ldots, S_r) \longrightarrow \mathrm{Sym}^r \Sigma\,, \quad C \longmapsto (z_1 + \ldots + z_r)\,, \tag{1}$$

where $\mathrm{Sym}^r \Sigma$ is the r-fold symmetric product of Σ and where $(z_1 + \ldots + z_r)$ is the non-ordered tuple of the singularities of C. Since each equisingular, in particular each equianalytic, deformation of a germ admits a unique singular section (cf. [Tei78]), the universal family over $V_h(S_1, \ldots, S_r)$,

$$\mathscr{U}_h(S_1,\dots,S_r) \hookrightarrow \Sigma \times V_h(S_1,\dots,S_r) \to V_h(S_1,\dots,S_r),$$

admits, locally at C, r singular sections. Composing these sections with the projections to Σ gives a local description of the map Φ_h and shows in particular that Φ_h is a well-defined morphism, even if $V_h(S_1,\dots,S_r)$ is not reduced.

We denote by $V_{h,\mathit{fix}}(S_1,\dots,S_r)$ the complex space consisting of the disjoint union of the fibres of Φ_h. Thus, each connected component of $V_{h,\mathit{fix}}(S_1,\dots,S_r)$ consists of curves with fixed positions of the singularities in Σ.

It follows from the universal property of $V_h(S_1,\dots,S_r)$ and from the above construction that $V_{h,\mathit{fix}}(S_1,\dots,S_r)$, together with the induced universal family on each fibre, represents the functor of equianalytic, resp. equisingular, families of given types $S_1,\dots,S_r$ along trivial sections.

Before formulating the main proposition relating the vanishing of cohomology to geometric properties of ESF, we introduce some notation: we write V_h to denote $V_h(S_1,\dots,S_r)$, resp. $V_{h,\mathit{fix}}(S_1,\dots,S_r)$, and $V_{|C|}$ to denote $V_{|C|}(S_1,\dots,S_r)$, resp. $V_{|C|,\mathit{fix}}(S_1,\dots,S_r)$. Moreover, we write

$$Z'(C) = \bigcup_{z\in \mathrm{Sing}(C)} Z'(C,z)$$

to denote one of the 0-dimensional schemes $Z^{ea}(C), Z^{ea}_{\mathit{fix}}(C), Z^{es}(C), Z^{es}_{\mathit{fix}}(C)$, where

- $Z^{ea}(C,z)$ is defined by the *Tjurina ideal*, that is, the ideal generated by a local equation $f\in\mathbb{C}\{u,v\}$ for $(C,z)\subset(\Sigma,z)$ and its partial derivatives;
- $Z^{ea}_{\mathit{fix}}(C,z)$ is defined by the ideal $I^{ea}_{\mathit{fix}}(f) = \langle f\rangle + \langle u,v\rangle\cdot\langle \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\rangle$;
- $Z^{es}(C,z)$ is defined by the *equisingularity ideal* $I^{es}(f)\subset\mathbb{C}\{u,v\}$ as introduced by Wahl [Wah];
- $Z^{es}_{\mathit{fix}}(C,z)$ is defined by the ideal

$$I^{es}_{\mathit{fix}}(f) := \left\{ g\in I^{es}(f) \;\middle|\; \begin{array}{c} f+\varepsilon g \text{ defines an equisingular deformation} \\ \text{of } (C,0) \text{ along the trivial section over } T_\varepsilon \end{array}\right\}$$

(here, T_ε denotes the fat point $(\{0\},\mathbb{C}[\varepsilon]/\langle\varepsilon^2\rangle)$).

We write $\mathcal{J}_{Z'(C)/C}$, resp. $\mathcal{J}_{Z'(C)/\Sigma}$, to denote the ideal sheaf of $Z'(C)$ in $\mathcal{O}_C$, resp. in $\mathcal{O}_\Sigma$, and $\mathcal{J}(C) := \mathcal{J}\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_\Sigma(C)$. Moreover, we write $\deg Z'(C)$ for the degree of $Z'(C)$ as a projective variety, that is,

$$\deg Z'(C) = \dim_{\mathbb{C}} \mathcal{O}_\Sigma/\mathcal{J}_{Z'(C)/\Sigma}\,.$$

Proposition 1 ([GL01, Prop. 2.6]). *Let $C\subset\Sigma$ be a reduced curve with Hilbert polynomial h and precisely r singularities $z_1,\dots,z_r$ of analytic or topological types $S_1,\dots,S_r$.*

(a) The Zariski tangent space of V_h at C is $H^0\big(\mathcal{J}_{Z'(C)/C}(C)\big)$, while the Zariski tangent space of $V_{|C|}$ at C is $H^0\big(\mathcal{J}_{Z'(C)/\Sigma}(C)\big)/H^0\big(\mathcal{O}_\Sigma\big)$.

(b) $h^0\big(\mathcal{J}_{Z'(C)/C}(C)\big) - h^1\big(\mathcal{J}_{Z'(C)/C}(C)\big) \;\leq\; \dim(V_h, C) \leq h^0\big(\mathcal{J}_{Z'(C)/C}(C)\big)$.

(c1) If $H^1\big(\mathcal{J}_{Z'(C)/C}(C)\big) = 0$ *then* V_h *is T-smooth at* C*, that is, smooth of the expected dimension* $h^0\big(\mathcal{O}_C(C)\big) - \deg Z'(C) = C^2 + 1 - p_a(C) - \deg Z'(C)$.

(c2) If $H^1\big(\mathcal{J}_{Z'(C)/\Sigma}(C)\big) = 0$ *then* $V_{|C|}$ *is T-smooth at* C*, that is, smooth of the expected dimension* $h^0\big(\mathcal{O}_\Sigma(C)\big) - 1 - \deg Z'(C)$.

(d) If $H^1\big(\mathcal{J}_{Z^{ea}(C)/C}(C)\big) = 0$ *then the natural morphism of germs*

$$\big(\mathrm{Hilb}^h_\Sigma,\, C\big) \longrightarrow \prod_{i=1}^{r} Def(C, z_i)$$

is smooth of fibre dimension $h^0\big(\mathcal{J}_{Z^{ea}(C)/C}(C)\big)$*. Here,* $\prod_{i=1}^r Def(C, z_i)$ *is the Cartesian product of the base spaces of the semiuniversal deformations of the germs* (C, z_i).

(e) Write $Z_{\mathit{fix}}(C)$ *for* $Z^{ea}_{\mathit{fix}}(C)$*, respectively* $Z^{es}_{\mathit{fix}}(C)$*. Then the vanishing of* $H^1\big(\mathcal{J}_{Z_{\mathit{fix}}(C)/C}(C)\big)$ *implies that the morphism of germs*

$$\Phi_h : \big(V_h(S_1, \ldots, S_r), C\big) \to \big(\mathrm{Sym}^r \Sigma, (z_1 + \ldots + z_r)\big)$$

is smooth of fibre dimension $h^0\big(\mathcal{J}_{Z_{\mathit{fix}}(C)/C}(C)\big)$.

We reformulate and strengthen Proposition 1 in the case of plane curves, which is of special interest. Of course, since $h^1\big(\mathcal{O}_{\mathbb{P}^2}\big) = h^2\big(\mathcal{O}_{\mathbb{P}^2}\big) = 0$, there is no difference whether we consider the curves in the linear system $|dH|$, H the hyperplane divisor, or curves with fixed Hilbert polynomial $h(z) = dz - (d^2 - 3d)/2$. We denote the corresponding varieties by $V_d = V_d(S_1, \ldots, S_r)$, respectively by $V_{d,\mathrm{fix}}(S_1, \ldots, S_r)$. Using the above notation, we obtain:

Proposition 2 ([GL01, Prop. 2.8]). *Let* $C \subset \mathbb{P}^2$ *be a reduced curve of degree* d *with precisely* r *singularities* $z_1, \ldots, z_r$ *of analytic or topological types* $S_1, \ldots, S_r$.

(a) $H^0\big(\mathcal{J}_{Z'(C)/\mathbb{P}^2}(d)\big)\big/H^0(\mathcal{O}_{\mathbb{P}^2})$ *is isomorphic to the Zariski tangent space of* V_d *at* C.

(b)
$$\begin{aligned} h^0\big(\mathcal{J}_{Z'(C)/\mathbb{P}^2}(d)\big) - h^1\big(\mathcal{J}_{Z'(C)/\mathbb{P}^2}(d)\big) - 1 \;&\leq\; \dim(V_d, C) \\ &\leq\; h^0\big(\mathcal{J}_{Z'(C)/\mathbb{P}^2}(d)\big) - 1. \end{aligned}$$

(c) $H^1\big(\mathcal{J}_{Z'(C)/\mathbb{P}^2}(d)\big) = 0$ *iff* V_d *is T-smooth at* C*, that is, smooth of the expected dimension* $d(d+3)/2 - \deg Z'(C)$.

(d) $H^1\big(\mathcal{J}_{Z^{ea}(C)/\mathbb{P}^2}(d)\big) = 0$ *iff the natural morphism of germs*

$$\big(\mathbb{P}\big(H^0\big(\mathcal{O}_{\mathbb{P}^2}(d)\big)\big),\, C\big) \to \textstyle\prod_{i=1}^r Def(C, z_i)$$

is smooth (hence surjective) of fibre dimension $h^0\big(\mathcal{J}_{Z^{ea}(C)/\mathbb{P}^2}(d)\big) - 1$.

