SOME REMARKS ON THE PLANAR KOUCHNIRENKO'S THEOREM

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ABSTRACT. In this note we consider different notions of non-degeneracy, as introduced by Kouchnirenko, Wall and Beelen-Pellikaan for plane curve singularities. It is known that non-degenerate singularities satisfy $\mu = \mu_N$ resp. $\delta = \delta_N$ where μ is the Milnor number, δ the delta-invariant and μ_N resp. δ_N integers depending only on the Newton diagram. We show that $\mu = \mu_N$ resp. $\delta = \delta_N$ is equivalent to inner resp. weak Newton non-degeneracy and give some applications and interesting examples related to the existence of "wild vanishing cycles". Although the results are new in any characteristic, the main difficulties arise in positive characteristic.

1. INTRODUCTION

Let K be an algebraically closed field, $K[[x]] = K[[x_1, \ldots, x_n]]$ the formal power series ring and $\mathfrak{m}$ its maximal ideal. Let us recall the definition of the Newton diagram and Wall's notion of a C-polytope (see [Wal99]). To each power series $f = \sum_\alpha c_\alpha x^\alpha \in K[[x]]$ we can associate its *Newton polyhedron* $\Gamma_+(f)$ as the convex hull of the set

$$\bigcup_{\alpha \in \mathrm{supp}(f)} (\alpha + \mathbb{R}^n_{\geq 0}).$$

where $\mathrm{supp}(f) = \{\alpha | c_\alpha \neq 0\}$ denotes the support of f. This is an unbounded polytope in $\mathbb{R}^n$. We call the union $\Gamma(f)$ of its compact faces the *Newton diagram* of f. By $\Gamma_-(f)$ we denote the union of all line segments joining the origin to a point on $\Gamma(f)$. We always assume that $f \in \mathfrak{m}$ if not explicitly stated otherwise.

If the Newton diagram of a singularity f meets all coordinate axes we call f *convenient.* However, not every isolated singularity is convenient, and one then has to enlarge the Newton diagram. A compact rational polytope P of dimension $n-1$ in the positive orthant $\mathbb{R}^n_{\geq 0}$ is called a *C-polytope* if the region above P is convex and if every ray in the positive orthant emanating from the origin meets P in exactly one point. The Newton diagram of f is a C-polytope iff f is convenient.

We first introduce the different notions of non-degeneracy. For this let $f = \sum_\alpha c_\alpha x^\alpha \in \mathfrak{m}$ be a power series, let P be a C-polytope and let Δ be a face of P.

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By $f_\Delta := \text{in}_\Delta(f) := \sum_{\alpha\in\Delta} c_\alpha x^\alpha$ we denote the initial form or principal part of f along Δ. Following Kouchnirenko we call f *non-degenerate* ND *along* Δ if the Jacobian ideal[1] $\mathrm{j}(f_\Delta)$ has no zero in the torus $(K^*)^n$. f is then said to be *Newton non-degenerate* NND if f is non-degenerate along each face (of any dimension) of the Newton diagram $\Gamma(f)$. We do not require f to be convenient.

To define inner non-degeneracy we need to fix two more notions. The face Δ is an *inner face* of P if it is not contained in any coordinate hyperplane. Each point $q \in K^n$ determines a coordinate hyperspace $H_q = \bigcap_{q_i=0}\{x_i = 0\} \subset \mathbb{R}^n$ in $\mathbb{R}^n$. We call f *inner non-degenerate* IND *along* Δ if for each zero q of the Jacobian ideal $\mathrm{j}(\text{in}_\Delta(f))$ the polytope Δ contains no point on H_q. f is called *inner Newton non-degenerate* INND *w.r.t. a C-polytope* P if no point of $\text{supp}(f)$ lies below P and f is IND along each inner face of P. We call f simply *inner Newton non-degenerate* INND if it is INND w.r.t some C-polytope.

Finally, we call f *weakly non-degenerate* WND *along* Δ if the Tjurina ideal[2] $tj(\text{in}_\Delta(f))$ has no zero in the torus $(K^*)^n$, and f is called *weakly Newton non-degenerate* WNND if f is weakly non-degenerate along each top-dimensional face of $\Gamma(f)$. Note that NND implies WNND while NND does not imply INND and vice versa. See [BGM10, Remark 3.1] for facts on and relations between the different types of non-degeneracy.

For any compact polytope Q in $\mathbb{R}^n_{\geq 0}$ we denote by $V_k(Q)$ the sum of the k-dimensional Euclidean volumes of the intersections of Q with the k-dimensional coordinate subspaces of $\mathbb{R}^n$ and, following Kouchnirenko, we then call

$$\mu_N(Q) = \sum_{k=0}^{n} (-1)^{n-k} k! V_k(Q)$$

the Newton number of Q. For a power series $f \in K[[x]]$ we define the *Newton number* of f to be

$$\mu_N(f) = \sup\{\mu_N(\Gamma_-(f_m)) | f_m := f + x_1^m + \ldots + x_n^m, m \geq 1\}.$$

If f is convenient then

$$\mu_N(f) = \mu_N(\Gamma_-(f)).$$

The following theorem was proved by Kouchnirenko in arbitrary characteristic. We recall that $\mu(f) := \dim K[[x,y]]/j(f)$ is the *Milnor number* of f.

Theorem 1.1. [Kou76] *For $f \in K[[x]]$ we have $\mu_N(f) \leq \mu(f)$, and if f is NND and convenient then $\mu_N(f) = \mu(f) < \infty$.*

[1]The Jacobian ideal $j(f)$ denotes the ideal generated by all partials of $f \in K[[x]]$.

[2]For $f \in K[[x]]$ we call $tj(f) = \langle f \rangle + j(f)$ the Tjurina ideal of f.

Since Theorem 1.1 does not cover all semi-quasihomogeneous singularities, Wall introduced the condition INND (denoted by NPND* in [Wal99]). Using Theorem 1.1, Wall proved the following theorem for $K = \mathbb{C}$ which was extended to arbitrary K in [BGM10].

Theorem 1.2. [Wal99], [BGM10] *If $f \in K[[x]]$ is INND, then*

$$\mu(f) = \mu_N(f) = \mu_N(\Gamma_-(f)) < \infty.$$

Kouchnirenko proved that the condition "convenient" is not necessary in Theorem 1.1 if $char(K) = 0$. The authors in [BGM10] show that in the planar case Kouchnirenko's result holds in arbitrary characteristic without the assumption that f is convenient (allowing $\mu(f) = \infty$):

Proposition 1.3. [BGM10, Proposition 4.5] *Suppose that $f \in K[[x, y]]$ is NND, then $\mu_N(f) = \mu(f)$.*

2. Milnor number

In the following we consider only the case of plane curve singularities. The main result of this section says that for $f \in K[[x, y]]$, the condition $\mu(f) = \mu_N(f) < \infty$ is equivalent to f being INND (Theorem 2.13). In characteristic zero this is also equivalent to f being NND and $\mu_N(f) < \infty$ (Corollary 2.17). However, in positive characteristic, this is in general not true as the following example shows.

