TROPICAL SURFACE SINGULARITIES

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Abstract. In this paper, we study tropicalisations of singular surfaces in toric threefolds. We completely classify singular tropical surfaces of maximal-dimensional type, show that they can generically have only finitely many singular points, and describe all possible locations of singular points. More precisely, we show that singular points must be either vertices, or generalized midpoints and baricenters of certain faces of singular tropical surfaces, and, in some case, there may be additional metric restrictions to faces of singular tropical surfaces.

1. Introduction

This paper studies singularities of tropical surfaces in $\mathbb{R}^3$. The question what the analogue of a singularity in the tropical world should be is quite natural to ask and has consequently interested several authors recently ([3], [4], [10]). The fact that this question is hard to answer in general makes it even more intriguing. We define a point p in a tropical surface S to be singular if there is an algebraic surface $\tilde{S}$, defined over the Puiseux-series, whose tropicalisation is S and which is singular at a point $\tilde{p} \in \tilde{S}$ that tropicalises to p. Given a non-degenerate lattice polytope $\Delta \in \mathbb{R}^3$, consider the family $\mathrm{Sing}(\Delta) \subset \mathbb{P}^{\#(\Delta\cap\mathbb{Z}^3)-1}$ of singular hypersurfaces in the toric threefold defined by Δ whose defining equations have Newton polytope Δ. We assume that Δ is non-defective, i.e. that $\mathrm{Sing}(\Delta)$ is a hypersurface in $\mathbb{P}^{\#(\Delta\cap\mathbb{Z}^3)-1}$, defined by a polynomial which is then called the discriminant of Δ. The tropicalisation $\mathrm{Trop}(\mathrm{Sing}(\Delta))$ of $\mathrm{Sing}(\Delta)$ has been studied in [3] and is called the tropical discriminant. While a general member of $\mathrm{Sing}(\Delta)$ has exactly one singular point, namely a node, an analogous statement is not true in tropical geometry. The reason is that for a given singular tropical surface, there can be several singular tropical surfaces tropicalising to it, but such that the resp. singular points tropicalise to different points in the tropical surface. Consequently, there are also tropical surfaces with infinitely many singularities. The subset of singular points of a tropical surface does not seem to have any nice structure however, in particular it is not a tropical subvariety. Examples 4.5 and 4.3 of [4] show tropical curves with infinitely many resp. two singular points. We concentrate on singular tropical surfaces of maximal-dimensional type (i.e. dual to a marked subdivision where all points are marked, i.e. those which can be drawn through $\#(\Delta \cap \mathbb{Z}^3) - 2$ generic points (see Subsection 2.3). If their dual subdivision is generic (i.e. it corresponds to a cone of codimension one of the secondary fan, see 2.3), such tropical surfaces have only finitely many singularities. We classify the possible locations of singular points.

2010 Mathematics Subject Classification: Primary: 14T05, Secondary: 51M20, 05B35.

Our study is closely related to [4], which deals with singular tropical hypersurfaces of any dimension. There, a more algebraic point of view is taken however: the main result is the description of tropical singular points in terms of Euler derivatives, i.e. tropical equations are given which a point must satisfy to be singular. We concentrate more on the geometry of singular tropical surfaces.

Our paper can be viewed as a sequel to [10], where we studied tropical plane curves with a singular point. The main result of [10] is the classification of singular tropical curves of maximal-dimensional type. A singular point of a tropical curve of maximal-dimensional type is either a "crossing" of two edges, or a three-valent vertex of multiplicity 3, or it is a point on an edge e of weight two which has equal distance to the two vertices of e (or which satisfies a similar metric condition respectively). To derive this result, we used the following methods: we considered the family of algebraic curves in a toric surface with a singularity in a fixed point. This family is defined by linear equations, and so its tropicalisation is a Bergman fan which can be described in terms of weight classes of flags of flats of the corresponding matroid ([6], [1]). We studied the possible weight classes and classified the corresponding tropical curves. Fixing a different point in the torus yields a shift of the Bergman fan (see Remark 3.1 and 3.2 of [10]).

Here, we apply the same methods to the family of algebraic surfaces in a toric threefold with a singularity in a fixed point. While the basic ideas we use are the same as in [10], the classification becomes much more complicated and we have to establish and use various facts about lattice polytopes. Also, we concentrate purely on tropical surfaces with only finitely many singularities (contrary to our classification in the curve case in [10]). Our main result is the classification in Theorem 1.2 below. Such a classification is not possible in higher dimensions (see Remark 1.7). Theorem 1.1 tells us for which tropical surfaces there are only finitely many singularities. For more details and notation, see Section 4.

Theorem 1.1
Let $\Delta \subset \mathbb{R}^n$ be a non-degenerate convex lattice polytope and denote by $\mathcal{A} = \Delta \cap \mathbb{Z}^n$ the lattice points of Δ. Let $F_u = \max_{(i,j,k)\in\mathcal{A}}\{u_{(i,j,k)} + ix + jy + kz\}$ define a generic (see Definition 3.6) singular tropical hypersurface S. Assume the dual marked subdivision corresponds to a cone of codimension c in the secondary fan. Then the set of singular points in S is a union of finitely many polyhedra of dimension $c-1$.

In the following classification below, we thus want to restrict to the case $c = 1$ of tropical surfaces S whose dual marked subdivision corresponds to a cone of codimension 1 in the secondary fan. It follows that the dual marked subdivision contains a unique circuit and that every marked polytope in the subdivision which does not contain the circuit is a simplex (see Remark 2.2). We can conclude from Lemma 3.1 of [4] that every singular point of S is contained in the cell of S dual to the circuit.

In addition, we make the assumption that the tropical surface is of maximal-dimensional type, i.e. that all lattice points are marked in the dual subdivision. This assumption is natural and gives us more control over the possible locations of the singular point.

Theorem 1.2
Let $F_u = \max_{(i,j,k)\in\mathcal{A}}\{u_{(i,j,k)} + ix + jy + kz\}$ *define a singular tropical surface* S. *We assume that* S *is generic (see Definition 3.6) and dual to a marked subdivision* $T = \{(Q_1, \mathcal{A}_1), \ldots, (Q_k, \mathcal{A}_k)\}$ *(see Subsection 2.3) of maximal-dimensional type (i.e. all points in* $\mathcal{A}$ *are marked). Assume the dual subdivision corresponds to a cone of codimension* 1 *in the secondary fan. Then every marked polytope* $(Q_i, \mathcal{A}_i)$ *in* T *which does not contain the circuit is a simplex, and* S *contains only finitely many singular points. Then their possible locations, classified up to integral unimodular affine transformations, are as follows:*

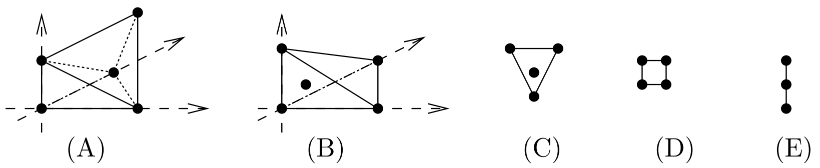

FIGURE 1. The possible circuits.

(a) *If the circuit is of dimension* 3 *(Cases (A) and (B) in Figure 1), the dual cell is a vertex* V *of* S *and this vertex is the only singular point.*

(a.1): *Either* V *is adjacent to six edges and nine* 2*-dimensional polyhedra. Then the dual polytope is a pentatope with vertices* $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ *and* $(1,p,q)$ *with* p *and* q *coprime (Case (A) in Figure 1, see also Figure 2).*

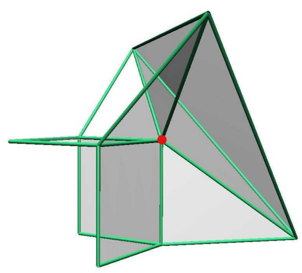

FIGURE 2. Case (a.1), a singular tropical surface dual to (A) with the singular point marked.