(e) Write $Z_{\mathit{fix}}(C)$ *for* $Z^{ea}_{\mathit{fix}}(C)$*, respectively* $Z^{es}_{\mathit{fix}}(C)$*. Then* $H^1\big(\mathcal{J}_{Z_{\mathit{fix}}(C)/\mathbb{P}^2}(d)\big)$ *vanishes iff the morphism of germs*

$$\Phi_d : \big(V_d(S_1,\ldots,S_r), C\big) \to \big(\mathrm{Sym}^r\mathbb{P}^2, (z_1+\ldots+z_r)\big)$$

is smooth of fibre dimension $h^0\big(\mathcal{J}_{Z_{\text{fix}}(C)/\mathbb{P}^2}(d)\big) - 1$. *In particular, the vanishing of* $H^1\big(\mathcal{J}_{Z_{\text{fix}}(C)/\mathbb{P}^2}(d)\big)$ *implies that arbitrarily close to* C *there are curves in* $V_d(S_1,\ldots,S_r)$ *whose singularities are in general position in* $\mathbb{P}^2$.

3 T-Smoothness

To show the T-smoothness of $V_{|C|}(S_1,\ldots,S_r)$ it suffices to show, according to Proposition 1 (c2), that $H^1\big(\mathcal{J}_{Z^{ea}(C)/\Sigma}(C)\big) = 0$ (in the case of analytic types), respectively $H^1\big(\mathcal{J}_{Z^{es}(C)/\Sigma}(C)\big) = 0$ (in the case of topological types). Note that for $\Sigma = \mathbb{P}^2$, these conditions are even equivalent to the T-smoothness of $V_d(S_1,\ldots,S_r)$ by Proposition 2 (c).

3.1 ESF of Plane Curves

The classical approach to the H^1-vanishing problem (based on Riemann-Roch and Serre duality) leads to sufficient conditions for the T-smoothness of ESF of plane curves such as the $3d$-condition (1) and its extensions mentioned above.

In the papers [GLS97, GLS00], we applied two different approaches to the H^1-vanishing problem, based on the Reider-Bogomolov theory of unstable rank 2 vector bundles (see also [CS97]), respectively on the Castelnuovo function of the ideal sheaf of a zero-dimensional scheme (see also [Dav86, Bar93a]). Both approaches lead to quadratic sufficient conditions for the T-smoothness of ESF of plane curves. Combining both approaches, we obtain:

Theorem 1 ([GLS00, GLS01]). *Let* $C \subset \mathbb{P}^2$ *be an irreducible curve of degree* $d > 5$ *having* r *singularities* $z_1,\ldots,z_r$ *of topological (respectively analytic) types* $S_1,\ldots,S_r$ *as its only singularities. Then* $V_d^{irr}(S_1,\ldots,S_r)$ *is T-smooth at* C *if*

$$\sum_{i=1}^{r} \gamma'(C,z_i) \le (d+3)^2, \tag{2}$$

$\gamma'(C,z_i) = \gamma^{es}(C,z_i)$ *for* S_i *a topological type, resp.* $\gamma'(C,z_i) = \gamma^{ea}(C,z_i)$ *for* S_i *an analytic type.*

Here, γ^{es} and γ^{ea} are new analytic invariants of singularities which are defined as follows (see [KL05] for a thorough discussion):

The $\boldsymbol{\gamma}$-invariant. Let $f \in \mathbb{C}\{u,v\}$ be a reduced power series, and let $I \subset \langle u,v\rangle \subset \mathbb{C}\{u,v\}$ be an ideal containing the Tjurina ideal $I^{ea}(f) = \langle f, \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\rangle$. For each $g \in \langle u,v\rangle \subset \mathbb{C}\{x,y\}$, we introduce $\Delta(f,g;I)$ as the minimum among $\dim_{\mathbb{C}} \mathbb{C}\{u,v\}/\langle I,g\rangle$ and $i(f,g) - \dim_{\mathbb{C}} \mathbb{C}\{u,v\}/\langle I,g\rangle$ (where $i(f,g)$ denotes the intersection multiplicity of f and g, $i(f,g) = \dim_{\mathbb{C}} \mathbb{C}\{u,v\}/\langle f,g\rangle)$. By [Shu97, Lemma 4.1], this minimum is at least 1 so that we may define

$$\gamma\big(f;I\big) := \max_{g\in\langle x,y\rangle} \left\{ \frac{(\dim_{\mathbb{C}} \mathbb{C}\{u,v\}/\langle I,g\rangle + \Delta(f,g;I))^2}{\Delta(f,g;I)} \right\},$$

and, finally,

$$\gamma^{ea}(f) := \max\big\{\gamma\big(f;I\big) \,\big|\, I \supset I^{ea}(f) \text{ a complete intersection ideal}\big\},$$
$$\gamma^{es}(f) := \max\big\{\gamma\big(f;I\big) \,\big|\, I \supset I^{es}(f) \text{ a complete intersection ideal}\big\}.$$

Note that $\gamma'(f) \le \big(\tau'_{ci}(f)+1\big)^2$, where $\tau'_{ci}(f)$ stands for one of

$$\tau_{ci}(f) := \max\big\{\dim_{\mathbb{C}} \mathbb{C}\{u,v\}/I \,\big|\, I \supset I^{ea}(f) \text{ a complete intersection ideal}\big\} \le \tau(f)\,,$$
$$\tau^{es}_{ci}(f) := \max\big\{\dim_{\mathbb{C}} \mathbb{C}\{u,v\}/I \,\big|\, I \supset I^{es}(f) \text{ a complete intersection ideal}\big\} \le \tau^{es}(f)\,.$$

Here, $\tau(f) = \dim_{\mathbb{C}} \mathbb{C}\{u,v\}/I^{ea}(f)$ is the *Tjurina number* of f and $\tau^{es}(f) = \dim_{\mathbb{C}} \mathbb{C}\{u,v\}/I^{es}(f)$ is the codimension of the μ-constant stratum in the semiuniversal deformation of f (see [GLS00, Lemma 4.2]).

In general, these bounds for the γ-invariant are far from being sharp. For instance, if f defines an *ordinary singularity of order* $m \ge 3$ (that is, the m-jet of f is a reduced homogeneous polynomial of degree m), then $\gamma^{es}(f) = 2m^2$ while $(\tau^{es}(f)+1)^2 = \frac{1}{2}m^4 + O(m)$. See [KL05] for the case of semiquasihomogeneous singularities.

For ESF of irreducible curves with nodes and cusps, respectively for ESF of irreducible curves with ordinary singularities, we obtain:

Corollary 1. *$V_d(n\cdot A_1,\, k\cdot A_2)$ is T-smooth or empty if*

$$4n + 9k \;\le\; (d+3)^2\,. \tag{3}$$

Corollary 2. *If $S_1,\dots,S_r$ are ordinary singularities of order $m_1,\dots,m_r$, then $V_d(S_1,\dots,S_r)$ is T-smooth or empty if*

$$4\cdot \#(\mathit{nodes}) + \sum_{m_i\ge 3} 2\cdot m_i^2 \;\le\; (d+3)^2. \tag{4}$$

In particular, it follows that Theorem 1 is asymptotically proper for ordinary singularities, since the inequality

$$\sum_{i=1}^{r} m_i(m_i-1) \;\le\; (d-1)(d-2)$$

is necessary for the existence of an irreducible curve with ordinary singularities of multiplicities $m_1,\dots,m_r$. More generally, by constructing series of ESF where the T-smoothness fails (see [Shu97, GLS97, Los02]), we proved the asymptotic properness of condition (2) in Theorem 1 for the case of semiquasihomogeneous singularities.

3.2 ESF of Curves on Smooth Algebraic Surfaces

Let Σ be a smooth projective surface and D_0 an effective divisor on Σ. In this situation, for lack of a generalization of the Castelnuovo function approach, so far only the Reider-Bogomolov approach leads to a quadratic sufficient condition for the T-smoothness of (non-empty) ESF $V_{|dD_0|}(S_1,\dots,S_r)$. Set

$$A(\Sigma,D_0) := \frac{(D_0.K_\Sigma)^2 - D_0^2 K_\Sigma^2}{4}\,,$$

where K_Σ is the canonical divisor on Σ. By the Hodge index theorem, this is a non-negative number if D_0 or K_Σ is ample.

Theorem 2 ([GLS97, GLS06a]). *Let $C \subset \Sigma$ be an irreducible curve with precisely r singular points $z_1,\dots,z_r$ of topological or analytic types $S_1,\dots,S_r$, ordered such that $\tau'_{ci}(C,z_1) \geq \ldots \geq \tau'_{ci}(C,z_r)$. Assume that C and $C-K_\Sigma$ are ample and that $C^2 \geq \max\{K_\Sigma^2, A(\Sigma,C)\}$. If*

$$\sum_{i=1}^{r} \tau'_{ci}(C,z_i) \;<\; \frac{(C-K_\Sigma)^2}{4} \tag{5}$$

and, for each $1 \leq s \leq r$,

$$\left(\sum_{i=1}^{s}\bigl(\tau'_{ci}(C,z_i)+1\bigr)\right)^2 \;<\; \sum_{i=1}^{s}\Bigl(C^2 - C.K_\Sigma\bigl(\tau'_{ci}(C,z_i)+1\bigr)\Bigr) - A(\Sigma,C)\,, \tag{6}$$

then $V_{|C|}(S_1,\dots,S_r)$ is T-smooth at C. Here, $\tau'_{ci}(S_i)$ stands for $\tau^{es}_{ci}(S_i)$ if S_i is a topological type and for $\tau_{ci}(S_i)$ if S_i is an analytic type.

