Mather-Yau Theorem in Positive Characteristic

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Abstract

The well-known Mather-Yau theorem says that the isomorphism type of the local ring of an isolated complex hypersurface singularity is determined by its Tjurina algebra. It is also well known that this result is wrong as stated for power series f in K[[x]] over fields K of positive characteristic. In this note we show that, however, also in positive characteristic the isomorphism type of an isolated hypersurface singularity f is determined by an Artinian algebra, namely by a "higher Tjurina algebra" for sufficiently high index, for which we give an effective bound. We prove also a similar version for the "higher Milnor algebra" considered as K[[f]]-algebra.

1 Introduction

Let K be an algebraically closed field of arbitrary characteristic and $K[[\mathbf{x}]] = K[[x_1, x_2, ..., x_n]]$ the formal power series ring over K with maximal ideal $\mathfrak{m}$. Let $f \in K[[\mathbf{x}]]$. We denote by

$j(f) = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n} \right\rangle$, the Jacobian ideal of f,

$M(f) = K[[\mathbf{x}]]/j(f)$, the Milnor algebra of f, and

$\mu(f) = dim_K M(f)$, the Milnor number of f.

Moreover, we call

$tj(f) = \left\langle f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n} \right\rangle$, the Tjurina ideal of f,

$T(f) = K[[\mathbf{x}]]/tj(f)$, the Tjurina algebra of f, and

$\tau(f) = dim_K T(f)$, the Tjurina number of f.

More generally, for $k \in \mathbb{N}$, set

$$T_k(f) = K[[\mathbf{x}]]/\left\langle f, \mathfrak{m}^k j(f)\right\rangle \text{ resp.}$$

$$M_k(f) = K[[\mathbf{x}]]/\mathfrak{m}^k j(f),$$

and call it the *k-th Tjurina* resp. *k-th Milnor algebra* of f.

Two power series f and g in $K[[\mathbf{x}]]$ are said to be *right equivalent*, denoted $f \overset{r}{\sim} g$, if there is an automorphism $\varphi \in Aut(K[[\mathbf{x}]])$ such that $g = \varphi(f)$. They are called *contact equivalent*, denoted $f \overset{c}{\sim} g$, if there are $\varphi \in Aut(K[[\mathbf{x}]])$ and a unit $u \in K[[\mathbf{x}]]^\star$ such that $g = u\varphi(f)$. If $f \overset{r}{\sim} g$ then the associated Milnor algebras are isomorphic and if $f \overset{c}{\sim} g$ then the associated Tjurina algebras are isomorphic. The theorem of Mather and Yau says:

Theorem 1.1 ([MY82 - Theorem 1])**.** *Let $f, g \in \mathfrak{m} \subset \mathbb{C}\{\mathbf{x}\}$ be such that $\tau(f) < \infty$. The following are equivalent:*
i) $f \overset{c}{\sim} g$
ii) $T(f) \cong T(g)$ as $\mathbb{C}$-algebras.

The theorem was slightly generalized in [GLS06 - Theorem 2.26] (without assuming isolated singularity):

Theorem 1.2. *Let $f, g \in \mathfrak{m} \subset \mathbb{C}\{\mathbf{x}\}$. The following are equivalent:*
i) $f \overset{c}{\sim} g$
ii) For all $k \geq 0$, $T_k(f) \cong T_k(g)$ as $\mathbb{C}$-algebras.
iii) There is some $k \geq 0$ such that $T_k(f) \cong T_k(g)$ as $\mathbb{C}$-algebras.
In particular, $f \overset{c}{\sim} g$ iff $T(f) \cong T(g)$ as $\mathbb{C}$-algebras.

However, the theorem is not true if K has characteristic $p > 0$ in general, as was already noted by Mather and Yau [MY81]. For $f = x^{p+1} + y^{p+1}$ and $g = f + x^p y$, we have that $g \overset{c}{\not\sim} f$ (they have different number of branches) but $T_k(f) = T_k(g)$ for $k = 0, 1$. However, $T_k(f) \ncong T_k(g)$ for $k \geq 2$, as one can check (for small p this can be verified by using Singular [DGPS12]).