(a.2): *Or* V *is adjacent to four edges and six* 2*-dimensional polyhedra, just as a smooth vertex (Case (B) in Figure 1, see also Figure 3). However, if we define the multiplicity of a vertex of a tropical hypersurface analogously to the case of tropical curves as the lattice volume of the corresponding polytope in the dual subdivision, then it follows that* V *is a vertex of higher multiplicity. More precisely, the multiplicity can be 4,5,7,11,13,17,19 or 20. The dual is a tetrahedron with vertices* $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ *and, resp.,* $(3,3,4)$, $(2,2,5)$, $(2,4,7)$, $(2,6,11)$, $(2,7,13)$, $(2,9,17)$, $(2,13,19)$, *or* $(3,7,20)$.

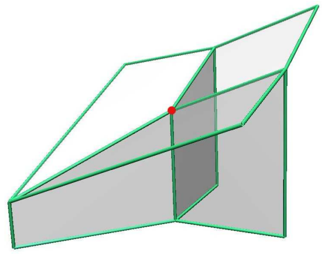

FIGURE 3. Case (a.2), a singular tropical surface dual to (B) with the singular point marked.

(b) *If the circuit is of dimension* 2 *(Cases (C) and (D) in Figure 1), the dual cell is an edge* E*. We have the following cases:*

(b.1): *E is dual to a triangle with vertices* $m_a = (0,0,0)$*,* $m_c = (0,1,2)$ *and* $m_d = (0,2,1)$*, i.e.* E *is adjacent to three* 2*-dimensional cells of* S *(Case (C) in Figure 1). Each end vertex of* E *is adjacent to four edges and six* 2*-dimensional polyhedra, just as a smooth vertex.*

(b.1.1): *If* E *is bounded, there can be a singularity at the midpoint of* E *or at points which divide* E *with the ratio* $3:1$ *(see Figure 6). Also, if one end vertex* V *of* E *is dual to a pyramid* Q *with the triangle as base and additional vertex* m_f *with* x*-coordinate* 3*, and if there is a vertex* V' *adjacent to* V *whose dual polytope is a pyramid with one face of* Q *as base and additional vertex* m_e *with* x*-coordinate* 1*, there can be a singularity on* E *whose distance to* V *is (see Figure 4)*

$$\frac{u_{m_a}}{3} - \left(\frac{u_{m_e}}{2} - \frac{u_{m_f}}{6}\right) - (u_{m_b} - u_{m_c}) \cdot \left(\frac{m_{ey}}{2} - \frac{m_{fy}}{6}\right) - (u_{m_b} - u_{m_d}) \cdot \left(\frac{m_{ez}}{2} - \frac{m_{fz}}{6}\right). \tag{1}$$

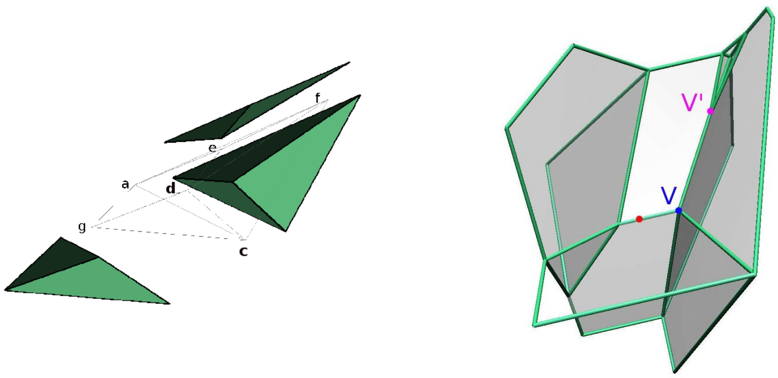

FIGURE 4. Case (b.1.1), a dual subdivision and the corresponding singular tropical surface with metric distance as in (1), with the singular point as well as V and V' marked.

(b.1.2): *If E is unbounded, then the dual to its unique end vertex V has to be a pyramid whose additional vertex has x-coordinate 3, and there has to be a vertex V' as above. The possible locations of singular points are as in (b.1.1) in this situation.*

(b.2): *E is dual to a quadrangle, i.e. adjacent to four 2-dimensional cells of S (Case (D) in Figure 1, see also Figure 5). E must be bounded and its end vertices are each adjacent to five edges and eight 2-dimensional cells. S contains a unique singular point which is the midpoint of E.*

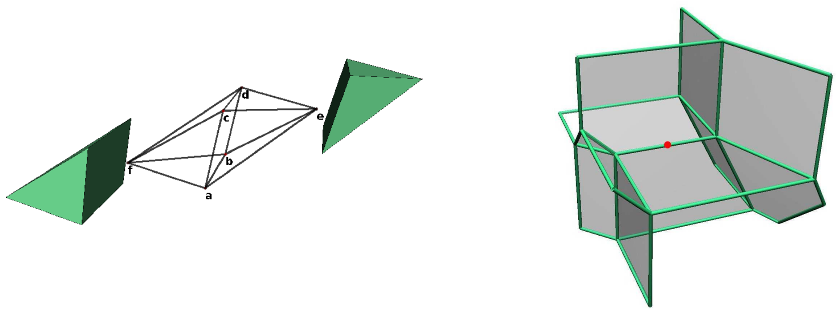

FIGURE 5. Case (b.2), a dual subdivision with circuit (D) and the corresponding singular tropical surface with the singular point marked.

(c) *If the circuit is of dimension 1 (Case (E) in Figure 1), then the dual is a 2-dimensional cell of S. Each singular point is at a* weighted barycenter *of the polytope, resp. at the* generalised midpoint *if the polytope contains two parallel edges, as specified in Subsection 4.5 below (for an image see Example 1.4).*

Subsection 4.5 referred to in statement (c) of Theorem 1.2 contains a classification of the possible shapes of the cell dual to the circuit and explains the terms *weighted barycenter* and *generalised midpoint*.

Remark 1.3
When taking the dual marked subdivision into account, we can make the statement of Case (b.1.1) of Theorem 1.2 more precise (see also Figure 6). The dual subdivision contains exactly two pyramids Δ_A and Δ_B containing the circuit as base. The lattice heights h_A and h_B of these pyramids can be one or three independently from each other. If they are both one, then S has a unique singularity at the midpoint of E. If they are both three, there can be a singularity at the midpoint, or at points whose distances from the end vertices of E depend on a neighbouring vertex V' satisfying the requirements from above. If $h_A = 1$ and $h_B = 3$, then there can be a singularity which divides E with ratio $3:1$ (closer to the vertex V_B dual to Δ_B), and there can be singularities whose distances from V_B depend on a vertex V' adjacent to V_B as before (see Subsection 4.3.1).