Note that, in general, the conditions (6) are not *quadratic* sufficient conditions in the above sense. But in many cases they are. For instance, if $-K_\Sigma$ is nef, then by applying the Cauchy inequality we deduce:

Corollary 3. *Let Σ be a smooth projective surface with $-K_\Sigma$ nef, D_0 an ample divisor on Σ such that $D_0^2 \geq A(\Sigma,D_0)$, and $d>0$ such that $d^2 \geq K_\Sigma^2/D_0^2$. If*

$$\sum_{i=1}^{r}\bigl(\tau'_{ci}(S_i)+1\bigr)^2 \;<\; \bigl(D_0^2 - A(\Sigma,D_0)\bigr)\cdot d^2 - 2(D_0.K_\Sigma)\cdot d\,, \tag{7}$$

then $V^{irr}_{|dD_0|}(S_1,\dots,S_r)$ is T-smooth or empty.

For ESF of curves with nodes and cusps, respectively for ESF of curves with ordinary singularities, the obvious estimates for τ' allow us to deduce the following corollaries:

Corollary 4. *With the assumptions of Corollary 3, let*

$$4n + 9k \;<\; \bigl(D_0^2 - A(\Sigma,D_0)\bigr)\cdot d^2 - 2(D_0.K_\Sigma)\cdot d \tag{8}$$

then $V^{irr}_{|dD_0|}(n\cdot A_1, k\cdot A_2)$ is T-smooth or empty.

Corollary 5. *With the assumptions of Corollary 3, let $S_1, \dots, S_r$ be ordinary singularities of order $m_1, \dots, m_r$. Then $V^{irr}_{|dD_0|}(S_1, \dots, S_r)$ is T-smooth or empty if*

$$4 \cdot \#(nodes) + \sum_{m_i \geq 3} \left\lfloor \frac{m_i^2 + 2m_i + 5}{4} \right\rfloor^2 < \big(D_0^2 - A(\Sigma, D_0)\big) d^2 - 2(D_0.K_\Sigma) d\,. \quad (9)$$

In our forthcoming book [GLS06a], special emphasis is put on two important examples, ESF of curves on a smooth hypersurface $\Sigma \subset \mathbb{P}^3$:

Theorem 3 ([GLS06a]). *Let $\Sigma \subset \mathbb{P}^3$ be a smooth hypersurface of degree $e \geq 5$, let $D_0 = (e-4)H$, H a hyperplane section, and let $d \in \mathbb{Q}$, $d > 1$. Suppose, moreover, that $\max_i \tau'_{ci}(S_i) \leq d-2$. If*

$$\sum_{i=1}^{r} \tau'_{ci}(S_i) \; < \; \frac{e(e-4)^2}{4} \cdot (d-1)^2 \quad (10)$$

and

$$\sum_{i=1}^{r} \frac{\big(\tau'_{ci}(S_i) + 1\big)^2}{1 - \frac{\tau'_{ci}(S_i)+1}{d}} \; < \; e(e-4)^2 \cdot d^2 \quad (11)$$

then $V^{irr}_{|dD_0|}(S_1, \dots, S_r)$ is T-smooth or empty.

For ESF of nodal curves, respectively for ESF of curves with only nodes and cusps, we can easily conclude the following quadratic sufficient conditions for T-smoothness:

Corollary 6. *Let $\Sigma \subset \mathbb{P}^3$ be a smooth hypersurface of degree $e \geq 5$, let $D_0 = (e-4)H$, H a hyperplane section, and let $d \in \mathbb{Q}$.*

(a) If $d \geq 3$ and

$$4n \; < \; e(e-4)^2 \cdot d(d-2) \quad (12)$$

then $V^{irr}_{|dD_0|}(n \cdot A_1)$ is T-smooth or empty.

(b) If $d \geq 4$ and

$$4n + 9k \; < \; e(e-4)^2 \cdot d(d-3) \quad (13)$$

then $V^{irr}_{|dD_0|}(n \cdot A_1, k \cdot A_2)$ is T-smooth or empty.

In the case of a quintic surface $\Sigma \subset \mathbb{P}^3$ ($e = 5$), Chiantini and Sernesi [CS97] provide examples of curves $C \subset \Sigma$, $C \equiv d \cdot H$, $d \geq 6$, having $(5/4) \cdot (d-1)^2$ nodes such that $V^{irr}_{|C|}(n \cdot A_1)$ is not T-smooth at C. In particular, these examples show that the exponent 2 for d in the right-hand side of (12) is the best possible. Actually, it even shows that for families of nodal curves on a quintic surface the condition (12) is asymptotically exact.

Performing more thorough computations in the proof of [GLS97, Thm. 1], Keilen improved the result of Theorem 2 for surfaces with Picard number one or two. For example, the following statement generalizes Theorem 3:

Theorem 4 ([Kei05]). *Let Σ be a surface with Neron-Severi group $L \cdot \mathbb{Z}$, L being ample, let $D = d \cdot L$, let $S_1, \ldots, S_r$ be topological or analytic singularity types, and let $K_\Sigma = k_\Sigma \cdot L$. Suppose that $d \geq \max\{k_\Sigma + 1, -k_\Sigma\}$, and*

$$\sum_{i=1}^{r} \gamma'(S_i) < \alpha \cdot (D - K_\Sigma)^2, \quad \alpha = \frac{1}{\max\{1, 1 + k_\Sigma\}}\,. \tag{14}$$

Then either $V^{irr}_{\Sigma,|D|}(S_1, \ldots, S_r)$ is empty or it is T-smooth.

It is interesting that the same invariant $\gamma'(S)$ comes out of the proof using the Bogomolov-Reider theory of unstable rank two vector bundles on surfaces instead of the Castelnuovo function theory (as done in the proof of Theorem 2).

3.3 ESF of Hypersurfaces

Sufficient Conditions for T-Smoothness. Denote by $V_d^n(S_1, \ldots, S_r)$ the set of hypersurfaces of degree d in $\mathbb{P}^n$, $n \geq 3$, whose singular locus consists of r isolated singularities of analytic types $S_1, \ldots, S_r$, respectively. The following theorem was proved independently by Shustin and Tyomkin and by Du Plessis and Wall (using different methods of proof):

Theorem 5 ([ST99, DPW00]). *Let $S_1, \ldots, S_r$ be analytic singularity types satisfying*

$$\sum_{i=1}^{r} \tau(S_i) \; < \; \begin{cases} 4d - 4 & \text{if } d \geq 5, \\ 18 & \text{if } d = 4, \\ 16 & \text{if } d = 3\;. \end{cases} \tag{15}$$

Then the variety $V_d^n(S_1, \ldots, S_r)$ is T-smooth, that is, smooth of the expected codimension $\sum_{i=1}^{r} \tau(S_i)$ in $\left|H^0\big(\mathcal{O}_{\mathbb{P}^n}(d)\big)\right|$.

A similar statement for topological types of singularities cannot be true in general, because, for some singularities, the $\mu = \text{const}$ stratum in a versal deformation base (being a local topological ESF) is not smooth [Lue87]. However, for semiquasihomogeneous singularities the $\mu = \text{const}$ stratum in a versal deformation base is smooth [Var82].

Here a hypersurface singularity $(W, z) \subset (\mathbb{P}^n, z)$ is called *semiquasihomogeneous* (*SQH*) if there are local analytic coordinates such that (W, z) is given by a power series

$$f = \sum_{\boldsymbol{w}\text{-deg}(\boldsymbol{\alpha}) \geq a} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \in \mathbb{C}\{\boldsymbol{x}\}\,, \tag{16}$$

(where $\boldsymbol{w}\text{-deg}(\boldsymbol{\alpha}) := \sum_{j=1}^{n} w_j \alpha_j$, $\boldsymbol{w} \in (\mathbb{Z}_{>0})^n$) such that the Newton polygon of f is convenient (that is, it intersects all coordinate axes), and such that the *principal part of f*,

$$f_0 = \sum_{\boldsymbol{w}\text{-deg}(\boldsymbol{\alpha}) = a} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}\,,$$

defines an isolated singularity at the origin. We introduce the ideal

$$I^{sqh}(f) = \langle \boldsymbol{x}^{\boldsymbol{\alpha}} \mid \boldsymbol{w}\text{-deg}(\boldsymbol{x}^{\boldsymbol{\alpha}}) \geq \boldsymbol{w}\text{-deg}(f_0) \rangle + I^{ea}(f) \subset \mathbb{C}\{\boldsymbol{x}\}\,.$$

Its codimension in $\mathbb{C}\{\boldsymbol{x}\}$ is an invariant of the topological type S of (W,z). We denote it by $\deg Z^{sqh}(S)$.

Using the smoothness of the $\mu = \text{const}$ stratum for SQH singularities, in [GLS06a] we prove the following extension of Theorem 5:

Theorem 6 ([GLS06a, ST99]). *Let $S_1,\dots,S_q$, $q<r$, be analytic singularity types, and let $S_{q+1},\dots,S_r$ be topological types of semiquasihomogeneous singularities. If*

$$\sum_{i=1}^{q} \tau(S_i) + \sum_{i=q+1}^{r} \deg Z^{sqh}(S_i) < \begin{cases} 4d-4 & \text{if } d \geq 5, \\ 18 & \text{if } d = 4, \\ 16 & \text{if } d = 3\,, \end{cases} \tag{17}$$

then the variety $V_d^n(S_1,\dots,S_r)$ is T-smooth.

Du Plessis and Wall [DPW00] consider also linear systems of hypersurfaces having a fixed intersection with a hyperplane, which does not pass through the singular points, and obtain the statement of Theorem 5 for such linear systems under the condition

$$\sum_{i=1}^{r} \tau(S_i) < \begin{cases} 3d-3 & \text{if } d \geq 4, \\ 8 & \text{if } d = 3\,. \end{cases}$$

Similarly, one can formulate an analogue of Theorem 6.

Non-T-Smooth ESF of Hypersurfaces. The (linear) condition (15) in Theorem 5 is not necessary, as already seen in the case $n=2$. In the following, we discuss which kind of sufficient conditions one might expect.