Example 2.1. $f = x^3 + xy + y^3$ in characteristic 3 satisfies $\mu(f) = \mu_N(f) = 1$ but f is not NND.

Remark 2.2. Let $f \in K[[x, y]]$ be convenient and $A_i = (c_i, e_i), i = 0, \dots, k$ the vertices of $\Gamma(f)$ with $c_0 = e_k = 0, c_i < c_{i+1}$ and $e_i > e_{i+1}$. Then

$$\mu_N(f) = 2V_2(\Gamma_-(f)) - c_k - e_0 + 1.$$

Lemma 2.3. *Let $f, g \in K[[x, y]]$ be convenient such that $\Gamma_-(f) \subset \Gamma_-(g)$. Then*

(a) $\mu_N(f) \leq \mu_N(g)$.
(b) *The equality holds if and only if $\Gamma_-(f) \cap \mathbb{R}^2_{\geq 1} = \Gamma_-(g) \cap \mathbb{R}^2_{\geq 1}$, where*

$$\mathbb{R}^2_{\geq 1} = \{(x, y) \in \mathbb{R}^2 | x \geq 1, y \geq 1\}.$$

Part (a) of the lemma was also shown in [Biv09, Coro. 5.6]. Let us denote by $\Gamma_1(f)$ the cone joining the origin with $\Gamma(f) \cap \mathbb{R}^2_{\geq 1}$. (cf. Fig. 1).

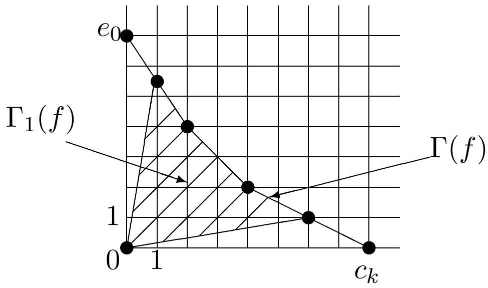

Fig. 1.

Proof. First, we prove that

$$\mu_N(f) = V_2(\Gamma_1(f)) + 1.$$

It is easy to see that $\Gamma_1(f)$ divides $\Gamma_-(f)$ into three parts whose volumes are $c_k/2, V_2(\Gamma_1(f))$ and $e_0/2$. Therefore

$$\mu_N(f) = 2V_2(\Gamma_-(f)) - c_k - e_0 + 1 = 2V_2(\Gamma_1(f)) + 1.$$

(a) Clearly, if $\Gamma_-(f) \subset \Gamma_-(g)$ then $\Gamma_1(f) \subset \Gamma_1(g)$ and hence

$$\mu_N(f) = V_2(\Gamma_1(f)) + 1 \leq 2V_2(\Gamma_1(g)) + 1 = \mu_N(g).$$

(b) follows easily from the formula $\mu_N(f) = 2V_2(\Gamma_1(f)) + 1$. □

We recall some classical notions. Let $f \in K[[x, y]]$ be irreducible. A couple $(x(t), y(t)) \in K[[t]]^2$ is called a *(primitive) parametrization* of f, if $f(x(t), y(t)) = 0$ and if the following universal factorization property holds: for each $(u(t), v(t)) \in K[[t]]^2$ with $f(u(t), v(t)) = 0$, there exists a unique series $h(t) \in K[[t]]$ such that $u(t) = x(h(t))$ and $v(t) = y(h(t))$.

If $g \in K[[x, y]]$ is irreducible and $(x(t), y(t))$ its parametrization, then the *intersection multiplicity* of any $f \in K[[x, y]]$ with g is given by $i(f, g) = \text{ord} f(x(t), y(t))$, and if u is a unit then $i(f, u) = 0$. The *intersection multiplicity* of f with a reducible power series $g = g_1 \cdot \ldots \cdot g_s$ is defined to be the sum $i(f, g) = i(f, g_1) + \ldots + i(f, g_s)$.

Proposition 2.4. [GLS06, Pro. 3.12] *Let $f, g \in K[[x, y]]$. Then*

$$i(f, g) = i(g, f) = \dim K[[x, y]]/\langle f, g\rangle.$$

The proof in [GLS06] was given for $K = \mathbb{C}$ but works in any characteristic.

Proposition 2.5. [Cam80, Lemma 3.4.3]
Let $f \in \mathfrak{m} \subset K[[x, y]]$ be irreducible and x-general of order m. Let $(x(t), y(t))$ be parametrization of f. Then $\text{ord}(y(t)) = m$.

Proof. Since f is x-general of order m, $f(x,0) = c \cdot x^m +$ terms of higher order. Then $i(f,y) = m$. Hence by definition and Proposition 2.4 we have

$$\mathrm{ord}(y(t)) = i(y,f) = i(f,y) = m.$$

□

Let $f = \sum_{i,j} c_{ij} x^i y^j \in K[[x,y]]$ and $\Gamma(f)$ be its Newton diagram. We call

$$f_{in} := \sum_{(i,j)\in\Gamma(f)} c_{ij} x^i y^j$$

the *initial part* of f.

Proposition 2.6. [BrK86, Lemma 3] *Let $f \in K[[x,y]]$ and let $E_i, i = 1,\dots,k$ be the edges of the Newton diagram of f. Then there is a factorization of f:*

$$f = monomial \cdot \bar{f}_1 \cdot \ldots \cdot \bar{f}_k$$

such that $\bar{f}_i$ is convenient, $f_{E_i} = monomial \times (\bar{f}_i)_{in}$ and

$$f_{in} = monomial \cdot (\bar{f}_1)_{in} \cdot \ldots \cdot (\bar{f}_k)_{in}.$$

In particular, if f is convenient then $f = \bar{f}_1 \cdot \ldots \cdot \bar{f}_k$.

A polynomial $f = \sum_{i,j} c_{ij} x^i y^j \in K[x,y]$ is called *weighted homogeneous* or *quasihomogeneous* of type $(n,m;d)$ if m,n,d are positive integers satisfying $ni + mj = d$, for each $(i,j) \in \mathrm{supp}(f)$.

Let $f \in K[[x,y]]$ be a formal power series and n,m positive integers. We can decompose f into a sum

$$f = f_d + f_{d+1} + \ldots,$$

where $f_d \neq 0$ and f_l is weighted homogeneous of type $(n,m;l)$ for $l \geq d$.