It is also known that over the complex numbers the Milnor algebra $M(f)$ determines f up to right equivalence, if we consider $M(f)$ as $\mathbb{C}\{t\}$-algebra where t acts by multiplication with f (but not as $\mathbb{C}$-algebra). The following result can be deduced from Theorem 1.2 (cf. [GLS06 - Theorem 2.28]):

Theorem 1.3. *Let $f, g \in \mathfrak{m} \subset \mathbb{C}\{\mathbf{x}\}$ be hypersurface singularities. Then the following are equivalent:*
i) $f \overset{r}{\sim} g$
ii) For all $k \geq 0$, $M_k(f) \cong M_k(g)$ as $\mathbb{C}\{t\}$-algebras.
iii) There is some $k \geq 0$ such that $M_k(f) \cong M_k(g)$ as $\mathbb{C}\{t\}$-algebras.
In particular, $f \overset{r}{\sim} g$ iff $M(f) \cong M(g)$ as $\mathbb{C}\{t\}$-algebras.

For the same f and g as in the example above we have $j(f) = j(g)$ and hence $M_k(f) = M_k(g)$ for k as $\mathbb{C}$-algebras. However, considered as $K[[f]]$- resp. $K[[g]]$-algebras, they are not isomorphic for $k \geq 2$.

Our aim is to see how far the Mather-Yau theorem and Theorem 1.3 hold in the case of positive characteristic. In order to do that we need some additional notions.

For $k \in \mathbb{N}$ we say that f is *right* (respectively *contact*) *k-determined* if it is right (respectively contact) equivalent to every $g \in K[[\mathbf{x}]]$ satisfying $g - f \in \mathfrak{m}^{k+1}$. We denote by $ord(f)$ the order (or multiplicity) of f, i.e. the maximal l such that $f \in \mathfrak{m}^l$, which is invariant under right and contact equivalence.

Note that the proof given by Mather and Yau (as well as in [GLS06]) uses integration of vector fields and can not be generalized to positive characteristic. To prove an appropriate generalization of Theorem 1.1 - 1.3 in positive characteristic we need the finite determinacy theorem proved in [BGM12].

Theorem 1.4 ([BGM12 - Theorem 2.1])**.** *Let $0 \neq f \in \mathfrak{m}^2$ and $k \in \mathbb{N}$.*
i) If $\mathfrak{m}^{k+2} \subset \mathfrak{m}^2 j(f)$ then f is right $(2k - ord(f) + 2)$-determined.
ii) If $\mathfrak{m}^{k+2} \subset \mathfrak{m}\langle f\rangle + \mathfrak{m}^2 j(f)$ then f is contact $(2k - ord(f) + 2)$-determined.

2 Results

Let us first consider the case of an arbitrary algebraically closed field K of characteristic 0.

Proposition 2.1. *Let K be an algebraically closed field of characteristic 0 and $f, g \in \mathfrak{m} \subset K[[\mathbf{x}]]$. Then Theorem 1.2 (resp. Theorem 1.3) holds if we replace $\mathbb{C}$ by K, $\mathbb{C}\{\mathbf{x}\}$ by $K[[\mathbf{x}]]$, and $\mathbb{C}\{t\}$ by $K[[t]]$.*

Proof. We sketch a proof which uses the Lefschetz principle. For details we refer to [Pham].