It would be natural to extend or generalize the corresponding results for plane curves to higher dimensions. Theorem 5 is a generalization of the $4d$-condition for plane curves (see, e.g., [GL96]). The classical $3d$-condition (1), however, cannot be extended to higher dimensions in the same form. This follows, since for plane curves it allows any number of nodes, while there are surfaces of degree $d \to \infty$ in $\mathbb{P}^3$ with $5d^3/12 + O(d^2)$ nodes [Chm92] as only singularities. For $d \gg 1$, these nodes must be dependent as $\dim \left|H^0\big(\mathcal{O}_{\mathbb{P}^3}(d)\big)\right| = d^3/6 + O(d^2)$. We also point out another important difference between the case of curves and the case of higher dimensional hypersurfaces. The quadratic numerical sufficient conditions for T-smoothness of ESF of plane curves are close to necessary conditions for the existence, which are quadratic in the degree d as well. In higher dimensions the situation is different. Namely, necessary conditions for the non-emptiness of $V_d^n(S_1,\dots,S_r)$, such as

$$\sum_{i=1}^{r} \mu(S_i) \leq (d-1)^n \ ,$$

are of order d^n in the right-hand side and, for a fixed n, there exist hypersurfaces with number of arbitrary singularities of order d^n (see Section 5.3 below). However, any possible sufficient condition for T-smoothness, in the form of an upper bound to the sum of certain positive singularity invariants, can have at most a quadratic function in d on the right-hand side. Indeed, the following lemma allows us to extend examples of analytic ESF of plane curves which are non-T-smooth to higher dimensions such that the degree of the hypersurface and the total Tjurina number are not changed.

Lemma 1 ([ST99]). *Let C be a reduced plane curve of degree $d > 2$. Then, for any $n > 2$, there exists a hypersurface $W \subset \mathbb{P}^n$ of degree d having only isolated singular points, such that $\tau(W) = \tau(C)$ and*

$$h^1\big(\mathcal{J}_{Z^{ea}(C)/\mathbb{P}^2}(d)\big) = h^1\big(\mathcal{J}_{Z^{ea}(W)/\mathbb{P}^n}(d)\big) \, .$$

Notice that, in view of the obstructed families given in [DPW00], this also yields that the inequality (15) cannot be improved by adding a constant.

4 Independence of Simultaneous Deformations

The T-smoothness problem for equisingular families is closely related to the independence of simultaneous deformations of isolated singular points of a curve on a surface, or of a hypersurface in a smooth projective algebraic variety, and we present here sufficient conditions for the independence of simultaneous deformations which are analogous to the T-smoothness criteria of Section 3.

4.1 Joint Versal Deformations

Let W be a hypersurface with r isolated singularities of analytic types $S_1, \ldots, S_r$, lying in a smooth projective algebraic variety X. The obstructions to the versality of the joint deformation of the singularities of W induced by the linear system $|W|$ lie in the group $H^1(\mathcal{J}_{Z^{ea}(W)/X}(W))$. In turn, the latter group is the obstruction to the T-smoothness of the germ at W of the ESF of all hypersurfaces $W' \in |W|$ having precisely r singularities of analytic types $S_1, \ldots, S_r$ (see Proposition 1 for the case that X is a surface).

Thus, the aforementioned sufficient conditions for T-smoothness, formulated in Theorems 1, 2, 3, 4, 5 and in Corollaries 1, 3, 4, 6, are also sufficient conditions for the versality of the joint deformations of the singular points:

Theorem 7. *Let W be a hypersurface with only isolated singular points $z_1, \ldots, z_r$ in a smooth projective algebraic variety X of dimension $n \geq 2$. Then the germ at W of the linear system $|W|$ induces a joint versal deformation of all the singularities of W, if one of the following conditions holds:*

(i) $X = \mathbb{P}^2$, *and* $W = C$ *is an irreducible curve of degree* d *such that (2) is satisfied with* $\gamma' = \gamma^{ea}$*; if* C *has* n *nodes and* k *cusps as only singularities, condition (2) can be replaced by (3);*

(ii) $X = \Sigma$ *is a surface with Neron-Severi group* $L \cdot \mathbb{Z}$, L *being ample, and with canonical divisor* $K_\Sigma = k_\Sigma \cdot L$*;* $W = C \sim d \cdot L$ *is an irreducible curve, satisfying* $d \geq \max\{k_\Sigma + 1, -k_\Sigma\}$ *and condition (14) with* $\gamma' = \gamma^{ea}$*;*

(iii) $X = \Sigma$ *is a surface,* $W = C$ *is an irreducible curve such that* C, $C - K_\Sigma$ *are ample and* $C^2 \geq \max\{K_\Sigma^2, A(\Sigma, C)\}$, *and* Σ, C *satisfy conditions (5) and (6) with* $\tau'_{ci} = \tau_{ci}$*;*

(iv) $X = \Sigma$ *is a surface with* $-K_\Sigma$ *nef,* $W = C$ *is an irreducible curve such that* $C \sim dD_0$, D_0 *an ample divisor on* Σ *such that* $D_0^2 \geq A(\Sigma, D_0)$, $d > 0$ *satisfying* $d^2 \geq K_\Sigma^2 / D_0^2$ *and condition (7) with* $\tau'_{ci} = \tau_{ci}$ *(the latter condition reduces to (8) if* C *has* n *nodes and* k *cusps as only singularities);*

(v) $X = \Sigma \subset \mathbb{P}^3$ *is a hypersurface of degree* $e \geq 5$, $W = C$ *is an irreducible curve such that* $C \sim d(e-4)H$, *where* H *is a hyperplane section,* $d \in \mathbb{Q}$, $d > 1$, *such that* $\max_i \tau_{ci}(C, z_i) \leq d - 2$, *and such that the conditions (10), (11) are satisfied with* $\tau'_{ci} = \tau_{ci}$ *(the latter two conditions turn into (13), if* C *has* n *nodes and* k *cusps as only singularities);*

(vi) $X = \mathbb{P}^n$, $n \geq 2$, W *is a reduced hypersurface of degree* d, *and condition (15) is fulfilled.*

4.2 Independence of Lower Deformations

Also the T-smoothness of a topological equisingular family has a deformation theoretic counterpart: the independence of lower deformations of isolated singularities.

Let W be a hypersurface with only isolated singular points $z_1, \ldots, z_r$ in a smooth projective algebraic variety X of dimension $n \geq 2$. For sake of simplicity, we assume that the singular points of W are all semiquasihomogeneous (SQH). That is, for each singular point z_i, there are local analytic coordinates $\boldsymbol{x} = (x_1, \ldots, x_n)$ on X such that the germ (W, z_i) is given by

$$F_i = \sum_{\ell_i(\boldsymbol{\alpha}) \geq a_i} A_{\boldsymbol{\alpha}}^{(i)} \boldsymbol{x}^{\boldsymbol{\alpha}} \in \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1, \ldots, x_n\}\,, \tag{18}$$

where $\ell_i(\boldsymbol{\alpha}) = \sum_{j=1}^n w_j^{(i)} \alpha_j$ for some $\boldsymbol{w}^{(i)} = (w_1^{(i)}, \ldots, w_n^{(i)}) \in (\mathbb{Z}_{>0})^n$, the Newton polygon of F_i is convenient and $F_{i,0} = \sum_{\ell_i(\boldsymbol{\alpha}) = a_i} A_{\boldsymbol{\alpha}}^{(i)} \boldsymbol{x}^{\boldsymbol{\alpha}}$ defines an isolated singularity at the origin. We call F_i a *SQH representative* of (W, z_i) with *principal part* $F_{i,0}$. Note that the class of SQH singularities includes all simple singularities.

We fix SQH representatives $F_1, \ldots, F_r$ for $(W, z_1), \ldots, (W, z_r)$. Then by a *deformation pattern* for (W, z_i) (respectively for F_i), we denote any affine hypersurface of $\mathbb{C}^n$ given by a polynomial

$$G_i = \sum_{0 \leq \ell_i(\boldsymbol{\alpha}) \leq a_i} A_{\boldsymbol{\alpha}}^{(i)} \boldsymbol{x}^{\boldsymbol{\alpha}} = F_{i,0} + \sum_{0 \leq \ell_i(\boldsymbol{\alpha}) < a_i} A_{\boldsymbol{\alpha}}^{(i)} \boldsymbol{x}^{\boldsymbol{\alpha}}$$

and having only isolated singularities. Note that the set of all deformation patterns for (W, z_i) can be identified with the subset

$$\mathcal{P}_i = \left\{ (A^{(i)}_{\boldsymbol{\alpha}})_{0\leq \ell_i(\boldsymbol{\alpha})<a_i} \;\middle|\; \begin{array}{c} A^{(i)}_{\boldsymbol{\alpha}} \in \mathbb{C} \text{ and the hypersurface } \{G_i = 0\} \\ \text{has only isolated singularities in } \mathbb{C}^n \end{array} \right\} \tag{19}$$

of the affine space parametrized by all lower coefficients.

After applying a weighted homothety $(x_1, \ldots, x_n) \mapsto (\lambda^{w_1^{(i)}} x_1, \ldots, \lambda^{w_n^{(i)}} x_n)$ (with $|\lambda|$ sufficiently large) to a given deformation pattern, we can assume that the coefficients $A^{(i)}_{\boldsymbol{\alpha}}$, $\ell_i(\boldsymbol{\alpha}) < a_i$, are sufficiently small such that, for a fixed large closed ball $B_i \subset \mathbb{C}^n$, the intersection $\{G_i = 0\} \cap \partial B_i$ is close to $\{F_{i,0} = 0\} \cap \partial B_i$. For each $i = 1, \ldots, r$, we choose a small closed regular neighbourhood V_i of z_i in X and a C^∞-diffeomorphism

$$\varphi_i : (\partial B_i, \{F_{i,0} = 0\} \cap \partial B_i) \to (\partial V_i, W \cap \partial V_i)$$

which is close to a weighted homothety in the coordinates $\boldsymbol{x}$.