For each series $\varphi(t) = c_1 t^{\alpha_1} + c_2 t^{\alpha_1} + \ldots$ with $c_1 \neq 0, \alpha_1 < \alpha_2 < \ldots$, we set

$$LT(\varphi(t)) := c_1 t^{\alpha_1} \text{ and } LC(\varphi(t)) := c_1.$$

Lemma 2.7. *Let m,n be two positive integers. Let $x(t), y(t) \in K[[t]]$ with $\mathrm{LT}(x(t)) = at^\alpha$ and $\mathrm{LT}(y(t)) = bt^\beta$ such that $\alpha : \beta = n : m$. Let $f = f_d + f_{d+1} + \ldots$, be a (n,m)-weighted homogeneous decomposition of $f \in K[[x,y]]$. Then $\mathrm{ord} f(x(t),y(t)) \geq \frac{d\alpha}{n}$. Equality holds if and only if $f_d(a,b) \neq 0$.*

Proof. We can write $x(t) = t^\alpha(a + u(t))$ and $y(t) = t^\beta(b + v(t))$, where $\mathrm{ord} u(t) > 0$ and $\mathrm{ord} v(t) > 0$. Then

$$\begin{aligned} f_l(x(t),y(t)) &= \sum_{ni+mj=l} c_{ij} (t^\alpha(a+u(t)))^i (t^\beta(b+v(t)))^j \\ &= t^{\frac{l\alpha}{n}} f_l(a+u(t), b+v(t)). \end{aligned}$$

Thus $\mathrm{ord} f_l(x(t),y(t)) \geq \frac{l\alpha}{n}$ and hence $\mathrm{ord} f(x(t),y(t)) \geq \frac{d\alpha}{n}$.

Since $f_d(a+u(t), b+v(t)) = f_d(a,b) + th(t)$ for some power series h,

$$\mathrm{ord} f(x(t), y(t)) = \mathrm{ord} f_d(x(t), y(t)) = \frac{d\alpha}{n}$$

iff $f_d(a,b) \neq 0$. □

Lemma 2.8. *Let $f \in K[[x,y]]$ be convenient such that $\Gamma(f)$ has only one edge. Let $m = \mathrm{ord} f(x,0), n = \mathrm{ord} f(0,y)$ and $f = \bar{f}_1 \cdot \ldots \cdot \bar{f}_r$ the factorisation of f into its branches (irreducible factors).*

(a) *Let $(x_j(t), y_j(t))$ be a parametrization of $\bar{f}_j, j = 1, \ldots, r$ with $\mathrm{LT}(x_j(t)) = a_j t^{\alpha_j}$ and $\mathrm{LT}(y_j(t)) = b_j t^{\beta_j}$. Then $f_{\mathrm{in}}(a_j, b_j) = 0$, $\alpha_j : \beta_j = n : m$ and $\alpha_1 + \cdots + \alpha_r = n$.*

(b) *Let $a, b \in K^*$ such that $f_{\mathrm{in}}(a,b) = 0$. Then there is a parametrization $(x(t), y(t))$ of a branch of f satisfying $\mathrm{LC}(x(t)) = a$ and $\mathrm{LC}(y(t)) = b$.*

Proof. Let $f = f_d + f_{d+1} + \ldots$ with $f_d \neq 0$ be the (n,m)-weighted homogeneous decomposition of f. Then $f_d = f_{\mathrm{in}}$.

(a) Clearly $f_{\mathrm{in}} = \prod (\bar{f}_j)_{\mathrm{in}}$ then $(\bar{f}_j)_{in}$ is also a (n,m)-weighted homogeneous polynomial of order some d_j. By Proposition 2.5, $\mathrm{ord}\bar{f}_j(x,0) = \beta_j$ and $\mathrm{ord}\bar{f}_j(0,y) = \alpha_j$, i.e. x^{β_j} and y^{α_j} are monomials of $(\bar{f}_j)_{\mathrm{in}}$. Thus $n\beta_j = d_j = m\alpha_j$ and hence $\alpha_j : \beta_j = n : m$.

Since $f(x_j(t), y_j(t)) = 0$, i.e. $\mathrm{ord} f(x_j(t), y_j(t)) = +\infty > \frac{d\alpha_j}{n}$, Lemma 2.7 yields that $f_d(a_j, b_j) = 0$, i.e. $f_{\mathrm{in}}(a_j, b_j) = 0$.

Now, by the definition of intersection multiplicity we have

$$n = i(f,x) = \sum_{j=1}^{r} \mathrm{ord} x_j(t) = \sum_{j=1}^{r} \alpha_j.$$

(b) Since $f = \bar{f}_1 \bar{f}_2$ implies $f_{\mathrm{in}} = (\bar{f}_1)_{\mathrm{in}} (\bar{f}_2)_{\mathrm{in}}$, it suffices to prove part (b) for the irreducible case.

Let $(\bar{x}(t), \bar{y}(t))$ be a parametrization of f. It follows from Proposition 2.5 that $\mathrm{LT}(\bar{x}(t)) = \bar{a} t^n$ and $\mathrm{LT}(\bar{y}(t)) = \bar{b} t^m$ for some $\bar{a}, \bar{b} \in K^*$ satisfying $f_{\mathrm{in}}(\bar{a}, \bar{b}) = 0$. Set

$$g(y) := f_{\mathrm{in}}(a, y) \text{ and } \sqrt[n]{a/\bar{a}} := \{\xi_i | i = 1, \ldots, n\}.$$

Then

$$g(\bar{b}\xi_i^m) = f_{\mathrm{in}}(a, \bar{b}\xi_i^m) = f_{\mathrm{in}}(\bar{a}\xi_i^n, \bar{b}\xi_i^m) = \xi_i^d f_{\mathrm{in}}(\bar{a}, \bar{b}) = 0.$$

On the other hand, it is easy to see that $\deg g = n$. This implies the set

$$\{\bar{b}\xi_i^m | i = 1, \ldots, n\}$$

contains all of roots of g. Since $g(b) = f_{\mathrm{in}}(a,b) = 0$, there is an index i_0 such that $b = \bar{b}\xi_{i_0}^m$. By putting

$$x(t) = \bar{x}(\xi_{i_0} t) \text{ and } y(t) = \bar{y}(\xi_{i_0} t)$$

we get $\mathrm{LC}(x(t)) = a$ and $\mathrm{LC}(y(t)) = b$. □

Definition 2.9. Let $f = \sum c_{ij} x^i y^j \in K[[x, y]]$ be such that $(0, n)$ is the vertex on the y-axis of $\Gamma(f)$. Let $(1, j_1)$ be the intersection point of $\Gamma(f)$ and the line $x = 1$. We define f to be *ND1 along* $(0, n)$ if either $char(K) = p = 0$ or if $p \neq 0$ then $p \not| n$ or $j_1 \in \mathbb{N}$ and the coefficient c_{1j_1} of xy^{j_1} in f is different from zero. ND1 along $(m, 0)$, with $(m, 0)$ the vertex on the x-axis of $\Gamma(f)$, is defined analogously.

f is called NND1 if f is convenient, ND along each inner face and ND1 along each vertex on the axes of $\Gamma(f)$.