Example 1.4
A tropical surface S can have several singularities, since there may be several singular surfaces tropicalising to S with different images for their singular point. We

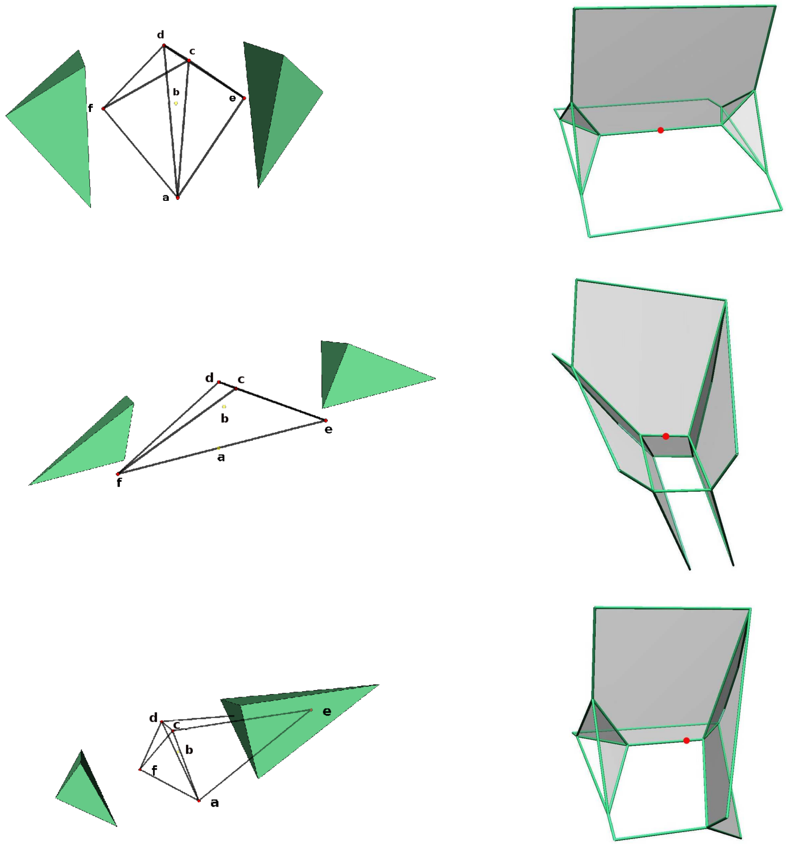

FIGURE 6. Case (b.1.1), the dual subdivision with Δ_A and Δ_B where $h_A : h_B$ is either $1 : 1$, or $3 : 3$, or $1 : 3$ leading to a singular point which divides the edge E either in the midpoint or the midpoint or with ratio $3 : 1$.

give here an example for this behaviour. Consider the polynomials

$$f = (1-3t^5-3t^8)+(-2+t^5)\cdot z+z^2+t^8\cdot\frac{1}{xy}+(t^5+t^8)\cdot y+(2t^5+t^8)\cdot x-t^5\cdot x^2yz$$

and

$$g = (1-3t^6+3t^8)-(2+t^8)\cdot z+z^2+t^8\cdot\frac{1}{xy}+(t^5-t^7)\cdot y+(t^5-2t^7)\cdot x+t^5\cdot x^2yz$$

over the field of Puiseux series. They both tropicalise to the tropical polynomial

$$F_u = \max\{0, z, 2z, -8-x-y, -5+y, -5+x, -5+2x+y+z\}$$

with $u = (0,0,0,-8,-5,-5,-5)$ and define thus the same tropical surface S. Moreover, both $V(f)$ and $V(g)$ are singular, however, $V(f)$ is singular in $(1,1,1)$ which tropicalises to $G = (0,0,0)$, while $V(g)$ is singular in $(t,t,1)$ which tropicalises to $H = (-1,-1,0)$. Thus S has two singular points on the quadrangle dual to the circuit formed by $(0,0,0)$, $(0,0,1)$, and $(0,0,2)$. The quadrangle is shown in Figure 7, and $G = \frac{1}{3}\cdot(A+B-E)$ and $H = \frac{1}{3}\cdot(C+D+E)$ are weighted barycentes of the vertices A and B respectively C and D with the virtual vertex E in the sense of Theorem 1.2 and Subsection 4.5 (see also Remark 3.7 and Examples 4.5 and 4.8).

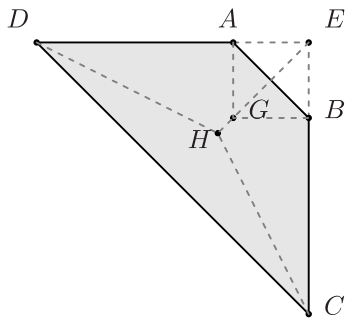

FIGURE 7. Two singular points on a tropical surface as weighted barycenters.

Theorem 1.2 gives necessary conditions for the geometry of a singular tropical surface. We can also formulate a sufficient condition, which follows immediatly from Lemma 2.4:

Theorem 1.5
Let $F_u = \max_{(i,j,k)\in\mathcal{A}}\{u_{(i,j,k)} + ix + jy + kz\}$ define a tropical surface S. We assume that S is dual to a marked subdivision of maximal-dimensional type and corresponding to a cone of codimension 1 in the secondary fan, i.e. it contains a unique circuit. Let $p \in S$ be a point in the cell dual to the circuit, and assume p satisfies conditions (a), (b) or (c) of Theorem 1.2 above.
Then S is the tropicalization of an algebraic surface with a singularity tropicalizing to p if and only if after shifting S such that p becomes the origin (and accordingly adding lineality vectors to the coefficients u such that they become equal along the circuit, see Section 3) the flag of subsets $\mathcal{F}(u)$ (see Subsection 2.4) satisfies the conditions of Lemma 2.4 resp. is in the boundary of such a flag.

Note that Theorems 1.2 and 1.5 together give a complete classification of maximal-dimensional tropical surfaces and their singular points, and both the necessary and sufficient criteria are easy to verify in any concrete example. For circuits of type (A), (B) and (D) (see Figure 1), the condition about the flag of subsets of Theorem 1.5 holds automatically. For circuits of type (C) and (E), the condition may impose extra non-local conditions. Non-local here means that they involve cells of the tropical surface which are not faces of the cell dual to the circuit.

Example 1.6
Let us consider the point configuration $\mathcal{A}$ with $m_a = (0,0,0)$, $m_b = (0,1,1)$, $m_c = (0,1,2)$, $m_d = (0,2,1)$, $m_e = (1,1,1)$, $m_f = (3,0,2)$ and $m_g = (-1,1,0)$, and a tropical surface S defined by a tropical polynomial F_u, $u = (u_a, u_b, \ldots, u_g)$, satisfying Condition (b.1.1) in Theorem 1.2. We assume that $u_a = u_b = u_c = u_d \geq u_e, u_f, u_g$, or equivalently, we assume that the edge E dual to the circuit satsifies $y = z = 0$. From Theorem 1.2 and Remark 1.3 we know that S can be singular

at a point p which divides E with ratio $3:1$, or at a point q whose position is determined by Equation (1). In the latter situation the position of the singular point is not *locally* determined, i.e. it is not determined purely by the linear forms in F_u corresponding to the part of the subdivision which is dual to to the edge E and its end points, but it involves the neighbouring vertex V' of S determined by the polytope m_a, m_d, m_e, m_f (see also Figure 4).

We now want to specify the sufficient conditions we observe in Theorem 1.5 in this situation in order to decide which of the points p or q is a singular point of the tropical surface. If we move p to the origin, this corresponds to adding the vector $\frac{u_e-u_f}{2}\cdot(0,0,0,0,1,3,-1)$ to the coefficient vector $(u_a,u_b,u_c,u_d,u_e,u_f,u_g)$ (see Section 3). The new coefficients satisfy the conditions of Lemma 2.4 if and only if the new g-coefficient is smaller than the new e and f-coefficients which became equal. This is the case if and only if $2u_e > u_g+u_f$. Thus the point p is a singular point of S if and only if $2u_e \geq u_g+u_f$. Moving the point q to the origin corresponds to adding the vector $\frac{u_g-u_f}{2}\cdot(0,0,0,0,1,3,-1)$ to the coefficient vector. The new coefficient vector satisfies the conditions of Lemma 2.4 if and only if $2u_e < u_g+u_f$. Thus q is a singular point of S if and only if $2u_e \leq u_g+u_f$. If $2u_e = u_g+u_f$ then $p=q$ and the coefficient vector is in the boundary of two weight classes satisfying the conditions of Lemma 2.4. In any case S has a unique singular point — exactly one of p and q is liftable, depending on the coefficients u.