We can embed a countable field extension K' of $\mathbb{Q}$, which contains the coefficients of f, g and of the automorphism of $K[[\mathbf{x}]]$ providing the isomorphism $T_k(f) \cong T_k(g)$, in $\mathbb{C}$ and apply Theorem 1.2 (resp. Theorem 1.3). We get $\varphi \in Aut(\mathbb{C}\{\mathbf{x}\})$ and $u \in \mathbb{C}\{\mathbf{x}\}^*$ such that $g = u\varphi(f)$ (resp. $g = \varphi(f)$). The equation $g = u\varphi(f)$ (resp. $g = \varphi(f)$) implies that we can in fact find $\varphi' \in Aut(\overline{K'}[[\mathbf{x}]])$ and $u' \in \overline{K'}[[\mathbf{x}]]^*$, with $\overline{K'}$ the algebraic closure of K', satisfying the same equation and hence showing that f and g are contact (resp. right) equivalent in $K[[\mathbf{x}]]$. □

Now we formulate our main results for K an algebraically closed field of any characteristic. Let $\lfloor x \rfloor = max\{n \in \mathbb{Z} \mid n \leq x\}$.

Theorem 2.2. *Let f, $g \in K[[\mathbf{x}]]$ be such that $ord(f) = s \geq 2$. Let $k \geq 2$. Then the following are equivalent:*
i) $f \overset{c}{\sim} g$
ii) $T_k(f) \cong T_k(g)$ as K-algebras for some (equivalently for all) k such that

$$\mathfrak{m}^{\left\lfloor \frac{k+2s}{2} \right\rfloor} \subset \mathfrak{m}\langle f \rangle + \mathfrak{m}^2 j(f).$$

Corollary 2.3. *Let $\tau(f) < \infty$. Then $f \overset{c}{\sim} g$ iff $T_k(f) \cong T_k(g)$ as K-algebras for some (equivalently for all) $k \geq 2\tau(f) - 2s + 4$.*

We set for any ideal $I \subset K[[\mathbf{x}]]$,

$$ord(I) := max\{l \in \mathbb{N} \mid I \subset \mathfrak{m}^l\}.$$

Theorem 2.4. *Let f, $g \in K[[\mathbf{x}]]$ be such that $ord(f) = s \geq 2$ and let $s' = ord(j(f))$. Then the following are equivalent:*
i) $f \overset{r}{\sim} g$
ii) $M_k(f) \cong M_k(g)$ as $K[[t]]$- algebras for some (equivalently for all) k such that

$$\mathfrak{m}^{\left\lfloor \frac{k+s+s'+1}{2} \right\rfloor} \subset \mathfrak{m}^2 j(f).$$

Corollary 2.5. *Let $\mu(f) < \infty$. Then $f \overset{r}{\sim} g$ iff $M_k(f) \cong M_k(g)$ as $K[[t]]$-algebras for some (equivalently for all) $k \geq 2\mu(f) - s - s' + 3$.*

Remark 2.6. (1) Since ideal-membership in power series rings can be effectively tested (e.g. by standard basis methods, cf. [GP07] and [DGPS12]) the bounds for k in Theorems 2.2 and 2.4 can be effectively computed. Corollaries 2.3 resp. 2.4 provide the simple bounds $k \geq 2\tau(f)$ resp. $k \geq 2\mu(f)$.

(2) It was proved in [Sho76], see also [Yau83], that for an isolated quasi-homogeneous singularity $f \in \mathfrak{m} \subset \mathbb{C}\{\mathbf{x}\}$ and any $g \in \mathfrak{m}$, $M(f) \cong M(g)$ as $\mathbb{C}$-algebras implies that $f \overset{r}{\sim} g$.

This theorem does not have an analogue in characteristic p, even not if we use the higher Milnor algebras. For example, for $f = x^{p+1} + y^{p+1}$ and $g = f + x^p y$ we have $M_k(f) = M_k(g)$ as K-algebras for all k, but $f \overset{r}{\not\sim} g$.