Proposition 2.10. *Let* $f = \sum c_{ij} x^i y^j \in K[[x, y]]$ *be convenient and let* $(0, n)$ *(resp.* $(m, 0)$*) be the vertex on the* y*-axis (resp. on the* x*-axis) of* $\Gamma(f)$*. Assume that* f *is not ND1 along the point* $(0, n)$ *or* $(m, 0)$ *then* $\mu(f) > \mu_N(f)$*.*

Proof. We consider only $(0, n)$ since $(m, 0)$ is analogous. Let $(1, j_1)$ be the intersection point of $\Gamma(f)$ and the line $x = 1$. The assumption that f is not ND1 along the point $(0, n)$ implies that $p|n$ and $c_{1j_1} = 0$. Putting $g(x, y) = f(x, y) - c_{0n} y^n$ one then has $\mu(f) = \mu(g)$ and $\Gamma_-(f) \subset \Gamma_-(g)$. On the other hand, it is easy to see that $(1, j_1) \in \Gamma_+(f) \setminus \Gamma_+(g)$. This means $\Gamma_-(f) \cap \mathbb{R}^2_{\geq 1} \subsetneqq \Gamma_-(g) \cap \mathbb{R}^2_{\geq 1}$. It hence follows from Lemma 2.3 that $\mu_N(g) > \mu_N(f)$. Thus

$$\mu(f) = \mu(g) \geq \mu_N(g) > \mu_N(f).$$

□

Proposition 2.11. *Let* $f \in K[[x, y]]$ *be convenient. If* f *is degenerate along some inner vertex of* $\Gamma(f)$ *then* $\mu(f) > \mu_N(f)$*.*

Proof. Assume that f is degenerate along some vertex (i_0, j_0) of $\Gamma(f)$ with $i_0 > 0$ and $j_0 > 0$. Then $p \neq 0$ and i_0 and j_0 are divisible by p. Put $g(x, y) = f(x, y) - c_{i_0 j_0} x^{i_0} y^{j_0}$. Then $\Gamma_+(g)$ does not contain the point (i_0, j_0). Thus

$$\Gamma_-(f) \cap \mathbb{R}^2_{\geq 1} \subsetneqq \Gamma_-(g) \cap \mathbb{R}^2_{\geq 1}.$$

Lemma 2.3 hence implies that $\mu_N(g) > \mu_N(f)$. We then have

$$\mu(f) = \mu(g) \geq \mu_N(g) > \mu_N(f).$$

□

Proposition 2.12. *Let* $f \in K[[x, y]]$ *be convenient. If* f *is degenerate along some edge of* $\Gamma(f)$ *then* $\mu(f) > \mu_N(f)$*.*

Proof. Let $f(x, y) = \sum c_{\alpha\beta} x^\alpha y^\beta$. Let f_x, f_y be the partials of f and put $h(x, y) := x f_x(x, y) + \lambda y f_y(x, y)$, where $\lambda \in K$ is generic. Then

$$h(x, y) = \sum (\alpha + \lambda\beta) c_{\alpha\beta} x^\alpha y^\beta.$$

Thus $\mathrm{supp}(h) = \mathrm{supp}(f) \setminus (p\mathbb{N})^2$ and if $p = 0$ then $\mathrm{supp}(h) = \mathrm{supp}(f)$. Hence $\Gamma_+(h) \subset \Gamma_+(f)$.

Case 1: f is ND along each vertex of $\Gamma(f)$.

Assume now that (i, j) is a vertex of $\Gamma(f)$. Since f is ND along (i, j), $p = 0$ or $p \neq 0$ and one of i, j is not divisible by p. Therefore $(i, j) \in \mathrm{supp}(f) \setminus (p\mathbb{N})^2 = \mathrm{supp}(h)$ and then $\Gamma_+(f) \subset \Gamma_+(h)$. Hence $\Gamma(h) = \Gamma(f)$.

Let $E_i, i = 1, \ldots, k$ be edges of $\Gamma(h)$. By Proposition 2.6, we can write $h = \bar{h}_1 \ldots \bar{h}_k$, where $\bar{h}_i$ are convenient and $h_{E_i}(x, y)) = \mathit{monomial} \times (\bar{h}_i)_{in}$. We denote by m_i and n_i the lengths of the projections of E_i on the horizontal and vertical axes.

Let $h = h_{d_i} + h_{d_i+1} + \ldots$ with $h_{d_i} \neq 0$ be the (n_i, m_i)-weighted homogeneous decomposition of h. Then $h_{d_i} = h_{E_i}$. Since E_i is also an edge of $\Gamma(f)$, $f = f_{d_i} + f_{d_i+1} + \ldots$ is the (n_i, m_i)-weighted homogeneous decomposition of f with $f_{d_i} = f_{E_i}$ and then

$$h_{d_i} = \sum_{n_i\alpha + m_i\beta = d_i} (\alpha + \lambda\beta) c_{\alpha\beta} x^\alpha y^\beta = x\frac{\partial f_{d_i}}{\partial x} + \lambda y \frac{\partial f_{d_i}}{\partial y}.$$

Let $yf_y = g_{d_i'} + g_{d_i'+1} + \ldots$ be the (n_i, m_i)-weighted homogeneous decomposition of yf_y. It is easy to see that $d_i' \geq d_i$ and $d_i' = d_i$ iff $y\frac{\partial f_{d_i}}{\partial y} \neq 0$.

Claim 1. *Let A_{i-1}, A_i be the vertices of the edge E_i and let $V_2(OA_{i-1}A_i)$ be the volume of triangle $OA_{i-1}A_i$. Then $d_i = 2V_2(OA_{i-1}A_i)$.*

Proof. Let (c_i, e_i) be the coordinates of $A_i, i = 0, \ldots, k$. Then $m_i = c_i - c_{i-1}$ and $n_i = e_{i-1} - e_i$. (cf. Fig. 2).

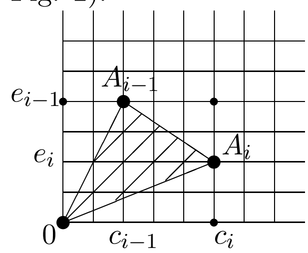

Fig. 2.