Remark 1.7
The classification is closely related to the study of Δ-equivalence classes of marked subdivisions (see Section 11.3 of [8]), since by Theorem 1.1 of [3], the tropical discriminant (which equals the codimension one subfan of the secondary fan that groups maximal dimensional cones of the secondary fan into Δ-equivalence classes) equals the Minkowski sum of the tropicalisation of the family of curves with a singularity in a fixed point and its lineality space. This explains why the dual marked subdivisions of maximal-dimensional singular tropical surfaces correspond to codimension one cones of the secondary fan which separate two non-Δ-equivalent maximal cones (see 11.3.10 of [8]): understanding the combinatorial types of singular tropical hypersurfaces is equivalent to understanding Δ-equivalence classes. Since understanding Δ-equivalence classes combinatorially is an open problem for dimension larger than 3, this connection restricts further generalizations of Theorems 1.2 and 1.5 to higher dimensions.

This paper is organised as follows. In the introduction the main result of the paper, Theorem 1.2, is stated. In Section 2 the basic notions will be introduced, most prominantly the tropicalisation $\mathrm{Trop}(\mathrm{Ker}(A))$ of the family of surfaces in a given toric threefold which are singular at $(1,1,1)$. We also explain how $\mathrm{Trop}(\mathrm{Ker}(A))$ comes in a natural way with a fan structure induced by the matroid associated to A, and we describe the full-dimensional cones of this fan as weight classes associated to flags of flats (see Lemma 2.4). It is well known from [3] that the secondary fan of the point configuration corresponding to A is the Minkowski sum of $\mathrm{Trop}(\mathrm{Ker}(A))$ and the lineality space. In Section 3 we reconsider how the Minkowski sum of a cone in $\mathrm{Trop}(\mathrm{Ker}(A))$ with the lineality space can lie in cones of the secondary fan, and we use this to introduce the notion of a generic singular surface as well as to prove Theorem 1.1. Section 4 is devoted to the classification of generic singular tropical surfaces of maximal-dimensional type, and the classification works along the classification of weight classes in Lemma 2.4. For the classification also polytopes with certain properties have to be classified, and the corresponding classification results can be found in Section 4 too.

1.1. **Acknowledgements.** We would like to thank Christian Haase for useful discussions. The images were obtained with the aid of Polymake [7], Javaview [13], jReality [9], tropicalinsect [2], xfig and texdraw. The authors were supported by the Hermann-Minkowski Minerva Center for Geometry at the Tel Aviv University, and by the DFG-grant MA 4797/3-1 as part of the priority program SPP 1489. The third author was also supported by the Israeli Science Foundation grant no. 448/09.

2. Notations and basic facts

In this section, we fix notations and collect basic properties of the family of surfaces with a singularity in a fixed point and its tropicalisation, the Bergman fan of the corresponding linear ideal. The content of this section is parallel to Sections 1, 2 and 3.1 of [10], only now we deal with surfaces instead of curves. We omit proofs in this section, since they are all straight-forward generalisations of the corresponding statements in [10].

2.1. **The family of surfaces with a singularity in a fixed point.** Fix a non-degenerate convex lattice polytope $\Delta \subset \mathbb{R}^3$ and denote by $\mathcal{A} = \Delta \cap \mathbb{Z}^3 = \{m_1, \ldots, m_s\}$ the lattice points of Δ. For any field $\mathbb{K}$ there is a toric threefold $\mathrm{Tor}_{\mathbb{K}}(\Delta)$ associated to Δ and it comes with the tautological line bundle $\mathcal{L}_\Delta$ generated by the global sections $\{x^i y^j z^k \ : \ (i,j,k) \in \mathcal{A}\}$. The torus $(\mathbb{K}^*)^3$ is embedded in $\mathrm{Tor}_{\mathbb{K}}(\Delta)$ via

$$\Psi_{\mathcal{A}} : (\mathbb{K}^*)^3 \longrightarrow \mathbb{P}^{\mathcal{A}}_{\mathbb{K}} : (x,y,z) \mapsto \big(x^i y^j z^k \mid (i,j,k) \in \mathcal{A}\big)$$

and inside the torus the elements in the linear system $|\mathcal{L}_\Delta|$ are defined by the equations

$$f_a = \sum_{(i,j,k)\in\mathcal{A}} a_{(i,j,k)} \cdot x^i \cdot y^j \cdot z^k = 0$$

with $a = (a_{(i,j,k)} \mid (i,j,k) \in \mathcal{A}) \in (\mathbb{P}^{\mathcal{A}}_K)^*$. $|\mathcal{L}_\Delta|$ contains a nonempty linear subsystem $\mathrm{Sing}_{\boldsymbol{p}}(\Delta)$ of surfaces with a singularity at the point $\boldsymbol{p} = (1,1,1)$. The equations for this subsystem are the linear equations

$$f_a(\boldsymbol{p}) = 0, \ \ \frac{\partial f_a}{\partial x}(\boldsymbol{p}) = 0, \ \ \frac{\partial f_a}{\partial y}(\boldsymbol{p}) = 0, \ \ \frac{\partial f_a}{\partial z}(\boldsymbol{p}) = 0,$$

or equivalently we can say that the family $\mathrm{Sing}_{\boldsymbol{p}}(\Delta)$ is the kernel of the $4 \times s$ matrix

$$A = \begin{pmatrix} 1 & \ldots & 1 \\ m_1 & \ldots & m_s \end{pmatrix}.$$

Notice that A is just the matrix of the point configuration $\mathcal{A}$, after raising the points to the $\{t = 1\}$-plane in $\mathbb{R}^4$, if we choose the coordinates (t,x,y,z) on $\mathbb{R}^4$.

2.2. **Tropicalisations.** For tropicalisations, we use an algebraically closed field $\mathbb{K}$ with a non-archimedean valuation $\mathrm{val} : \mathbb{K}^* \longrightarrow \mathbb{R}$ whose value group is dense in $\mathbb{R}$, e.g. the algebraic closure $\overline{\mathbb{C}(t)}$ of the field of rational functions over $\mathbb{C}$, or $\mathbb{C}\{\{t\}\}$ the field of Puiseux series, or a field of generalised Puiseux series as in [11]. In each of these cases the elements of the field can be represented by generalised power series of the form

$$p = a_1 t^{q_1} + a_2 t^{q_2} + \ldots$$

with complex coefficients and real exponents, and the valuation maps p to the least exponent q_1 whose coefficient a_1 is non-zero.

For an ideal $I \subset \mathbb{K}[x_1^{\pm}, \ldots, x_n^{\pm}] = \mathbb{K}[\boldsymbol{x}^{\pm}]$ determining a variety $V = V(I) \subset (\mathbb{K}^*)^n$ we define the *tropicalisation* of V to be

$$\mathrm{Trop}(V) := \overline{\{(-\mathrm{val}(x_1), \ldots, -\mathrm{val}(x_n)) \mid (x_1, \ldots, x_n) \in V(I)\}},$$

i.e. we map V componentwise with the negative of the valuation map and take the topological closure in $\mathbb{R}^n$. If the ideal I is generated by homogeneous polynomials we may alternatively consider $V(I)$ inside $\mathbb{P}^n_{\mathbb{K}}$ and we, consequently, should consider $\mathrm{Trop}(V)$ modulo the linear space spanned by $(1, \ldots, 1)$, i.e. we should identify $\mathrm{Trop}(V)$ with its image in $\mathbb{R}^n/(1, \ldots, 1)$.