If $p_1, \ldots, p_s$ are the singular points of $\{G_i = 0\}$ in B_i then, for each j, we choose the topological[4] or the analytic equivalence relation, and by the germ of the *equisingularity stratum in* $\mathcal{P}_i$ *at* G_i we mean the germ of the set of all deformation patterns G'_i having singular points $p'_1, \ldots, p'_s$ close to $p_1, \ldots, p_s$ such that (G_i, p_i) and (G'_i, p'_i) have the same type (with respect to the chosen equivalence). We call the deformation pattern defined by G_i *transversal* if the germ at G_i of the equisingularity stratum in the space $\mathcal{P}_i$ is T-smooth.

Given a one-parameter deformation $W_t \in |W|$, $t \in (\mathbb{C}, 0)$, of the hypersurface $W = W_0$ such that $W_t \cap V_i$, $t \neq 0$, is equisingular (with respect to the fixed equivalences), we say that $\{W_t\}$ *matches the deformation pattern* G_i for (W, z_i) if, for $t \neq 0$, there is a homeomorphism $\psi_{i,t}$ of $(B_i, \{G_i = 0\} \cap B_i)$ onto $(V_i, W_t \cap V_i)$. Moreover, at those singular points p_j of $\{G_i = 0\}$ where the analytic equivalence was chosen, we additionally require that $\psi_{i,t}$ induces an analytic isomorphism in a neighbourhood of p_j.

The main result of [Shu99] is

Theorem 8. *Let* W *be a hypersurface in a smooth projective algebraic variety* X *of dimension* $n \geq 2$ *with only isolated semiquasihomogeneous singular points* $z_1, \ldots, z_r$*. If the germ at* W *of the topological ESF in the linear system* $|W|$ *is T-smooth, then, for each tuple of independently prescribed transversal deformation patterns for* $(W, z_1), \ldots, (W, z_r)$*, there exists a one-parameter deformation* $W_t \in |W|$*,* $t \in (\mathbb{C}, 0)$*, of* W *which matches the given patterns.*

Moreover, if all the given data are defined over the reals, then the deformation W_t *and the matching homeomorphisms* $\psi_{i,t}$ *can be chosen over the reals, too.*

[4] When talking about topological types, we always assume $n = 2$.

The proof is constructive and is based on a version of the patchworking method, which is one of the main tools used for finding sufficient conditions for the existence of hypersurfaces with prescribed singularities (see Section 5 below).

Various T-smoothness criteria for topological ESF immediately imply numerical sufficient conditions for the independence of one-parameter deformations matching given deformation patterns. Skipping the linear conditions given in [GK89, GL96, Shu87, Shu91a, Vas90], we collect in the following theorem the outcome of the criteria in Section 3. Notice that the case that W has only simple singularities is already covered by Theorem 7.

Theorem 9. *Let W be a hypersurface in a smooth projective algebraic variety X of dimension $n \geq 2$ with only isolated semiquasihomogeneous singular points $z_1, \ldots, z_r$. Moreover, let $G_1, \ldots, G_r$ define transversal deformation patterns for $(W, z_1), \ldots, (W, z_r)$. Then there exists a one-parameter deformation $W_t \in |W|$, $t \in (\mathbb{C}, 0)$, of W which matches these deformation patterns, if one of the following conditions is satisfied:*

(i) $X = \mathbb{P}^2$ *and* $W = C$ *is an irreducible curve of degree* d *such that (2) holds with* $\gamma' = \gamma^{es}$*; if* C *has only ordinary singularities of order* $m_1, \ldots, m_r$*, condition (2) can be replaced by (4);*

(ii) $X = \Sigma$ *is a surface with Neron-Severi group* $L \cdot \mathbb{Z}$*,* L *being ample, and with canonical divisor* $K_\Sigma = k_\Sigma \cdot L$*;* $W = C \sim d \cdot L$ *is an irreducible curve, satisfying* $d \geq \max\{k_\Sigma + 1, -k_\Sigma\}$ *and condition (14) with* $\gamma' = \gamma^{es}$*;*

(iii) $X = \Sigma$ *is a surface,* $W = C$ *is an irreducible curve such that* C*,* $C - K_\Sigma$ *are ample and* $C^2 \geq \max\{K_\Sigma^2, A(\Sigma, C)\}$*, and* Σ, C *satisfy conditions (5) and (6) with* $\tau'_{ci} = \tau^{es}_{ci}$ *(conditions (5), (6) can be replaced by (9) if* C *has only ordinary singularities of order* $m_1, \ldots, m_r$*);*

(iv) $X = \Sigma$ *is a surface with* $-K_\Sigma$ *nef,* $W = C$ *is an irreducible curve such that* $C \sim dD_0$*,* D_0 *an ample divisor on* Σ *with* $D_0^2 \geq A(\Sigma, D_0)$*,* $d > 0$ *satisfying* $d^2 \geq K_\Sigma^2 / D_0^2$ *and condition (7) with* $\tau'_{ci} = \tau^{es}_{ci}$*;*

(v) $X = \Sigma \subset \mathbb{P}^3$ *is a hypersurface of degree* $e \geq 5$*,* $W = C$ *is an irreducible curve such that* $C \sim d(e-4)H$*, where* H *is a hyperplane section,* $d \in \mathbb{Q}$*,* $d > 1$*, such that* $\max_i \tau^{es}_{ci}(C, z_i) \leq d - 2$*, and such that the conditions (10), (11) are satisfied with* $\tau'_{ci} = \tau^{es}_{ci}$*;*

(vi) $X = \mathbb{P}^n$*,* $n \geq 2$*,* W *is a reduced hypersurface of degree* d*, and condition (17) is fulfilled with* $q = 0$*.*

5 Existence

In the following, we describe two methods which lead to general numerical sufficient conditions for the existence of projective hypersurfaces with prescribed singularities. Both approaches are based on the reduction of the existence problem to an H^1-vanishing problem for the ideal sheaves of certain zero-dimensional schemes associated with topological, respectively analytic, types of singularities. One way is to associate directly a zero-dimensional scheme corresponding to the r prescribed singularity types $S_1, \dots, S_r$ (fixing the position of the singular points) and to produce a sufficient condition by using an appropriate H^1-vanishing criterion. Another way is to construct, first, a projective hypersurface with ordinary singularities (in general position) and then to deform it into a hypersurface with the prescribed singularities using the patchworking construction ([Shu98, Shu05]). The sufficient conditions obtained by each of the two approaches do not cover the conditions obtained by the other approach in general. Hence, both methods are needed.

5.1 ESF of Plane Curves

With a reduced plane curve germ $(C, z) \subset (\mathbb{P}^2, z)$ we associate the following zero-dimensional schemes of $\mathbb{P}^2$ (with support $\{z\}$):

- $Z^s(C, z)$, the *singularity scheme*, defined by the ideal

  $$I^s(C,z) := \left\{ g \in \mathcal{O}_{\mathbb{P}^2,z} \;\middle|\; \operatorname{mt} \widehat{g}_{(q)} \geq \operatorname{mt} \widehat{C}_{(q)} \text{ for each } q \in \mathcal{T}^*(C,z) \right\},$$

  where $\mathcal{T}^*(C, z)$ denotes the tree of essential infinitely near points, and $\widehat{g}_{(q)}$ (resp. $\widehat{C}_{(q)}$) is the total transform of g (resp. of (C, z)) at q (see [GLS98a] or [GLS06] for details);
- $Z^s_{st}(C, z) := Z^s(CL, z)$, where L is a curve which is smooth and transversal to (the tangent cone of) C at z;
- $Z^a(C, z)$, the scheme defined by the ideal $I^a(C, z) \subset \mathcal{O}_{\mathbb{P}^2,z}$ encoding the analytic type (see [GLS00] for a definition);
- $Z^a_{st}(C, z)$, the scheme defined by the ideal $\mathfrak{m}_z I^a(C, z)$, where $\mathfrak{m}_z$ denotes the maximal ideal of $\mathcal{O}_{\mathbb{P}^2,z}$.

H^1-Vanishing Approach. The following proposition allows us to deduce the existence of plane curves with prescribed singularities from an H^1-vanishing statement:

Proposition 3 ([Shu04]). *(1) Given a zero-dimensional scheme $Z \subset \mathbb{P}^2$, a point $z \in \mathbb{P}^2$ outside the support of Z and a reduced curve germ $(C, z) \subset (\mathbb{P}^2, z)$ satisfying*

$$H^1\big(\mathcal{J}_{Z \cup Z^s_{st}(C,z)/\mathbb{P}^2}(d)\big) = 0\,. \tag{20}$$

Then there exists a curve $D \in \left|H^0\big(\mathcal{J}_{Z \cup Z^s(C,z)/\mathbb{P}^2}(d)\big)\right|$ such that the germ of D at z is topologically equivalent to (C, z). Moreover, these curves D form a dense open subset in $\left|H^0\big(\mathcal{J}_{Z \cup Z^s(C,z)/\mathbb{P}^2}(d)\big)\right|$.