Considering the rectangle $(0, 0); (c_i, 0); (c_i, e_{i-1}); (0, e_{i-1})$ we have

$$\begin{aligned} 2V_2(OA_{i-1}A_i) &= 2c_ie_{i-1} - c_ie_i - c_{i-1}e_{i-1} - m_in_i \\ &= (c_{i-1} + m_i)e_{i-1} + c_i(e_i + n_i) - c_ie_i - c_{i-1}e_{i-1} - m_in_i \\ &= m_ie_{i-1} + c_in_i - m_in_i = m_i(e_i + n_i) + c_in_i - m_in_i \\ &= m_ie_i + n_ic_i = d_i \end{aligned}$$

This proves Claim 1.

Claim 2. *$i(\bar{h}_i, yf_y) \geq d_i$, and if f is degenerate along E_i then $i(\bar{h}_i, yf_y) > d_i$.*

Proof. Let $(x_j(t), y_j(t)), j = 1, \ldots, r$ be parametrizations of the branches $\bar{h}_{i,j}$ of $\bar{h}_i$. Then by Lemma 2.8, we have $\mathrm{LT}(x_j(t)) = a_j t^{\alpha_j}$ and $\mathrm{LT}(y_j(t)) = b_j t^{\beta_j}$, where $a_j, b_j \in K^*, \bar{h}_i(a_j, b_j) = 0, \alpha_j : \beta_j = n_i : m_i$ for all $j = 1, \ldots, r$ and $\alpha_1 + \ldots + \alpha_r = n_i$. It follows from Lemma 2.7 that $\mathrm{ord}(yf_y)(x_j(t), y_j(t)) \geq \frac{d_i{}' \alpha_j}{n_i}$ for all $j = 1, \ldots, r$. Thus

$$i(\bar{h}_i, yf_y) = \sum_{j=1}^{r} \mathrm{ord}(yf_y)(x_j(t), y_j(t)) \geq \sum_{j=1}^{r} \frac{d_i{}' \alpha_j}{n_i} = d_i{}' \geq d_i.$$

Assume that f is degenerate along E_i then there exist $a, b \neq 0$ such that

$$x\frac{\partial f_{d_i}}{\partial x}(a,b) = y\frac{\partial f_{d_i}}{\partial y}(a,b) = 0.$$

Therefore $h_{d_i}(a,b) = 0$. Lemma 2.8 implies that there is a parametrization of a branch of $\bar{h}_i$ such that $\mathrm{LT}(\bar{x}(t)) = at^{\alpha}$ and $\mathrm{LT}(\bar{y}(t)) = bt^{\beta}$. We may assume that $(\bar{x}(t), \bar{y}(t))$ is a parametrization of the branch $\bar{h}_{i,1}$. Then $\alpha = \alpha_1$ and $\beta = \beta_1$.

To show $i(\bar{h}_i, yf_y) > d_i$, we may restrict to the case that $d_i{}' = d_i$, because of the inequality $i(\bar{h}_i, yf_y) \geq d_i{}' \geq d_i$. As $d_i{}' = d_i$ then $g_{d_i{}'}(a,b) = y\frac{\partial f_{d_i}}{\partial y}(a,b) = 0$. Lemma 2.7 yields

$$\mathrm{ord}(yf_y)(\bar{x}(t), \bar{y}(t)) > \frac{d_i \alpha_1}{n_i}.$$

Thus

$$\begin{aligned} i(\bar{h}_i, yf_y) &= \mathrm{ord}(yf_y)(\bar{x}(t), \bar{y}(t)) + \sum_{j=2}^{r} \mathrm{ord}(yf_y)(x_j(t), y_j(t)) \\ &> \frac{d_i \alpha_1}{n_i} + \sum_{j=2}^{r} \frac{d_i{}' \alpha_j}{n_i} = d_i. \end{aligned}$$

This proves Claim 2.

It now follows from Claim 1 and Claim 2 that

$$i(h, yf_y) \geq \sum_{i=1}^{k} 2V_2(OA_{i-1}A_i) = 2V_2(\Gamma_-(f)).$$

Hence

$$\begin{aligned} \mu(f) = i(f_x, f_y) &= i(h, yf_y) - i(x, f_y) - i(f_x, y) - 1 \\ &\geq 2V_2(\Gamma_-(f)) - (e_0 - 1) - (c_k - 1) - 1 \\ &= \mu_N(f). \end{aligned}$$

Moreover, if f is degenerate along some edge of $\Gamma(f)$ then $\mu(f) > \mu_N(f)$ by Claim 1 and 2. This proves of Case 1.

Case 2: In the general case, by propositions 2.10, 2.11 we may assume that f is ND along each inner vertex and ND1 along the two vertice on the axes of $\Gamma(f)$. For m sufficiently large and $p \not| m$, we put

$$\bar{f}_m(x,y) = \sum_{(\alpha,\beta)\notin(p\mathbb{N})^2} c_{\alpha\beta}x^\alpha y^\beta + x^m + y^m.$$

Then

$$\mu(\bar{f}_m) = \mu(f_m) = \mu(f) \text{ and } \mu_N(\bar{f}_m) \geq \mu_N(f_m) = \mu_N(f),$$

where the inequality follows from Lemma 2.3.

Claim 3. *$\bar{f}_m$ is degenerate along a some edge of $\Gamma(\bar{f})$.*

Proof. By the assumption f is degenerate along some edge E of $\Gamma(f)$. If E is also an edge of $\Gamma(\bar{f}_m)$ then $j(\text{in}_E(\bar{f}_m) = j(\text{in}_E(f))$ and hence $\bar{f}_m$ is degenerate along E. If E is not an edge of $\Gamma(\bar{f}_m)$, then E must meet the axes since f is ND along each inner vertex of $\Gamma(f)$. We may assume that $(0,n)$ is a vertex of E. We will show that

$$\sharp(\text{supp}(\bar{f}_m) \cap E) \geq 2.$$

Let $(1, j_1)$ be the intersection point of E and the line $x = 1$. Since f is ND1 along $(0,n)$, either $(0,n) \in \text{supp}(\bar{f}_m) \cap E$ or $(1, j_1) \in \text{supp}(\bar{f}_m) \cap E$, i.e. $\text{supp}(\bar{f}_m) \cap E \neq \emptyset$ On the other hand, it is easy to see that $\sharp(\text{supp}(\bar{f}_m) \cap E) \neq 1$ since f is degenerate along the edge E. Hence $\sharp(\text{supp}(\bar{f}_m) \cap E) \geq 2$. Let us denote by $\bar{E}$ the convex hull of the set $\text{supp}(\bar{f}_m) \cap E$. Then $\bar{E}$ is an edge of $\Gamma(\bar{f}_m)$ and $j(\text{in}_{\bar{E}}(\bar{f}_m) = j(\text{in}_E(f))$. Thus $\bar{f}_m$ is degenerate along $\bar{E}$ since f is degenerate along the edge E, which proves Claim 3.