3 Proofs

Proof of Theorem 2.2. i) $\Rightarrow$ ii). By definition of contact equivalence, there are $\varphi \in Aut(K[[\mathbf{x}]])$ and $u \in K[[\mathbf{x}]]^*$ such that $g = u\varphi(f)$. For arbitrary k, we have

$$\left\langle g, \mathfrak{m}^k j(g) \right\rangle = \left\langle u\varphi(f), \mathfrak{m}^k j\left[u\varphi(f)\right] \right\rangle = \left\langle \varphi(f), \mathfrak{m}^k j(\varphi(f)) \right\rangle =$$

$$= \left\langle \varphi(f), \mathfrak{m}^k \varphi(j(f)) \right\rangle = \varphi(\left\langle f, \mathfrak{m}^k j(f) \right\rangle).$$

ii) $\Rightarrow$ i). Suppose that for some k such that $\mathfrak{m}^{\left\lfloor \frac{k+2s}{2} \right\rfloor} \subset \mathfrak{m}\left\langle f \right\rangle + \mathfrak{m}^2 j(f)$, φ is an isomorphism of the K-algebras in ii). Then by the Lifting Lemma [GLS06 - Lemma 1.23], φ lifts to an isomorphism $\tilde{\varphi}$: $K[[\mathbf{x}]] \to K[[\mathbf{x}]]$ with $\tilde{\varphi}(\left\langle f, \mathfrak{m}^k j(f) \right\rangle) = \left\langle g, \mathfrak{m}^k j(g) \right\rangle$. Since

$$\tilde{\varphi}\left(\left\langle f, \mathfrak{m}^k j(f) \right\rangle\right) = \left\langle \tilde{\varphi}(f), \mathfrak{m}^k \tilde{\varphi}(j(f)) \right\rangle = \left\langle \tilde{\varphi}(f), \mathfrak{m}^k j(\tilde{\varphi}(f)) \right\rangle,$$

we may assume that

$$\left\langle f, \mathfrak{m}^k j(f) \right\rangle = \left\langle g, \mathfrak{m}^k j(g) \right\rangle.$$

This implies $f = h_1 g + H$ for some $h_1 \in K[[\mathbf{x}]]$ and $H \in \mathfrak{m}^k j(g)$. Since $f \in \mathfrak{m}^s$, $j(f) \in \mathfrak{m}^{s-1}$ and hence $\mathfrak{m}^k j(f) \subset \mathfrak{m}^{k+s-1} \subset \mathfrak{m}^s$ so that $g \in \mathfrak{m}^s$ and then $H \in \mathfrak{m}^{k+s-1}$. Consider two cases:
Case 1: h_1 is a unit. Since $\mathfrak{m}^{\left\lfloor \frac{k+2s}{2} \right\rfloor} \subset \mathfrak{m}\left\langle f \right\rangle + \mathfrak{m}^2 j(f)$, by Theorem 1.4, f is contact $(2\left\lfloor \frac{k+2s}{2} \right\rfloor - s - 2)$-determined. Since $f - h_1 g \in \mathfrak{m}^{k+s-1}$ and $k + s - 1 > 2\frac{k+2s}{2} - s - 2 \geq 2\left\lfloor \frac{k+2s}{2} \right\rfloor - s - 2$, we have $h_1 g \overset{c}{\sim} f$. Moreover, since h_1 is a unit, $g \overset{c}{\sim} h_1 g$. Hence, $g \overset{c}{\sim} f$.
Case 2: h_1 is not a unit. Since $g = h_2 f + G$ for some $h_2 \in K[[\mathbf{x}]]$ and $G \in \mathfrak{m}^k j(f)$, we have

$$f = h_1 g + H = h_1(h_2 f + G) + H = h_1 h_2 f + h_1 G + H.$$

Since $h_1 \in \mathfrak{m}$ and $G \in \mathfrak{m}^{k+s-1}$, this implies

$$(1 - h_1h_2)f = h_1G + H \in \mathfrak{m}^{k+s-1}.$$

Hence, $f \in \mathfrak{m}^{k+s-1}$ since $1-h_1h_2$ is a unit. On the other hand since $ord(f) = s$, $f \notin \mathfrak{m}^{s+1}$. Therefore, $k \leq 1$, a contradiction. □