We use these conventions in two situations:

- *The tropicalisation of* $\mathrm{Sing}_{\boldsymbol{p}}(\Delta) = \ker(A)$*:* The linear space $V = \ker(A)$ is defined by linear equations over $\mathbb{Q}$. This is thus an example of the *constant coefficient case*, where the ideal $I = I(V)$ defining a variety V is generated by polynomials in $\mathbb{C}[x_1, \ldots, x_s]$. In the constant coefficient case, $\mathrm{Trop}(V)$ is always a fan, and since $V = \ker(A)$ is a linear space, $\mathrm{Trop}(V)$ is the socalled *Bergman fan* of I ([6], [1]). We will study the Bergman fan $\mathrm{Trop}(\ker(A))$ further in Subsection 2.4. Note, since the linear generators of A are homogeneous, we will consider $\mathrm{Trop}(V)$ modulo the vector space spanned by $(1, \ldots, 1)$ as mentioned above. That is, we consider $\mathrm{Trop}(\mathrm{Sing}_{\boldsymbol{p}}(\Delta)) = \mathrm{Trop}(\mathrm{Ker}(A))$ as a fan in $\mathbb{R}^{s-1} = \mathbb{R}^{\mathcal{A}}/(1, \ldots, 1)$.
- *The tropicalisation of a surface* $V(f_a)$ *with* $a \in \mathrm{Sing}_{\boldsymbol{p}}(\Delta)$*:* This is an example of a tropical hypersurface. If V is a hypersurface defined by $f = \sum a_m \boldsymbol{x}^m$, then its tropicalisation equals the locus of non-differentiability of the *tropical polynomial*

  $$\mathrm{trop}\, f : \mathbb{R}^n \longrightarrow \mathbb{R} : \boldsymbol{x} \mapsto \max\{-\mathrm{val}(a_m) + m \cdot \boldsymbol{x}\}$$

  by Kapranov's Theorem (see [5, Theorem 2.1.1]).

Let us first study the hypersurface case more closely.

2.3. **Tropical hypersurfaces and dual marked subdivisions.** Tropical hypersurfaces are dual to marked subdivisions, so we begin by repeating shortly some basic definitions about marked subdivisions. For more details, see [8, Chapter 7] or [12].

A *marked polytope* is a d-dimensional convex lattice polytope Q in $\mathbb{R}^d$ together with a subset $\mathcal{A}$ of the lattice points in $Q \cap \mathbb{Z}^d$ containing the vertices of Q.

A *marked subdivision* of a polytope Δ is a collection of marked polytopes, $T = \{(Q_1, \mathcal{A}_1), \ldots, (Q_k, \mathcal{A}_k)\}$, such that

- $\Delta = \bigcup_{i=1}^{k} Q_i$,
- $Q_i \cap Q_j$ is a face (possibly empty) of Q_i and of Q_j for all $i, j = 1, \dots, k$,
- $\mathcal{A}_i \cap (Q_i \cap Q_j) = \mathcal{A}_j \cap (Q_i \cap Q_j)$ for all $i, j = 1, \dots, k$.

We do not require that $\bigcup_{i=1}^{k} \mathcal{A}_i = \Delta \cap \mathbb{Z}^d$.

We define the *type* of a marked subdivision to be the subdivision, i.e. the collection of the Q_i, without the markings.

For a finite subset $\mathcal{A}$ of the lattice $\mathbb{Z}^d$ we denote by $\mathbb{R}^{\mathcal{A}}$ the set of vectors indexed by the lattice points in $\mathcal{A}$. A point $u \in \mathbb{R}^{\mathcal{A}}$ induces a *marked subdivision* of Δ by considering the convex hull of

$$\{(m, u_m) \mid m \in \mathcal{A}\} \subset \mathbb{R}^d \times \mathbb{R} \tag{2}$$

in $\mathbb{R}^{d+1}$, and projecting the upper faces onto $\mathbb{R}^d$. A lattice point m is marked if the point (m, u_m) is contained in one of the upper faces. Marked subdivisions of Δ obtained in this way are called *regular* or *coherent*. We say two points u and u' in $\mathbb{R}^{\mathcal{A}}$ are equivalent if and only if they induce the same regular marked subdivision of Δ. This defines an equivalence relation on $\mathbb{R}^{\mathcal{A}}$ whose equivalence classes are the relative interiors of convex cones. The collection of these cones is the *secondary fan* of Δ.

Regular marked subdivisions of Δ are dual to tropical hypersurfaces (see e.g. [12, Prop. 3.11]). Given a point $u \in \mathbb{R}^{\mathcal{A}}$ it defines a tropical hypersurface S_F as the locus of non-differentiability of the tropical polynomial

$$F_u = \max\{u_m + m \cdot \boldsymbol{x} \mid m \in \mathcal{A}\},$$

and it defines a regular subdivision of Δ. Each k-dimensional polytope in the subdivision is dual to a $d-k$-dimensional orthogonal polyhedron of the tropical hypersurface.

For tropical surfaces dual to a marked subdivision of a polytope in $\mathbb{R}^3$, this means more precisely:

- each 3-dimensional polytope in the subdivision is dual to a vertex of the tropical surface;
- each 2-dimensional face in the subdivision is dual to an edge of the tropical surface, which is perpendicular to the plane spanned by the 2-dimensional face;
- each edge of the subdivision is dual to a perpendicular 2-dimensional polyhedron of the tropical surface. The weight of a 2-dimensional polyhedron of the tropical surface is defined to be $\#(e \cap \mathbb{Z}^2) - 1$, where e is the dual edge in the marked subdivision.

The duality implies that we can deduce the type of the marked subdivision from the tropical hypersurface S_F, but not the markings. To deduce the markings, we need to know the coefficients u_m.

Obviously, the vector $(1, \dots, 1)$ is contained in the lineality space of the secondary fan. Therefore we can mod out this vector and consider the resulting fan in $\mathbb{R}^{s-1} = \mathbb{R}^{\mathcal{A}}/(1, \dots, 1)$ with $s = \#\mathcal{A}$. We have seen above that every point u in $\mathbb{R}^{\mathcal{A}}$ defines a tropical hypersurface via the tropical polynomial $F_u = \max\{u_m + m \cdot \boldsymbol{x}\}$. Of course, adding 1 to each coefficient u_m does not change the tropical hypersurface associated to this polynomial. Hence if we consider $\mathbb{R}^{\mathcal{A}}$ as a parametrising space for tropical hypersurfaces, it makes sense to mod out the linear space spanned by $(1,\ldots,1)$, and we will do so in what follows. By abuse of notation, we call the fan in $\mathbb{R}^{s-1}$ that we get from the secondary fan in this way also the *secondary fan*.

The identification of $\mathbb{R}^{\mathcal{A}}$ with $\mathbb{R}^s$, $s = \#\mathcal{A}$, is done by fixing an ordering of the elements of $\mathcal{A}$, say $m_1,\ldots,m_s$. When referring to an element $u \in \mathbb{R}^{\mathcal{A}} = \mathbb{R}^s$ we will sometimes refer to the coordinates of u as u_m with $m \in \mathcal{A}$ and sometimes simply as u_i with $i = 1,\ldots,s$. This should not lead to any ambiguity.

Let $T = \{(Q_l, \mathcal{A}_l) \mid l = 1,\ldots,k\}$, be a marked subdivision of Δ.

$$\ker(A) = \left\{(a_m) \in \mathbb{R}^{\mathcal{A}} \;\middle|\; \sum\nolimits_m a_m \cdot m = 0,\ \sum\nolimits_m a_m = 0\right\}$$

is the space of affine relations among the lattice points m of Δ. For any l, let

$$L_{\mathcal{A}_l} = \{(a_m) \in L \mid a_m = 0 \text{ for } m \notin \mathcal{A}_l\}$$

be the space of affine relations among the elements of $\mathcal{A}_l$, and let L_T be their sum.

Lemma 2.1
The codimension of the cone of the secondary fan corresponding to the marked subdivision T equals $\dim(L_T)$.

In particular, a cone in the secondary fan corresponding to a marked subdivision is top-dimensional if and only if the marked subdivision is a triangulation, i.e. all polytopes Q_i are triangles and in each Q_i no other point besides the vertices is marked.

For a proof, see [8, Corollary 2.7].