(2) In the previous notation, let

$$H^1\big(\mathcal{J}_{Z \cup Z^a_{st}(C,z)/\mathbb{P}^2}(d)\big) = 0\,. \tag{21}$$

Then there exists a curve $D \in \big|H^0\big(\mathcal{J}_{Z\cup Z^a(C,z)/\mathbb{P}^2}(d)\big)\big|$ *such that the germ of* D *at* z *is analytically equivalent to* (C,z)*. These curves* D *form a dense open subset in* $\big|H^0\big(\mathcal{J}_{Z\cup Z^a(C,z)/\mathbb{P}^2}(d)\big)\big|$*.*

Together with the H^1-vanishing theorem for generic zero-dimensional schemes given in [Shu04] (using a Castelnuovo function approach), Proposition 3 yields:

Theorem 10 ([Shu04]). *Let* $(C_1,z_1),\dots,(C_r,z_r)$ *be reduced plane curve germs, let* n *be the number of nodes,* k *the number of cusps and* t *the number of* A_{2m} *singularities,* $m \geq 2$*, among the singularities* (C_i,z_i)*,* $i=1,\dots,r$*.*

(1) If

$$6n + 10k + \frac{49}{6}t + \frac{625}{48} \sum_{(C_i,z_i)\neq A_1,A_2} \delta(C_i,z_i) \;\leq\; d^2 - 2d + 3\,, \tag{22}$$

then there exists a reduced, irreducible plane curve of degree d *having* r *singular points topologically equivalent to* $(C_1,z_1),\dots,(C_r,z_r)$*, respectively, as its only singularities.*

(2) If

$$6n + 10k + \sum_{(C_i,z_i)\neq A_1,A_2} \frac{\big(7\mu(C_i,z_i)+2\delta(C_i,z_i)\big)^2}{6\mu(C_i,z_i)+3\delta(C_i,z_i)} \;\leq\; d^2 - 2d + 3\,, \tag{23}$$

then there exists a reduced, irreducible plane curve of degree d *having* r *singular points analytically equivalent to* $(C_1,z_1),\dots,(C_r,z_r)$*, respectively, as its only singularities.*

See [Shu04] for a slightly stronger result. Note that condition (23) can be weakened to the following simple form (in view of $\delta \leq 3\mu/4$ for reduced plane curve singularities different from nodes):

$$\sum_{i=1}^{r} \mu(C_i,z_i) \leq \frac{1}{9}(d^2 - 2d + 3)\,. \tag{24}$$

Next, we pay special attention to the case of curves with exactly one singular point, because such curves are an essential ingredient for the patchworking approach to the existence problem.

Curves with one Singular Point and Order of T-existence. Let (C,z) be a reduced plane curve singularity. Denote by $e^s(C,z)$, resp. $e^a(C,z)$, the minimal degree d of a plane curve $D \subset \mathbb{P}^2$ whose singular locus consists of a unique point w such that (D,w) is topologically (resp. analytically) equivalent to (C,z) and which satisfies the condition

$$H^1\big(\mathcal{J}_{Z^{es}(D)/\mathbb{P}^2}(d-1)\big) = 0\,, \text{ resp. } H^1\big(\mathcal{J}_{Z^{ea}(D)/\mathbb{P}^2}(d-1)\big) = 0\,. \tag{25}$$

We call $e^s(C,z)$ (resp. $e^a(C,z)$) the *order of T-existence* for the topological (resp. analytic) singularity type represented by (C,z).

Lemma 2. *(1) Let D be a plane curve as in the definition of the order of T-existence, and let L be a straight line which does not pass through the singular point w of D. Then the germ at D of the family of curves of degree d having in a neighbourhood of w a singular point which is topologically (respectively analytically) equivalent to (C,z) is smooth of the expected dimension, and it intersects transversally the linear system*

$$\Big\{G \in \big|H^0\big(\mathcal{O}_{\mathbb{P}^2}(d)\big)\big| \;\Big|\; G \cap L = D \cap L\Big\} .$$

(2) Let $L \subset \mathbb{P}^2$ be a straight line. Then the set of d-tuples $(z_1, \dots, z_d)$ of distinct points on L for which there is a curve D of degree d as in the definition of the order of T-existence satisfying $D \cap L = \{z_1, \dots, z_d\}$ is Zariski open in $\mathrm{Sym}^d(L)$.

Combining Theorem 10 and the existence result for plane curves with simple singularities in [Los99], we get the following estimates for e^s and e^a:

Theorem 11. *If (C,z) is a simple plane curve singularity then*

$$e^s(C,z) = e^a(C,z) \begin{cases} \leq 2\lfloor\sqrt{\mu+5}\rfloor & \text{if } (C,z) \text{ of type } A_\mu,\ \mu \geq 1\,, \\ \leq 2\lfloor\sqrt{\mu+7}\rfloor + 1 & \text{if } (C,z) \text{ of type } D_\mu,\ \mu \geq 4\,, \\ = \lfloor\mu/2\rfloor + 1 & \text{if } (C,z) \text{ of type } E_\mu,\ \mu = 6,7,8. \end{cases}$$

If (C,z) is not simple, then

$$e^s(C,z) \leq \frac{25}{4\sqrt{3}}\sqrt{\delta(C,z)} - 1\,,$$

$$e^a(C,z) \leq \frac{7\mu(C,z) + 2\delta(C,z)}{\sqrt{6\mu(C,z) + 3\delta(C,z)}} - 1 \leq 3\sqrt{\mu(C,z)} - 1\,.$$

For simple singularities with small Milnor number, these estimates are far from being sharp. For instance, it is well-known that

$$e^s(A_\mu) = \begin{cases} 4 & \text{if } 3 \leq \mu \leq 7\,, \\ 5 & \text{if } 8 \leq \mu \leq 13\,, \\ 6 & \text{if } 14 \leq \mu \leq 19\,, \end{cases} \qquad e^s(D_\mu) = \begin{cases} 3 & \text{if } \mu = 4\,, \\ 4 & \text{if } \mu = 5\,, \\ 5 & \text{if } 6 \leq \mu \leq 10\,, \\ 6 & \text{if } 11 \leq \mu \leq 13\,. \end{cases}$$

Moreover, $e^s(E_6) = e^s(E_7) = 4$, $e^s(E_8) = 5$.

Curves with many Singular Points (Patchworking Approach). The following proposition is a special case of Proposition 5 below, which is proved by a reasoning based on patchworking:

Proposition 4. *Let $(C_1,z_1),\dots,(C_r,z_r)$ be reduced plane curve singularities. Let $m_i := e^s(C_i,z_i)$ (resp. $m_i := e^a(C_i,z_i)$), and assume that*

$$H^1\big(\mathcal{J}_{Z(\boldsymbol{m})/\mathbb{P}^2}(d-1)\big) = 0\,, \tag{26}$$

where $Z(\boldsymbol{m})$ is the fat point scheme supported at the (generic) points $p_1,\dots,p_r$ of $\mathbb{P}^2$ and defined by the ideals $\mathfrak{m}_{p_i}^{m_i}$. Moreover, let

$$d > \max_{1\le i\le r} e^s(C_i,z_i)\,, \quad \big(\textit{resp. } d > \max_{1\le i\le r} e^a(C_i,z_i)\big)\,.$$

Then there exists a reduced, irreducible plane curve $D\subset\mathbb{P}^2$ of degree d with $\mathrm{Sing}(D)=\{p_1,\dots,p_r\}$ *such that each germ (D,p_i) is topologically (resp. analytically) equivalent to (C_i,z_i), $i=1,\dots,r$.*

Applying the H^1-vanishing criterion of [Xu95, Thm. 3], we immediately derive

Corollary 7. *Let $(C_1,z_1),\dots,(C_r,z_r)$ be reduced plane curve singularities such that $e^s(C_1,z_1)\ge\ldots\ge e^s(C_r,z_r)$. If*

$$\begin{aligned} e^s(C_1,z_1)+e^s(C_2,z_2) &\le d-1\,, \quad \textit{as } r\ge 2\,,\\ e^s(C_1,z_1)+\ldots+e^s(C_5,z_5) &\le 2d-2\,, \quad \textit{as } r\ge 5\,,\\ \sum_{i=1}^{r}(e^s(C_i,z_i)+1)^2 &< \frac{9}{10}(d+2)^2\,, \end{aligned}$$

then there exists a reduced, irreducible plane curve C of degree d with exactly r singular points $p_1,\dots,p_r$, such that each germ (C,p_i) is topologically equivalent to (C_i,z_i), $i=1,\dots,r$.

The same statement holds true if we replace e^s by e^a and the topological equivalence relation by the analytic one.

Comparing the sufficient conditions of Theorem 10 and of Corollary 7, we see that the existence criterion of Theorem 10 is better for non-simple singularities, whereas the criterion obtained from Corollary 7 and the estimates in Theorem 11 is better for simple singularities.

5.2 ESF of Curves on Smooth Projective Surfaces

In [KT02], the following sufficient criterion for the existence of curves with prescribed singularities is proved:

Proposition 5 ([KT02]). *Let Σ be a smooth projective algebraic surface, D a divisor on Σ, and $L\subset\Sigma$ a very ample divisor. Let $(C_1,z_1),\dots,(C_r,z_r)$ be reduced plane curve singularities. Let*

$$H^1\big(\mathcal{J}_{Z(\boldsymbol{m})/\Sigma}(D-L)\big) = 0\ , \tag{27}$$

$$\max_{1\le i\le r} m_i < L(D-L-K_\Sigma)-1\ , \tag{28}$$

where $Z(\boldsymbol{m}) \subset \Sigma$ *is the fat point scheme supported at some (generic) points* $p_1, \ldots, p_r \in \Sigma$ *and defined by the ideals* $\mathfrak{m}_{p_i}^{m_i}$, *with* $m_i = e^s(C_i, z_i)$, *respectively* $m_i = e^a(C_i, z_i)$, $i = 1, \ldots, r$. *Then there exists an irreducible curve* $C \in |D|$ *such that* $\operatorname{Sing}(C) = \{p_1, \ldots, p_r\}$ *and each germ* (C, p_i) *is topologically, resp. analytically, equivalent to* (C_i, z_i), $i = 1, \ldots, r$.