Proof of Corollary 2.3. If $f \overset{c}{\sim} g$ then the two K-algebras $T_k(f)$ and $T_k(g)$ are isomorphic for all k by the proof of Theorem 2.2. Conversely, suppose that $T_k(f) \cong T_k(g)$ for $k \geq 2\tau(f) - 2s + 4$. The fact that $dim_K T(f) = \tau$ implies $\mathfrak{m}^\tau \subset tj(f)$ so that

$$\mathfrak{m}^{\tau+2} \subset \mathfrak{m}^2 \langle f \rangle + \mathfrak{m}^2 j(f) \subset \mathfrak{m} \langle f \rangle + \mathfrak{m}^2 j(f).$$

On the other hand, since $k \geq 2\tau - 2s + 4$ we have $\frac{k+2s}{2} \geq \tau + 2$ so that $\left\lfloor \frac{k+2s}{2} \right\rfloor \geq \tau + 2$. This implies $\mathfrak{m}^{\left\lfloor \frac{k+2s}{2} \right\rfloor} \subset \mathfrak{m}^{\tau+2} \subset \mathfrak{m} \langle f \rangle + \mathfrak{m}^2 j(f)$. By Theorem 2.2, we get $f \overset{c}{\sim} g$. □

Proof of Theorem 2.4. *i)* $\Rightarrow$ *ii)*. By assumption, there is $\phi \in Aut(K[[\mathbf{x}]])$ such that $g = \phi(f)$. Then for arbitrary k, we have

$$\phi(\mathfrak{m}^k j(f)) = \mathfrak{m}^k j(\phi(f)) = \mathfrak{m}^k j(g).$$

The K-algebra automorphism ϕ induces the K-algebra isomorphism

$$\bar{\phi} : M_k(f) \to M_k(g), \quad \bar{h} \mapsto \overline{\phi(h)},$$

which is a $K[[t]]$-algebra isomorphism since $\bar{\phi}(\bar{f}) = \overline{\phi(f)} = \bar{g}$.
ii) $\Rightarrow$ *i)*. Using the lifting lemma, the $K[[t]]$-algebra isomorphism

$$\bar{\phi} : M_k(f) \to M_k(g)$$

lifts to a K-algebra isomorphism $\phi \in Aut(K[[\mathbf{x}]])$ such that $\phi(\mathfrak{m}^k j(f)) = \mathfrak{m}^k j(g)$. Since $\bar{\phi}(\bar{f}) = \bar{g}$ and $\bar{\phi}(\bar{f}) = \overline{\phi(f)}$ in $M_k(g)$, we get $\phi(f) - g \in \mathfrak{m}^k j(g) = \mathfrak{m}^k j(\phi(f))$. Since $ord(f)$, $ord(j(f))$, and the degree of the right determinacy of f are invariant under right equivalence, we reduce to the situation $f - g \in \mathfrak{m}^k j(f)$. This implies $f - g \in \mathfrak{m}^{k+s'}$. Moreover, since $\mathfrak{m}^{\left\lfloor \frac{k+s+s'+1}{2} \right\rfloor} \subset \mathfrak{m}^2 j(f)$, by Theorem 1.4, f is right $(2\left\lfloor \frac{k+s+s'+1}{2} \right\rfloor - s - 2)$-determined. Since $k + s' > 2\left\lfloor \frac{k+s+s'+1}{2} \right\rfloor - s - 2$, we get $f \overset{r}{\sim} g$. □

Proof of Corollary 2.5. We proved in Theorem 2.4 that if $f \overset{r}{\sim} g$ then $M_k(f)$ and $M_k(g)$ are isomorphic as $K[[t]]$-algebras for all k. Conversely, since $dim_K M(f) = \mu$, $\mathfrak{m}^{\mu+2} \subset \mathfrak{m}^2 j(f)$. Since $k \geq 2\mu - s - s' + 3$, we have $\left\lfloor \frac{k+s+s'+1}{2} \right\rfloor \geq \mu + 2$ so that

$$\mathfrak{m}^{\left\lfloor \frac{k+s+s'+1}{2} \right\rfloor} \subset \mathfrak{m}^{\mu+2} \subset \mathfrak{m}^2 j(f).$$

By Theorem 2.4, we get $f \overset{r}{\sim} g$. □

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