FB Mathematik — Wintersemester 2003/04, Set 14
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Linear Algebra I

Due date: Monday, 16/02/2004, 13h00

Exercise 53: Let V be a finite-dimensional K-vector space, $f : V \to V$ a K-linear map. Show that the following statements are equivalent:

a. $V = \mathrm{Ker}(f) \oplus \mathrm{Im}(f)$,

b. $\mathrm{Ker}(f^2) = \mathrm{Ker}(f)$.

Exercise 54: Let K be a field. We define the *trace* of a square matrix $A = \left(a_{ij}\right)_{i,j} \in \mathrm{Mat}(n \times n, K)$ as the sum of the diagonal elements of A, i. e.

$$\mathrm{Spur}(A) := \sum_{i=1}^{n} a_{i,i}.$$

Prove the following statements:

a. $\mathrm{Spur}(AB) = \mathrm{Spur}(BA)$ for $A, B \in \mathrm{Mat}(n \times n, K)$.

b. $\mathrm{Spur}(A) = \mathrm{Spur}(B^{-1}AB)$ for $A \in \mathrm{Mat}(n \times n, K)$ and $B \in \mathrm{Gl}_n(K)$.

c. If V is a finite-dimensional K-vector space and $f : V \to V$ a K-linear map, then for two bases E and F of V

$$\mathrm{Spur}\left(M_E^E(f)\right) = \mathrm{Spur}\left(M_F^F(f)\right).$$

In particular we may define $\mathrm{Spur}(f) := \mathrm{Spur}\left(M_E^E(f)\right)$, i. e. the definition is independent of the choses basis E.

d. If in c. there is an $r \in \mathbb{N}$ with $f^r = 0$, then $\mathrm{Spur}(f) = 0$.

e. If there is an $r \in \mathbb{N}$ with $A^r = 0$, then $\mathrm{Spur}(A) = 0$.

Hint to the proof of d.: Do induction on $n = \dim_K(V)$. Show first that $\mathrm{Ker}(f) \neq \{0\}$ and use then induction on $f_{V/\,\mathrm{Ker}(f)}$ – you will need the result of exercise 51 c.

Exercise 55: Let R be a commutative ring with 1 and let $a \in R$ be given. Calculate the determinant of the following matrix:

$$\begin{pmatrix} 1 & a & a^2 & \dots & a^{n-1} \\ a^{n-1} & 1 & a & \dots & a^{n-2} \\ a^{n-2} & a^{n-1} & 1 & \dots & a^{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ a & a^2 & a^3 & \dots & 1 \end{pmatrix} \in \mathrm{Mat}(n \times n, R).$$