Remark 2.2
A cone in the secondary fan is of codimension one if it contains exactly one circuit. Here, a *circuit* is a set of lattice points that is affinely dependent but such that each proper subset is affinely independent. A circuit in 3-space consists either of the 5 vertices of a pentatope such that each subset of 4 vertices spans the space (A), or of the four vertices of a simplex and an interior point (B), or of 4 points in a plane as in (C) / (D), or of 3 points on a line (E), as depicted in Figure 8.

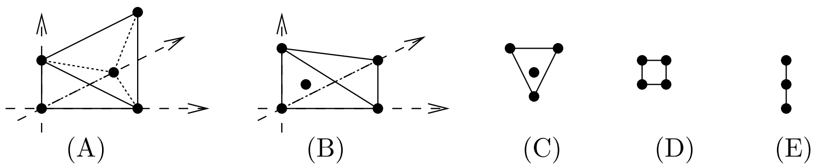

FIGURE 8. Circuits in 3-space.

Given a tropical surface S, we have seen above that it is dual to a type $\alpha = \{Q_1,\ldots,Q_k\}$ of a marked subdivision. We call α also the *type of the tropical surface*. We can parametrise all tropical surfaces of a given type by an unbounded polyhedron in $\mathbb{R}^{3\cdot v}$ where v denotes the number of vertices of S. We associate a point in $\mathbb{R}^{3\cdot v}$ to a tropical surface by collecting all coordinates of vertices. The polyhedron is defined by equations and inequalities that we can deduce from the type and that tell us which vertices are connected by an edge of which direction. We define the *dimension* $\dim(\alpha)$ of a type α to be the dimension of this parametrising polyhedron.

For the following lemma recall that we consider the secondary fan of Δ as a fan in $\mathbb{R}^{\mathcal{A}}/(1,\ldots,1)$.

Lemma 2.3
Given a marked subdivision $T = \{(Q_l, \mathcal{A}_l)\}$ of Δ of type α, we have

$$\dim(\alpha) \leq \dim(C_T),$$

where C_T denotes the cone of the secondary fan corresponding to T. Equality holds if and only if in T all lattice points of Δ are marked, i.e. if $\bigcup_l \mathcal{A}_l = \Delta \cap \mathbb{Z}^3$.

The proof is analogous to Lemma 2.5 of [10].

Since many tropical polynomials can induce the same tropical surface, the secondary fan is not the parameter space for tropical surfaces. If we consider a subfan of the secondary fan of some pure dimension, then the union of cones corresponding to marked subdivisions with all lattice points marked as above will serve as parameter space. Since $\mathrm{Trop}(\mathrm{Sing}(\Delta))$ is a subfan of the secondary fan (see Section 3), we can conclude that singular tropical surfaces such that all lattice points in the dual marked subdivision are marked form the parameter space. In particular, such a singular tropical surface can be fixed by imposing $\#(\Delta \cap \mathbb{Z}^3) - 2$ point conditions. This explains our interest for singular tropical surfaces dual to marked subdivisions with all lattice points marked. We say they are of *maximal-dimensional type*.

2.4. **The tropicalisation of $\mathrm{Sing}_{\boldsymbol{p}}(\Delta) = \ker(A)$.** We use the following known results about the tropicalisation of linear spaces ([18], § 9.3, [6], [1]). The tropicalisation of the linear space $\ker(A)$ depends only on the matroid M associated to A as follows: we define M by its collection of circuits, which are minimal sets $\{i_1, \ldots, i_r\} \subset \{1, \ldots, s\}$ such that the columns $b_{i_1}, \ldots, b_{i_r}$ of a Gale dual B of A are linearly dependent. A Gale dual is a matrix B whose rows span the kernel of A. Given $u \in \mathbb{R}^s$, let $\mathcal{F}(u)$ denote the unique *flag of subsets*

$$\emptyset =: F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_k \subsetneq F_{k+1} := \{1, \ldots, s\}$$

such that

$$u_i < u_j \iff \exists\, l \,:\, i \in F_{l-1} \text{ and } j \notin F_{l-1}.$$

In particular,

$$u_i = u_j \iff \exists\, l \,:\, i, j \in F_l \setminus F_{l-1}.$$

The *weight class* of a flag $\mathcal{F}$ is the set of all u such that $\mathcal{F}(u) = \mathcal{F}$. We say that a flag of subsets $\mathcal{F}(u)$ is in the boundary of a flag of subsets $\mathcal{F}(v)$ if $\mathcal{F}(u)$ can be obtained from $\mathcal{F}(v)$ by removing some of the F_i. Note that the weight class of a flag is an open cone, and $\mathcal{F}(u)$ is in the boundary of $\mathcal{F}(v)$ if and only if u is in the boundary of the weight class of $\mathcal{F}(v)$.

A flag $\mathcal{F}$ is a *flag of flats* of the Gale dual B of A respectively of the associated matroid M if the linear span of the vectors $\{b_j \mid j \in F_i\}$ contains no b_k with $k \notin F_i$. As before, the vectors b_j denote the columns of B. It follows from Theorem 1 of

[1] resp. Theorem 4.1 of [6] that the Bergman fan of a matroid M is the set of all weight classes of flags of flats of M.

As a consequence, we can study Trop(ker(A)) by studying weight classes of flags of flats of a Gale dual of A. Note that since A is a $4 \times s$-matrix, maximal flags of flats can be identified with flags of $s-4$ subspaces $V_i \subset \mathbb{R}^{s-4}$:

$$\{0\} \subsetneqq V_1 \subsetneqq \ldots \subsetneqq V_{s-4},$$

where each V_i is generated by a subset of the column vectors b_j of the Gale dual B of A indexed by the set F_i, and the vectors $\{b_j \mid j \in F_i\}$ are all the column vectors of the Gale dual that are contained in the subspace V_i. In particular, $F_{s-4} = \{1, \ldots, s\}$. We set $F'_i := F_i \setminus F_{i-1}$. Each F'_i must of course consist of at least one element j. Since we have s vectors in total, we have 4 "extra" vectors that can a priori belong to any of the F'_i. In the next lemma, we show how the four extra vectors can be spread.

Lemma 2.4
With the notation from above, for each flag of flats $\mathcal{F} = \mathcal{F}(u)$ of a Gale dual B of A we have either

(a) $\#F'_i = 1$ *for all* $i = 1, \ldots, s-5$ *and* $\#F'_{s-4} = 5$, *or*
(b) $\#F'_{s-4} = 4$ *and there is a* $j \in \{1, \ldots, s-5\}$ *with* $\#F'_j = 2$, *or*
(c) $\#F'_{s-4} = 3$ *and there is a* $j \in \{1, \ldots, s-5\}$ *with* $\#F'_j = 3$, *or*
(d) $\#F'_{s-4} = 3$ *and there are* $i < j \in \{1, \ldots, s-5\}$ *with* $\#F'_i = \#F'_j = 2$.

In each case, the lattice points corresponding to the indices in F'_{s-4} form a circuit. In the first case, this is a circuit of type (A) or (B) as in Remark 2.2, in the second case of type (C) or (D), and in the third and fourth case of type (E).

In the second case, all points m_r with $r \in F'_l$, $l > j$, are on the same plane as the four points of F'_{s-4}, and none of the points with $r \in F'_j$ is on this plane.

In the third case, all points m_r with $r \in F'_l$, $l > j$, are on the same line as the three points of F'_{s-4}, and each choice of two of the points in F'_j spans the space together with the three points of F'_{s-4}.

In the fourth case, all points m_r with $r \in F'_l$, $l > j$, are on the same line as the three points of F'_{s-4}, and all points m_r with $r \in F'_l$, $j > l > i$, are on the same plane as the three points of F'_{s-4} and the two points of F'_j, and the two points of F'_i do not lie on this plane.