Now, by definition, $\bar{f}_m$ is ND along each vertex of $\Gamma(\bar{f}_m)$. Since $\bar{f}_m$ is degenerate along a some edge of $\Gamma(\bar{f}_m)$, applying the first case to $\bar{f}_m$, we get $\mu(\bar{f}_m) > \mu_N(\bar{f}_m)$. Hence

$$\mu(f) = \mu(\bar{f}_m) > \mu_N(\bar{f}_m) \geq \mu_N(f).$$

This proves Proposition 2.12. □

Theorem 2.13. *Let $f \in \mathfrak{m} \subset K[[x,y]]$ and let $f_m = f + x^m + y^m$. Then the following are equivalent*

(i) $\mu(f) = \mu_N(f) < \infty$.
(ii) $\mu(f) < \infty$ *and f_m is NND1 for some large integer number m.*
(iii) *f is INND.*

Proof. *(i) $\Rightarrow$ (ii)* : Since $\mu(f) = \mu_N(f) < \infty$ we have by definition of $\mu_N(f)$

$$\mu(f_m) = \mu(f) = \mu_N(f) = \mu_N(f_m) < \infty.$$

Combining Propositions 2.10, 2.11 and 2.12 we get the claim.

$(ii) \Rightarrow (iii)$: Assume that $\mu(f) < \infty$ and f_m is NND1. Firstly, it is easy to see that there is an $M \in \mathbb{N}$ such that $\Gamma(f) \subset \Gamma(f_M)$. It suffices to show f is INND w.r.t. $\Gamma(f_m)$ for all $m > M$. We argue by contradiction. Suppose that it is not true. Then f is not IND along some edge Δ of $\Gamma(f_m)$ which meets the axes, since f_m is NND1. We may assume that Δ meets the axes at $(0, n)$. Let (k, l) be the second vertex of Δ. We consider two cases:

• If $l = 0$, i.e. $\Gamma(f_m)$ has only one edge Δ. Then Δ is also a unique edge of $\Gamma(f)$ and $\text{in}_\Delta(f) = \text{in}_\Delta(f_m)$. Since f is not IND along Δ, there exists $(a, b) \in K \setminus \{(0,0)\}$ which is a zero point of $\text{j}(\text{in}_\Delta(f))$. Beside, since f_m is ND along Δ, either $a = 0$ or $b = 0$. Assume that $a = 0$ and $b \neq 0$. We will show that f_m is not ND1 along $(0, n)$. Firstly, we write $\text{in}_\Delta(f_m) = c_{0n}y^n + x \cdot g(x, y)$, then $\dfrac{\partial \text{in}_\Delta(f_m)}{\partial y} = ny^{n-1} + x \cdot \dfrac{\partial g}{\partial y}$. Thus

$$\frac{\partial \text{in}_\Delta(f_m)}{\partial y}(0, b) = ny^{n-1} = 0 \Rightarrow p \neq 0 \text{ and } p|n.$$

We now write $\text{in}_\Delta(f_m) = c_{0n}y^n + c_{1j}xy^j + x^2 \cdot h(x, y)$, then

$$\frac{\partial \text{in}_\Delta(f_m)}{\partial x} = c_{1j}y^j + 2x \cdot h(x, y) + x^2 \cdot \frac{\partial h}{\partial x}.$$

Since $\dfrac{\partial \text{in}_\Delta(f_m)}{\partial x}(0, b) = 0$, $c_{1j} = 0$. Hence f_m is not ND1 along $(0, n)$, a contradiction.

• Assume that $l > 0$. If Δ is also an edge of $\Gamma(f)$ then $\text{in}_\Delta(f) = \text{in}_\Delta(f_m)$. Since f is not IND along Δ, there exists $(a, b) \in K \times K^*$ being a zero of $\text{j}(\text{in}_\Delta(f))$. Since f_m is ND along Δ, $a = 0$. Analogously as above f_m is not ND1 along $(0, n)$ and we get a contradiction. Assume now that Δ is not an edge of $\Gamma(f)$, i.e. $m = n$ and $x|f(x, y)$. Let P be the end point of $\Gamma(f)$ closest to y-axis. It follows from $\Gamma(f) \subset \Gamma(f_M)$ and $m > M$ that P must be a vertex of Δ, i.e. $P = (k, l)$. This implies $f = x^k \cdot h(x, y)$. Since $\mu(f) < \infty$, $k = 1$. Then $\text{in}_\Delta(f) = c_{0n}y^n + c_{1l}xy^l$ and clearly f is always IND along Δ, a contradiction. Hence f is INND w.r.t. $\Gamma(f_m)$ and then it is INND.

$(iii) \Rightarrow (i)$: See Theorem 1.2. □

Corollary 2.14. *Let $f \in K[[x, y]]$ and let $M \in \mathbb{N}$ such that $\Gamma(f) \subset \Gamma(f_M)$. Then f is INND if and only if it is INND w.r.t. $\Gamma(f_m)$ for some (equivalently for all) $m > M$.*

Proof. One direction is obvious, it remains to show f is INND $\Rightarrow$ f is INND w.r.t. $\Gamma(f_m)$ for all $m > M$. We take $m_1 > M$ satisfying Theoren 2.13 and then

$$f \text{ is INND } \Rightarrow \mu(f) < \infty \text{ and } f_{m_1} \text{ is NND1 } \Rightarrow f \text{ is INND w.r.t. } \Gamma(f_{m_1}).$$

For each inner face Δ_m of $\Gamma(f_m)$, since $m, m_1 > M$, there is an inner face Δ_{m_1} of $\Gamma(f_{m_1})$ such that $\text{in}_{\Delta_m}(f) = \text{in}_{\Delta_{m_1}}(f)$. Thus f is IND along Δ_m since it is IND along Δ_{m_1}. Hence f is INND w.r.t. $\Gamma(f_m)$. □

Corollary 2.15. *Let $M \in \mathbb{N}$ be such that $\Gamma(f) \subset \Gamma(f_M)$. Then Theorem 2.13 holds for each $m > M$.*

Remark 2.16. Let $\mu(f) < \infty$. Then M can be chosen as the maximum of n_1 and m_1, where $n_1 = n$ if $\Gamma(f) \cap \{x = 0\} = \{(0, n)\}$ and $n_1 = 2i_1$ if $\Gamma(f) \cap \{x = 0\} = \emptyset$ and $\Gamma(f) \cap \{x = 1\} = \{(1, i_1)\}$. Similarly we define m_1 with x replaced by y. This remark and the previous corollaries are important for concrete computation.