Combining this with the H^1-vanishing criterion of [KT02, Cor. 4.2] and with the estimates for e^s, e^a in Theorem 11, we get the following explicit numerical existence criterion:

Theorem 12 ([GLS06a]). *Let* Σ *be a smooth projective algebraic surface,* D *a divisor on* Σ *with* $D - K_\Sigma$ *nef, and* $L \subset \Sigma$ *a very ample divisor. Let* $(C_1, z_1), \ldots, (C_r, z_r)$ *be reduced plane curve singularities, among them* n *nodes and* k *cusps.*

(1) If

$$18n + 32k + \frac{625}{24} \sum_{\delta(C_i,z_i)>1} \delta(C_i, z_i) \;\le\; (D - K_\Sigma - L)^2 \,, \tag{29}$$

$$\frac{25}{4\sqrt{3}} \max_{1\le i\le r} \sqrt{\delta(C_i, z_i)} + 1 \;<\; (D - L - K_\Sigma).L \,, \tag{30}$$

and, for each irreducible curve B *with* $B^2 = 0$ *and* $\dim |B|_a > 0$,

$$\frac{25}{4\sqrt{3}} \max_{1\le i\le r} \sqrt{\delta(C_i, z_i)} \;<\; (D - K_\Sigma - L).B + 1 \,, \tag{31}$$

then there exists a reduced, irreducible curve $C \in |D|$ *with* r *singular points topologically equivalent to* $(C_1, z_1), \ldots, (C_r, z_r)$, *respectively, as its only singularities.*

(2) If

$$18n + 32k + 18 \sum_{\mu(C_i,z_i)>2} \mu(C_i, z_i) \;\le\; (D - K_\Sigma - L)^2 \,, \tag{32}$$

$$3 \max_{1\le i\le r} \sqrt{\mu(C_i, z_i)} + 1 \;<\; (D - L - K_\Sigma).L \,, \tag{33}$$

and, for each irreducible curve B *with* $B^2 = 0$ *and* $\dim |B|_a > 0$,

$$3 \max_{1\le i\le r} \sqrt{\mu(C_i, z_i)} \;<\; (D - K_\Sigma - L).B + 1 \,, \tag{34}$$

then there exists a reduced, irreducible curve $C \in |D|$ *with* r *singular points analytically equivalent to* $(C_1, z_1), \ldots, (C_r, z_r)$, *respectively, as its only singularities.*

Here $|B|_a$ means the family of curves algebraically equivalent to B. A discussion of the hypotheses of Theorem 12 for specific classes of surfaces as well as concrete examples can be found in [KT02].

5.3 ESF of Hypersurfaces in $\mathbf{P}^n$

For singular hypersurfaces in $\mathbb{P}^n$, $n \geq 3$, no general asymptotically proper sufficient condition for the existence of hypersurfaces with prescribed singularities (such as (24) in the case of plane curves) is known. But, restricting ourselves to the case of only simple singularities, in [SW04] even an asymptotically optimal condition is given.

To formulate this result, we need some notation: let $\mathcal{S}$ be a finite set of analytic types of isolated hypersurface singularities in $\mathbb{P}^n$. Define

$$\alpha_n(\mathcal{S}) = \lim_{d\to\infty} \sup_{W_d} \frac{\tau(W_d)}{d^n} ,$$

where W_d runs over the set of all hypersurfaces $W_d \subset \mathbb{P}^n$ of degree d whose singularities are of types $S \in \mathcal{S}$ and which belong to the T-smooth component of the corresponding ESF. Here, $\tau(W_d)$ stands for the sum of the Tjurina numbers $\tau(W_d, z)$ over all points $z \in \mathrm{Sing}(W_d)$. By $\alpha_n^{\mathbb{R}}(\mathcal{S})$ we denote the respective limit taken over hypersurfaces having only real singular points of real singularity types $S \in \mathcal{S}$. Clearly, $\alpha_n^{\mathbb{R}}(\mathcal{S}) \leq \alpha_n(\mathcal{S}) \leq 1/n!$.

Theorem 13 ([SW04, Wes03, Wes04]). *Let $n \geq 2$.*

(1) For each finite set $\mathcal{S}$ of simple hypersurface singularities in $\mathbb{P}^n$, we have

$$\alpha_n^{\mathbb{R}}(\mathcal{S}) = \alpha_n(\mathcal{S}) = \frac{1}{n!} .$$

(2) For each finite set $\mathcal{S}$ of analytic types of isolated hypersurface singularities of corank 2 in $\mathbb{P}^n$, we have

$$\alpha_n(\mathcal{S}) \geq \alpha_n^{\mathbb{R}}(\mathcal{S}) \geq \frac{1}{9n!} .$$

The proof exploits again the patchworking construction. It is based on the following fact: for each simple singularity type S and each $n \geq 2$, there exist an n-dimensional convex lattice polytope $\Delta_n(S)$ of volume $\mu(S)/n!$ and a polynomial $F \in \mathbb{C}[x_1, \ldots, x_n]$ (resp. $F \in \mathbb{R}[x_1, \ldots, x_n]$) with Newton polytope $\Delta_n(S)$ which defines a hypersurface in the toric variety $\mathrm{Tor}(\Delta_n(S))$ (associated to $\Delta_n(S)$) having precisely one singular point of type S in the torus $(\mathbb{C}^*)^n$ (resp. in the torus $(\mathbb{R}^*)^n$) and being non-singular and transverse along the toric divisors in $\mathrm{Tor}(\Delta_n(S))$.

6 Irreducibility

The question about the irreducibility of ESF $V_{|C|}(S_1, \ldots, S_r)$ is more delicate than the existence and smoothness problem, in particular, if one tries to find sufficient conditions for the irreducibility. The results are by far not that complete as for the other two problems. The irreducibility problem is of special topological interest, since it is connected with the problem of having within the same ESF different fundamental groups of the complement of a plane algebraic curve.

As pointed out in the introduction, even the case of plane nodal curves (Severi's conjecture) appeared to be very hard. The examples of reducible ESF listed below indicate that for more complicated singularities, beginning with cusps, possible numerical sufficient conditions for the irreducibility should be rather different with respect to their asymptotics to the necessary existence conditions (as discussed in Section 5).

Approaches to the Irreducibility Problem. (1) One possible approach (for ESF of plane curves) consists of building for any two curves in the ESF a connecting path, using explicit equations of the curves, respectively of projective transformations. This method works for small degrees only. Besides the classical case of conics and cubics, this method has been used to prove that all ESF of quartic and quintic curves are irreducible (cf. [BG81, Wal96]). But for degrees $d > 5$, this is no longer true and the method is no more efficient (except for some very special cases).

(2) Arbarello and Cornalba [AC83] suggested another approach. It consists of relating the ESF to the moduli space of plane curves of given genus, which is known to be irreducible (cf. [DM69]). This gave some particular results on families of plane nodal curves and plane curves with nodes and cusps. Namely, Kang [Kan89] proved that the variety $V_d^{irr}(n\cdot A_1, k\cdot A_2)$ is irreducible whenever

$$\frac{d^2-4d+1}{2} \le n \le \frac{(d-1)(d-2)}{2}, \qquad k \le \frac{d+1}{2}. \tag{35}$$

(3) Harris introduced a new idea to the irreducibility problem, which completed the case of plane nodal curves ([Har85a]). This new idea was to proceed inductively from rational plane nodal curves (whose family is classically known to be irreducible) to any family of plane nodal curves of a given genus. Further development of this idea lead to new results by Ran [Ran89] and by Kang [Kan89a]: if O_m denotes an ordinary singularity of order $m \ge 2$, then Ran showed that, for each $n \ge 0$, the variety $V_d^{irr}(n\cdot A_1, 1\cdot O_m)$ is irreducible (or empty). Kang's result says that, for each $n \ge 0$, $k \le 3$, the variety $V_d^{irr}(n\cdot A_1, k\cdot A_2)$ is irreducible (if non-empty).

However, the requirement to study all possible deformations of the considered curves does not allow to extend such an approach to more complicated singularities, or to a large number of singularities different from nodes.

(4) Up to now, there is mainly one approach which is applicable to equisingular families of curves of any degree with any quantity of arbitrary singularities (and even to projective hypersurfaces of any dimension). The basic idea is to find an irreducible analytic space $\mathcal{M}(S_1,\dots,S_r)$ and a dominant morphism

$$V_{|D|}(S_1,\dots,S_r) \longrightarrow \mathcal{M}(S_1,\dots,S_r)$$

with equidimensional and irreducible fibres. It turns out, that in such a way proving the irreducibility of $V_{|D|}(S_1,\dots,S_r)$ can be reduced to an H^1-vanishing problem: let $Z(C)=Z^s(C)$ (resp. $Z(C)=Z^a(C)$) be the zero-dimensional schemes encoding the topological (resp. analytic) type of the singularities (see Section 5). Then the variety $V_{|D|}(S_1,\dots,S_r)$ is irreducible if $H^1\bigl(\mathcal{J}_{Z(C)/\Sigma}(D)\bigr)=0$ for each $C\in V_{|D|}(S_1,\dots,S_r)$.

For a detailed discussion of the latter approach, we refer to [GLS00, Kei03]. Combining this approach with [Xu95, Thm. 3] and another H^1-vanishing theorem based on the Castelnuovo function approach, we obtain:

Theorem 14 ([GLS00]). *Let $S_1,\dots,S_r$ be topological or analytic types of plane curve singularities, and d an integer. If* $\max_{i=1..r}\tau'(S_i)\le(2/5)d-1$ *and*

$$\frac{25}{2}\cdot\#(nodes)+18\cdot\#(cusps)+\frac{10}{9}\cdot\sum_{\tau'(S_i)\ge 3}\bigl(\tau'(S_i)+2\bigr)^2 \;<\; d^2,$$

then $V_d^{irr}(S_1,\dots,S_r)$ is non-empty and irreducible. Here, $\tau'(S_i)$ stands for $\tau^{es}(S_i)$ if S_i is a topological type and for $\tau(S_i)$ if S_i is an analytic type.