The proof is a straight-forward generalisation of Lemma 3.7 of [10]. Note that with this Lemma we describe only interior points of cones corresponding to weight classes of top dimension in Trop(ker(A)). The analogous statement to Remark 3.8 of [10] holds true as well: for any circuit and any choice of points satisfying the affine dependencies as above we can find a corresponding weight class in Trop(ker(A)). That means that whenever the coefficients of a tropical polynomial meet one of the above conditions, it lifts to a polynomial over $\mathbb{K}$ defining a surface with singularity at (1,1,1).

3. The tropical discriminant revisited

For $\boldsymbol{x} \in \mathbb{R}^n$ arbitrary, denote by $\boldsymbol{p_x} \in (\mathbb{K}^*)^n$ a point with $\mathrm{val}(\boldsymbol{p_x}) = \boldsymbol{x}$, and consider the family $\mathrm{Sing}_{\boldsymbol{p_x}}(\Delta)$ of surfaces with a singularity in $\boldsymbol{p_x}$. Its tropicalisation $\mathrm{Trop}(\mathrm{Sing}_{\boldsymbol{p_x}}(\Delta))$ does not depend on the choice of $\boldsymbol{p_x}$. Moreover, it follows from Remark 3.2 of [10] that it is a shift of $\mathrm{Trop}(\mathrm{Sing}_{\boldsymbol{p}}(\Delta)) = \mathrm{Trop}(\ker(A))$ by a vector which we denote by $v(\boldsymbol{x})$ whose coordinates in $\mathbb{R}^s/(1,\ldots,1)$ are given by the scalar products of the $m \in \mathcal{A}$ with $\boldsymbol{x}$.

If we let $\boldsymbol{x}$ vary over all points in $\mathbb{R}^n$, it follows that $v(\boldsymbol{x})$ varies over all points in the rowspace of the matrix A in $\mathbb{R}^s/(1,\ldots,1)$. In the following, we denote the rowspace of A in $\mathbb{R}^s/(1,\ldots,1)$ by L. Notice that L also equals the lineality space of the secondary fan.

Notation 3.1
Let $v : \mathbb{R}^n \to L$ denote the linear map sending $\boldsymbol{x}$ to $v(\boldsymbol{x}) = (m \cdot \boldsymbol{x})_{m\in\mathcal{A}}$ as above. Notice that v is a bijective linear map between vector spaces of dimension n.

This illustrates the equality $\mathrm{Trop}(\ker(A)) + \mathrm{rowspace}(A) = \mathrm{Trop}(\mathrm{Sing}(\Delta))$ which is proved in Theorem 1.1 of [3]. Since we assume that Δ yields a non-defective point configuration, it follows from [8],11.3.9, that $\mathrm{Trop}(\mathrm{Sing}(\Delta))$ is a subfan of the codimension-one-skeleton of the secondary fan. Therefore it comes with a natural fan structure given by the secondary fan. Since it equals $\mathrm{Trop}(\ker(A)) + \mathrm{rowspace}(A)$, it also comes with a natural fan structure by weight classes of the lattice of flats of the matroid of A. Let us study the relation of these two fan structures.

Notation 3.2
For a point $u \in \mathbb{R}^s/(1,\ldots,1)$, we set $C(u)$ the unique cone of the secondary fan with $u \in \mathrm{relint}(C(u))$. Notice that $C(u) = C(u + l)$ for every $l \in L$.

Definition 3.3
We call a weight class C, i.e. a cone of $\mathrm{Trop}(\ker(A))$, *defective* if there exists a point $u \in C + L$ with $\dim(C + L) < \dim(C(u))$.

This notion is related to the notion of a defective point configuration of course: if the point configuration is defective then every weight class in $\mathrm{Trop}(\ker(A))$ is also defective. In fact, we can view the defectiveness of a weight class as a "local defectiveness": if we restrict to the set of points in the union of all F_i' with $\#F_i' > 1$ (notation from Lemma 2.4), then this point configuration is defective.

Remark 3.4
If C is a weight class and $u \in C$ such that $C(u)$ has codimension one in the secondary fan, then C is defective if and only if $\mathrm{span}(C) \cap L \neq \{0\}$.

Example 3.5
We consider the point configuration $\mathcal{A} = \{m_a, m_b, \ldots, m_h\}$ with

$$m_a = (0,0,0), \quad m_b = (0,0,1), \quad m_c = (0,0,2), \quad m_d = (0,1,0),$$
$$m_e = (0,-1,0), \quad m_f = (1,0,0), \quad m_g = (1,1,0), \quad m_h = (-1,0,0)$$

and we consider the weight class

$$C = \{x_{m_a} = x_{m_b} = x_{m_c} > x_{m_d} = x_{m_e} > x_{m_f} = x_{m_g} > x_{m_h}\}.$$

The corresponding subdivision of the polytope Δ is shown in Figure 9. For a point

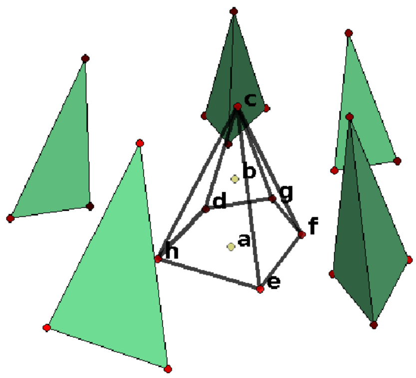

FIGURE 9. A subdivision corresponding to a defective weight class.

u in the weight class C, the corresponding cone $C(u)$ in the secondary fan is of codimension one. However, the intersection of span(C) with the lineality space in $\mathbb{R}^8/(1,\ldots,1)$ is 1-dimensional, since it contains the vector $(0,0,0,0,0,-1,-1,1)$. This shows that the weight class is defective.

Indeed, the weigth class C shares a facet with each of the two weight classes

$$C' = \{x_{m_a} = x_{m_b} = x_{m_c} > x_{m_d} = x_{m_e} > x_{m_f} = x_{m_h} > x_{m_g}\}$$

and

$$C'' = \{x_{m_a} = x_{m_b} = x_{m_c} > x_{m_d} = x_{m_e} > x_{m_g} = x_{m_h} > x_{m_f}\}.$$

The span of each of these two weight classes intersects the lineality space transversally. The cone $C(u)$ from above is just the union

$$(C + L) \cup (C' + L) \cup (C'' + L),$$

where actually $C + L$ is not needed, since it is a face of both $C' + L$ and $C'' + L$. This is thus an example that a full-dimensional weight class in Trop(Ker(A)) may lead to a lower dimensional cone in the tropical discriminant of A which lies in the interior of a full-dimensional cone of the tropical discriminant.

Note that in this example the point configuration $\mathcal{A}$ itself is not defective, however the subset consisting of points $m_a, m_b, m_c, m_d, m_e, m_f$ is.

Assume C is a non-defective weight class, then $C + L$ is contained in cones of the secondary fan of dimension equal to $\dim(C + L)$ or less. The set of all $u \in C$ with $\dim(C + L) > \dim(C(u))$ is obviously of smaller dimension than $\dim C$.

Definition 3.6
We call a point $u \in \text{Trop}(\text{Ker}(A)) + L \subseteq \mathbb{R}^s/(1,\dots,1)$ in the tropical discriminant of $\mathcal{A}$ *generic* if it cannot be written as $u = v + l$ with $l \in L$ in the lineality space and $v \in C$ in a cone of Trop(Ker(A)) such that either C is defective, or not of top dimension in Trop(Ker(A)), or $\dim(C + L) > \dim(C(v))$. The singular tropical hypersurface defined by the tropical polynomial F_u is then also called *generic*.

From the above, it is obvious that the set of generic points in the tropical discriminant is of top dimension.