Proof of Corollary 2.15. Clearly, the equivalence $(i) \Leftrightarrow (iii)$ does not depend on m and as in the proof of Theorem 2.13 the impication $(ii) \Rightarrow (iii)$ holds for all $m > M$. It remains to show that f is INND $\Rightarrow f_m$ is NND1. By Corollary 2.14, it suffices to show that f is INND w.r.t. $\Gamma(f_m) \Rightarrow f_m$ is NND1. By contradiction, suppose that f is INND w.r.t. $\Gamma(f_m)$ and f_m is not NND1. Then f is not ND1 along some vertex of $\Gamma(f_m)$ in the axes. Assume that f is not ND1 along $(0, n) \in \Gamma(f_m)$. Then

$$p \neq 0, p|n \text{ and } \Gamma(f_m) \cap \{x = 1\} \cap \text{supp}(f_m) = \emptyset,$$

i.e. $(f_m)_{in} = c_{0n}y^n + x^2 \cdot h(x, y)$. This implies $\mu((f_m)_{in}) = \infty$. By Theorem 1.2, $(f_m)_{in}$ is not INND and then f_m is also not INND, a contradiction. □

Corollary 2.17. *Let K is a field of characteristic zero and $f \in \mathfrak{m} \subset K[[x, y]]$. Then the following are equivalent*

(i) $\mu(f) = \mu_N(f) < \infty$.
(ii) *f is INND.*
(iii) *f is NND and $\mu_N(f) < \infty$.*

In particular, if f is convenient then (i)-(iii) are equivalent to

(iv) *f is NND.*

Proof. The implications $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (i)$ follow from Theorem 2.13 and Proposition 1.3. It remains to prove $(ii) \Rightarrow (iii)$.

Assume that f is INND. Then by Theorem 2.13, $\mu_N(f) < \infty$. We will show that f is ND along each vertex and each edge of $\Gamma(f)$. Since $char(K) = 0$, f is ND along each vertex of $\Gamma(f)$. Let Δ be an edge of $\Gamma(f)$. Clearly, it is an inner edge of $\Gamma(f_m)$, where m sufficiently large. Since f is INND, by Corollary 2.14 f is INND w.r.t. $\Gamma(f_m)$. Then f is IND along Δ hence it is also ND along Δ. This implies f is NND. □

Corollary 2.18. *If f is NND and $\mu_N(f) < \infty$ then f is INND.*

Proof. This follows from Proposition 1.3 and Theorem 2.13. □

Note that $char(K) = 0$ is only used to assure that f is ND along each vertex of $\Gamma(f) \cap (\{0\} \times \mathbb{N} \cup \mathbb{N} \times \{0\})$. Hence, the last corollary holds also if $p > 0$ and $p \not| n$ if $(0, n) = \Gamma(f) \cap \{0\} \times \mathbb{N}$ and $p \not| m$ if $(m, 0) = \Gamma(f) \cap \mathbb{N} \times \{0\}$. Example 2.1 shows that this condition is necessary.

3. δ-Invariant

We consider now another important invariant of plane curve singularities, the δ-invariant and its combinatorial counterpart, the Newton δ-invariant. We show that both coincide iff f is weakly non-degenerate.

Let $f \in \mathfrak{m} \subset K[[x, y]]$ be a power series and suppose that the Newton diagram of f has k facets $E_1, \dots, E_k$. By $l(E_i)$ we denote the lattice length of E_i, i.e. the number of lattice points on E_i minus one.

We fix a minimal resolution of the singularity computed via successively blowing up points, denote by $Q \to 0$ that Q is the origin or an infinitely near point of the origin and by m_Q the multiplicity of f or of the strict transform of f at Q. We set

(a) $\delta(f) := \sum_{Q \to 0} \frac{m_Q(m_Q-1)}{2}$ the *delta invariant* of f.

(b) $\nu(f) := \sum_{Q \text{ special}} \frac{m_Q(m_Q-1)}{2}$, where an infinitely near point Q is *special* if it is the origin or the origin of the corresponding chart of the blowing up.

(c) If f is convenient, we define

$$\delta_N(f) := V_2(\Gamma_-(f)) - \frac{V_1(\Gamma_-(f))}{2} + \frac{\sum_{i=1}^k l(E_i)}{2},$$

and otherwise we set $\delta_N(f) := \sup\{\delta_N(f_m) | f_m := f + x^m + y^m, m \in \mathbb{N}\}$ and call it the *Newton δ-invariant* of f.

Proposition 3.1. [BGM10, Lemma 4.8] *If $f \in K[[x, y]]$ then $\delta_N(f) = \nu(f)$.*

The relation between $\nu(f)$ and $\delta(f)$ was studied in [BeP00] and [BGM10]. The delta invariant $\delta(f)$ equals also $\dim_K(\bar{R}/R)$ where $R = K[[x, y]]/\langle f \rangle$ and $\bar{R}$ is the integral closure of of R in its total ring of fractions.

Proposition 3.2. [BGM10, Prop. 4.9]
If $f \in K[[x, y]]$ is WNND then $\delta_N(f) = \nu(f) = \delta(f)$.

Note that $\nu(f) \le \delta(f)$ always holds. We prove now the converse of Proposition 3.2.

Theorem 3.3. *Let $f \in K[[x, y]]$ be reduced. Then $\delta(f) = \delta_N(f)$ if and only if f is WNND.*

Proof. The sufficiency of the condition WNND of the theorem follows from Proposition 3.2. We now assume f is not WND along some edge E of $\Gamma(f)$. It can always factorize

$$f_E = monomial \times \prod_{i=1}^{K}(a_i x^{m_0} - b_i y^{n_0})^{r_i},$$

where $a_i, b_i \in K^*$, $(a_i : b_i)$ pairwise distinct, $\gcd(m_0, n_0) = 1$, $r_i \geq 1$. Since f is not WND along E, there is $x_0, y_0 \in K^*$ such that

$$f_E(x_0, y_0) = \frac{\partial f_E}{\partial x}(x_0, y_0) = \frac{f_E}{\partial y}(x_0, y_0) = 0.$$

Then there exists an index i_0 such that $a_{i_0} x_0^{m_0} - b_{i_0} y_0^{n_0} = 0$. We will show that $r_{i_0} > 1$. In fact, if this is not true then $f_E(x, y) = (a_{i_0} x^{m_0} - b_{i_0} y^{n_0}) \cdot h(x, y)$ with $h(x_0, y_0) \neq 0$. Since

$$\frac{\partial f_E}{\partial x}(x_0, y_0) = \frac{f_E}{\partial y}(x_0, y_0) = 0,$$

this is impossible if $p = 0$ and implies that p divides m_0 and n_0 if $p > 0$ which contradicts the hypothesis $\gcd(m_0, n_0) = 1$.