In particular,

Corollary 8. *Let $d\ge 8$. Then $V_d^{irr}(n\cdot A_1,k\cdot A_2)$ is irreducible if*

$$\frac{25}{2}n+18k \;<\; d^2. \tag{36}$$

Corollary 9. *Let $S_1,\dots,S_r$ be ordinary singularities of order $m_1,\dots,m_r$, and assume that* $\max m_i\le(2/5)\,d$. *Then $V_d^{irr}(S_1,\dots,S_r)$ is non-empty and irreducible if*

$$\frac{25}{2}\cdot\#(nodes)+\sum_{m_i\ge 3}\frac{m_i^2(m_i+1)^2}{4} \;<\; d^2. \tag{37}$$

Reducible Equisingular Families. In [GLS00], we apparently gave the first series of reducible ESF of plane cuspidal curves, where the different components cannot be distinguished by the fundamental group of the complement of the corresponding curves. If this happens, we say that the ESF has components which are *anti-Zariski pairs*.

The following proposition gives infinitely many ESFs with anti-Zariski pairs:

Proposition 6 ([GLS00]). *Let p,d be integers satisfying*

$$p\ge 15,\quad 6p\;<\;d\;\le\;12p-\tfrac{3}{2}-\sqrt{35p^2-15p+\tfrac{1}{4}}\;. \tag{38}$$

Then the variety $V_d^{irr}(6p^2 \cdot A_2)$ *of irreducible plane curves of degree* d *with* $6p^2$ *cusps has components of different dimensions.*

Moreover, for all curves $C \in V_d^{irr}(6p^2 \cdot A_2)$ *the fundamental group of the complement is* $\pi_1(\mathbb{P}^2 \setminus C) = \mathbb{Z}/d\mathbb{Z}$.

For instance, the variety $V_{91}^{irr}(1350 \cdot A_2)$ is reducible, and it has components which are anti-Zariski pairs.

Further, in [GLS01], we gave a series of reducible ESF of plane curves with only ordinary singularities having components which are anti-Zariski pairs:

Proposition 7 ([GLS01]). *Let* $m \geq 9$. *Then there is an integer* $\ell_0 = \ell_0(m)$ *such that for each* $\ell \geq \max\{\ell_0, m\}$ *and for each* s *satisfying*

$$\frac{\ell-1}{2} \;\leq\; s \;\leq\; \ell\left(1-\sqrt{\frac{2}{m}}\right) - \frac{3}{2}$$

the variety $V_{\ell m+s}^{irr}(\ell^2 \cdot O_m)$ *of plane irreducible curves of degree* $\ell m+s$ *having* ℓ^2 *ordinary singularities of order* m *as only singularities is reducible.*

More precisely, $V_{\ell m+s}^{irr}(\ell^2 \cdot O_m)$ *has at least two components, one regular component (of the expected dimension) and one component of higher dimension. And, for each curve* C *belonging to any of the components, we have* $\pi_1(\mathbb{P}^2 \setminus C) = \mathbb{Z}/(\ell m+s)\mathbb{Z}$.

7 Open Problems and Conjectures

Though some results discussed above are sharp, others seem to be far from a final form, and here we start with a discussion and conjectures about the expected progress in the geometry of families of singular curves. Further discussion concerns possible generalizations of the methods and open questions.

7.1 ESF of Curves

Existence of Curves with Prescribed Singularities. A natural question about the existence results for algebraic curves given in Section 3 concerns a possible improvement of the asymptotically proper conditions to asymptotically optimal ones:

How to improve the constant coefficients in the general sufficient conditions for the existence ?

Concerning our method based on H^1-vanishing for the ideal sheaves, a desired improvement would come from finding better H^1-vanishing conditions for generic zero-dimensional schemes. For instance, from proving the Harbourne-Hirschowitz conjecture, which gives (if true) the best possible H^1-vanishing criterion for ideal sheaves of generic fat schemes $Z(\boldsymbol{m})$.

Another type of questions concerning curves with specific singularities is the following: the known sufficient and necessary conditions for the existence of singular plane curves are formulated as bounds to sums of singularity invariants. But it seems that there cannot be a general condition of this type which is sufficient *and* necessary at the same time. The simplest question of such kind is:

Are there k and d such that a curve of degree d with k cusps does exist, but with $k' < k$ cusps does not ?

A candidate could be Hirano's series of cuspidal curves mentioned in the introduction.

T-Smoothness and Versality of Deformations. The following conjecture about the asymptotic properness of the sufficient conditions for the T-smoothness of topological ESF of plane curves given in Section 3 seems to be quite realistic (and holds for semiquasihomogeneous singularities):

Conjecture 1. There exists an absolute constant $A > 0$ such that for each topological singularity type S there are infinitely many pairs $(r, d) \in \mathbb{N}^2$ such that $V_d^{irr}(r \cdot S)$ is empty or non-smooth or has dimension greater than the expected one and $r \cdot \gamma(S) \ \leq \ A \cdot d^2$.

We propose a similar conjecture for analytic ESF of plane curves, though it is confirmed only for simple singularities (in which case it coincides with the conjecture for topological ESF).

A closely related question, belonging to local singularity theory, concerns the γ-invariant:

Find an explicit formula, or an algorithm to compute $\gamma^{es}(f)$, $\gamma^{ea}(f)$. Find (asymptotically) close lower and upper bounds for these invariants. Is γ^{es} a topological invariant ?

Irreducibility Problem. Our sufficient irreducibility conditions seem to be far from optimal ones. We state the problem:

Find asymptotically proper sufficient conditions for the irreducibility of ESF of plane curves (or show that the conditions in Section 6 are asymptotically proper).

We also rise the following important question:

Does there exist a pair of plane irreducible algebraic curves of the same degree with the same collection of singularities, which belong to different components of an ESF but are topologically isotopic in $\mathbb{P}^2$ (anti-Zariski pair) ?

The examples in Section 6 provide candidates for this – reducible ESF, whose members have the same (Abelian) fundamental group of the complement.

7.2 Hypersurfaces in Higher-Dimensional Varieties

One can formulate the existence, T-smoothness, and irreducibility problems for families of hypersurfaces with isolated singularities, belonging to (very) ample linear systems on projective algebraic varieties. To find a relevant approach to these problems is the most important question. For hypersurfaces of dimension > 1 there exists no infinitesimal deformation theory for topological types. So, we restrict ourselves to analytic types here.

Constructions of curves with prescribed singularities as presented in Section 5 can, in principle, be generalized to higher dimensions. An expected analogue of the results for curves could be

Conjecture 2. Given a very ample linear system $|W|$ on a projective algebraic variety X of dimension n, there exists a constant $A = A(X, W) > 0$ such that, for each collection $S_1, \ldots, S_r$ of singularity types and for each positive integer d satisfying $\sum_{i=1}^{r} \mu(S_i) < Ad^n$, there is a hypersurface $W_d \in |dW|$ with exactly r isolated singularities of types $S_1, \ldots, S_r$, respectively.

In view of the patchworking approach, to prove the conjecture, it is actually enough to consider the case of ordinary singularities and to answer the following analogue of one of the above questions affirmatively:

Does there exist some number $A(n) > 0$ such that, for each analytic type S of isolated hypersurface singularities in $\mathbb{P}^n$, there exists a hypersurface of $\mathbb{P}^n$ of degree $d \leq A(n) \cdot \mu(S)^{1/n}$ which has a singularity of type S and no non-isolated singularities ?

This is known only for simple singularities (see [Wes03, Shu04]).

Hypersurfaces with specific singularities (such as nodes) attract the attention of many researches, mainly looking for the maximal possible number of singularities (see, for instance, [Chm92]). We would like to raise the question about an analogue of the Chiantini-Ciliberto theorem for nodal curves on surfaces (see [CC99]) as a natural counterpart, concerning the domain with regular behaviour of ESF:

Given a projective algebraic variety X and a very ample linear system $|W|$ on it with a non-singular generic member. Prove that, for any $r \leq \dim|W|$ there exists a hypersurface $W_r \in |W|$ with r nodes as its only singularities such that the germ at W of the corresponding ESF is T-smooth.

7.3 Related Problems

Enumerative Problems. Recently, the newly founded theories of moduli spaces of stable curves and maps, Gromov-Witten invariants, quantum cohomology, as well as deeply developed methods of classical algebraic geometry and algebraic topology have led to a remarkable progress in enumerative geometry, notably for the enumeration of singular algebraic curves (see, for example, [KM94, CH98, CH98a, GP98] for the enumeration of rational nodal curves on rational surfaces, see [Ran89a, CH98b] for the enumeration of plane nodal curves of any genus, see [Kaz03, Liu00] for counting curves with arbitrary singularities). We point out that the questions to which this survey has been devoted, such as on the existence of certain singular algebraic curves, on the expected dimension and on the transversality of the intersection of ESF, are unavoidable in all of the above approaches to enumerative geometry. The affirmative answers to such questions are necessary for attributing an enumerative meaning to the computations in the aforementioned works.

We pose the problem to find links between the methods discussed above and the methods of enumerative geometry, and we expect that this would lead to a solution for new enumerative problems and to a better understanding of known results. As an example, we mention the tropical enumerative geometry [Mik03, Mik05, Shu05], in which the patchworking construction and, more generally, the deformation theory play an important role.

Non-Isolated Singularities. None of the problems discussed above is even well-stated for non-reduced curves, or hypersurfaces with non-isolated singularities. We simply mention this as a direction for further study.

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