As in Proposition 2.6 we can deduce that $(a_i x^{m_0} - b_i y^{n_0})$ is a factor of f_{in} of multiplicity r_{i_0}. Then by Lemma 3.4, $Q := (b_{i_0} : a_{i_0})$ is an infinitely near point of the origin of multiplicity $r_Q = r_{i_0} > 1$. Clearly, Q is not special. It hence follows from Proposition 3.1 that $\delta(f) > \nu(f) = \delta_N(f)$.

□

Lemma 3.4. *Let $f \in K[[x, y]]$ be reduced and f_{in} its initial part w.r.t. $\Gamma(f)$. Let*

$$f_{in}(x, y) = \prod_{\substack{1 \leq i \leq s \\ 1 \leq j \leq k}} (a_{ij} x^{m_j} - b_{ij} y^{n_j})^{r_{ij}},$$

where $(a_{ij} : b_{ij}) \in K\mathbb{P}^1$ are pairwise distinct, $m_j, n_j > 0$ and $(m_j, n_j) = 1$ for all $i = 1, \ldots, s$, $j = 1, \ldots, k$. If there are i_0, j_0 such that $r_{i_0 j_0} > 1$ then $Q_{i_0 j_0} := (b_{i_0 j_0} : a_{i_0 j_0})$ is an infinitely near point of the origin of multiplicity $r_{i_0 j_0}$.

Proof. We set $m := \mathrm{mt}(f)$, $J := \{1, \ldots, k\}$, $J_+ := \{j \in J | m_j > n_j\}$,

$$J_- := \{j \in J | n_j > m_j\} \text{ and } J_0 := J \setminus (J_+ \cup J_-).$$

Then for each $j \in J_0$, $m_j = n_j = 1$ and then $a_{ij} x^{m_j} - b_{ij} y^{n_j}$ is a linear factor of f_{in}. We will prove the lemma by induction on

$$n(f) := \text{the number of blowing ups needed to resolve the singularity of} f.$$

Assume that $n(f) = 1$, i.e. the strict transform $\tilde{f}$ of f has multiplicity one. We first show that $j_0 \in J_0$.

If $j_0 \in J_+$, considering the local equation at the point $(0:1)$ in Chart 1: $\tilde{f}(u,v) = \dfrac{f(u,uv)}{u^m}$, we have then

$$(\tilde{f})_{in}(u,v) = \prod_{\substack{j\in J_+\\ 1\le i\le s}} (a_{ij}u^{\tilde{m}_j} - b_{ij}v^{\tilde{n}_j})^{r_{ij}},$$

where $\tilde{m}_j = m_j - n_j > 0$ and $\tilde{n}_j = n_j$. Since $j_0 \in J_+$, $\mathrm{mt}(\tilde{f}) \ge r_{i_0 j_0} > 1$, a contradiction. This implies $j_0 \notin J_+$. Similarly, $j_0 \notin J_-$. Thus $j_0 \in J_0$, which means that the factor $(a_{i_0j_0}x - b_{i_0j_0}y)$ is a tangent of f of multiplicity $r_{i_0j_0}$. Hence $Q_{i_0j_0} = (b_{i_0j_0} : a_{i_0j_0})$ is an infinitely near point of the origin of multiplicity $r_{i_0j_0}$.

Now we show the induction step. Firstly, if $j_o \in J_0$ then the lemma holds as above. If $j_o \notin J_0$, we may assume without loss the generality that $j_0 \in J_+$. Let $q = (\alpha : 1)$ be an infinitely near point of the origin. Then the strict transform $\tilde{f}$ of f has the local equation in Chart 1 at q: $\tilde{f}(u,v) = \dfrac{f(u,u(v+\alpha))}{u^m}$. It is easy to verify that

$$(\tilde{f})_{in}(u,v) = \prod_{\substack{j\in J_+\\ 1\le i\le s}} (a_{ij}u^{\tilde{m}_j} - b_{ij}v^{\tilde{n}_j})^{r_{ij}},$$

where $\tilde{m}_j = m_j - n_j > 0$ and $\tilde{n}_j = n_j$. The induction step is proven by applying the induction thesis to $\tilde{f}$. □

Note that NND implies WNND and that both are equivalent for $f \in K[[x,y]]$ and $char(K) = 0$.

Example 3.5. Let $f(x,y) = (x^2+y^2)(x-y) \in K[[x,y]]$ with $char(K) = 3$. Then $\delta_N(f) = \delta(f) = 3$ and f is WNND but not NND.

The example shows that NND is sufficient but not necessary for $\delta(f) = \delta_N(f)$.

If $char(K) = 0$ we have Milnor's famous formula $\mu(f) = 2\delta(f) - r(f) + 1$, where $r(f)$ is the number of branches of f. The formula is wrong in general if $char(K) > 0$ but still holds if f is NND by [BGM10, Thm. 4.13]. Using the general inequality

$$\mu_N(f) = 2\delta_N(f) - r(f) + 1 \le 2\delta(f) - r(f) + 1 \le \mu(f)$$

from [BGM10], Theorem 2.13 and Theorem 3.3 imply

Corollary 3.6. *Let $f \in K[[x,y]]$ be reduced. Then f is INND if and only if f is WNND and $\mu(f) = 2\delta(f) - r(f) + 1$.*

Remark 3.7. (1) The difference $\mathrm{wvc}(f) := \mu(f) - 2\delta(f) + r(f) - 1$ counts the number of *wild vanishing cycles* of (the Milnor fiber) of f (cf. [Del73], [MHW01], [BGM10]), which vanishes if $char(K) = 0$ or if f is INND.

(2) wvc(f) is computable for any given f. This follows since $\mu(f)$ is computable by a standard basis computation w.r.t. a local ordering (cf. [GP08]) and $\delta(f)$ and $r(f)$ are computable by computing a Hamburger-Noether expansion (cf. [Cam80]). Both algorithms are implemented in Singular (cf. [GPS05]).

Example 3.8. Consider $f = x(x-y)^2 + y^7$ and $g = x(x-y)^2 + y^7 + x^6$ and char $(K) = 3$. Using Singular we compute $\mu(f) = 8, \delta(f) = 5, r(f) = 3$ and $\mu(g) = 8, \delta(f) = 4, r(g) = 2$. We have $\mathrm{wvc}(f) = 0, \mathrm{wvc}(g) = 1, \Gamma(f) = \Gamma(g)$ and f is not INND. This shows

- INND is sufficient but not necessary for the absence of wild vanishing cycles,
- the Newton diagram can not distinguish between singularities which have wild vanishing cycles and those which have not.

Although we can compute the number of wild vanishing cycles, it seems hard to understand them. We like to pose the following

Problem. *Is there any "geometric" way to understand the wild vanishing cycles, distinguishing them from the ordinary vanishing cycles counted by $2\delta - r + 1$? Is there at least a "reasonable" characterization of those singularities without wild vanishing cycles?*

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