Proof of Theorem 1.1:
Let $u \in \text{Trop}(\text{Sing}(\Delta))$ be generic. It follows from the definition of genericity that we can write u as a sum $v + l$ with $v \in \text{Trop}(\ker(A))$ and $l \in L$, such that the weight class C of Trop(ker(A)) which contains v in its relative interior is top-dimensional and satisfies $\dim(C + L) = \dim(C(v))$. Assume $C(v) = C(u)$ is a cone of codimension c of the secondary fan. Notice that the representation of u as a sum as above is not unique. Firstly, there might be several weight classes C in Trop(ker(A)) such that we can write u as the sum of a vector in C and a vector in L. Secondly, even if we fix one cone C, there might be several representations of u as the sum of a vector in this C and a vector in L. For now, let us fix one weight class C which allows a representation of u as $u = v + l$ with $v \in C$ and $l \in L$.

Since $\dim \text{Trop}(\ker(A)) = s - 1 - (n + 1)$ (where $s = \#\mathcal{A}$) and $v \in C$ is in a top-dimensional weight class, we have $\dim(C+L) = \dim(C)+\dim(L)-\dim(\text{span}(C)\cap L) = s-1-(n+1)+n-\dim(\text{span}(C)\cap L) = \dim(C(v)) = s-1-c$, where span($C$) denotes the smallest linear space containing C. It follows that $\dim(\text{span}(C)\cap L) = c-1$. Therefore there exists a $c-1$-dimensional polyhedron in $H \subset C$ such that for all $h \in H$ we have $v+h \in C$. We can thus write v also as $v = (v+h)-h$, where the first summand is in C and the second summand is in L, and these are all possibilities to represent v as a sum of a vector in C plus a vector in L. Consequently, we can write u as $u = (v+h)+(l-h)$ and again, these are all possibilities to represent u as a sum of a vector in C and a vector in L. It follows that F_u defines a tropical surface which is singular at all points $\boldsymbol{x}_{l-h}$, where $\boldsymbol{x}_{l-h} \in \mathbb{R}^n$ denotes the preimage of the bijective linear map sending $\boldsymbol{x} \in \mathbb{R}^n$ to $v(\boldsymbol{x}) = (m\cdot\boldsymbol{x})_{m\in\mathcal{A}}$ from Notation 3.1. Since the map v^{-1} maps the $c-1$-dimensional polyhedron $l - H$ to a $c-1$-dimensional polyhedron, it follows that all singular points of the surface of F_u that we get by decomposing u as a sum of a vector in C and a vector in L lie in a $c-1$-dimensional polyhedron. As we have seen above there may be several (but finitely many) weight classes C in Trop(ker(A)) such that we can write u as the sum of a vector in C and a vector in L, and it thus follows that the set of singular points of the tropical surface defined by F_u is a finite union of $c-1$-dimensional polyhedra. □

Remark 3.7
Recall again Example 1.4 where we had a surface S with two singular points. These two singular points arise because we can interpret the coefficient vector u of the tropical polynomial defining S in two ways as a sum of a vector in a weight class of Trop(ker(A)) and a vector in the lineality space. The two weight classes are

different. The point configuration in question corresponds to the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and the singular point $G = (0,0,0)$ on F_u with

$$u = (0,0,0,-8,-5,-5,-5) \in \mathrm{Trop}(\mathrm{Ker}(A))$$

comes from the weight class containing u. However, we can also write u as

$$u = (0,0,0,-6,-6,-6,-8) + (0,0,0,-2,1,1,3) = v + l$$

where

$$l = (0,0,0,-1,0,1,2) + (0,0,0,-1,1,0,1) = (0,0,0,-2,1,1,3)$$

belongs to the lineality space of the secondary fan of the point configuration and v belongs to some other weight class. The corresponding singular point on S is $H = (-1,-1,0)$, since we have added once the vector of x-coordinates and once the vector of y-coordinates to the weight vector v in the weight class in order to get u. This corresponds to shifting the whole surface (determined by F_v, which is singular at 0) by $(-1,-1,0)$ (see also 3.1 and before). Examples 4.5 and 4.8 give further explanations concerning this example.

This shows that even if the point u in the tropical discriminant is generic, the surface corresponding to u may have more than one singular point.

4. The classification

Now we would like to use the preparation from Section 2 to prove Theorem 1.2, i.e. to classify singular points of tropical surfaces which are of maximal-dimensional type and have only finitely many singularities. That is, we restrict ourselves to points $u \in \mathrm{Trop}(\mathrm{Sing}(\Delta))$ which are generic in the sense of Definition 3.6, and which in addition satisfy that $\dim(C(u)) = s-2$, where $C(u)$ is as in Notation 3.2. In addition, we want to restrict ourselves to the situation where the dual marked subdivision as in Subsection 2.3 has all lattice points marked (see Lemma 2.3). Since we can always write $u = v + l$ for some $a \in \mathrm{Trop}(\ker(A))$ and $l \in \mathrm{rowspace}(A)$, just as in the proof of Theorem 1.1 above, we can classify the singularities of the tropical surface defined by F_v with $v \in \mathrm{Trop}(\ker(A))$ first, and then investigate how the shift to F_u effects the location of the singular points. We thus have to consider all different types of weight classes as in Lemma 2.4, and the corresponding possible types of circuits. It turns out that in most cases we do not have to worry about the shift when passing from F_v to F_u, since we describe the location of the singular point relative to other points in the surface, e.g. as the midpoint of an edge. This midpoint is of course shifted accordingly.

4.1. **Weight class as in Lemma 2.4(a), circuit (A) of Remark 2.2.** Let $u \in \mathrm{Trop}(\ker(A))$ be in a weight class as in Lemma 2.4(a), and assume $F'_{s-4} = \{a,b,c,d,e\}$. Consider the marked subdivision defined by u as in Subsection 2.3. As the heights of the points m_a, m_b, m_c, m_d and m_e are biggest, it follows that the convex hull spanned by these points is a polytope of the subdivision. Let us first assume that this polytope is a circuit of type (A) as in Remark 2.2. The vertex of the tropical surface dual to this pentatope is at the point (x, y, z) where the maximum is attained by the corresponding five terms of $\mathrm{trop}\{u_m + m \cdot (x, y, z)\}$, in particular the five terms are equal at this vertex. That means, we can set the five terms equal and solve for x, y and z to get the position of the vertex. But since the coefficients are all equal, we get $x = y = z = 0$ when solving. Notice that $(0, 0, 0)$ is the tropicalisation of the singular point $(1, 1, 1)$.

Since we require that all lattice points are marked, this polytope cannot contain any lattice point besides these five. By Theorem 3.5 of [17], a pentatope which does not contain any lattice point besides its five vertices can be brought by an integral unimodular affine transformation to the following form: its vertices are at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(1, p, q)$ with p and q coprime. It is a bipyramid.

It follows that in this situation the node of the tropical surface is at a vertex with six adjacent edges and nine adjacent 2-dimensional polyhedra.

This settles case (a.1) of Theorem 1.2.

4.2. **Weight class as in Lemma 2.4(a), circuit (B) of Remark 2.2.** As above, it follows that the singular point $(0, 0, 0)$ is dual to the convex hull of m_a, m_b, m_c, m_d and m_e. This is a vertex of the tropical surface with four adjacent edges and six 2-dimensional polyhedra, just as a smooth vertex. However, if we define the multiplicity of a vertex of a tropical hypersurface analogously to the case of tropical curves as the lattice volume of the corresponding polytope in the dual subdivision, then it follows that the singular point is a vertex of higher multiplicity. More precisely, the multiplicity can be 4,5,7,11,13,17,19 or 20. This follows from the classification of 3-dimensional tetrahedra with one interior lattice point (and no other lattice points besides the vertices) (see [16], Theorem 7). Since we require that all lattice points are marked, the tetrahedron which is the convex hull of m_a, m_b, m_c, m_d and m_e has to be of this form. The classification states that up to integral unimodular affine transformation, such a tetrahedron is one of the following 8: it has vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and, respectively, $(3, 3, 4)$, $(2, 2, 5)$, $(2, 4, 7)$, $(2, 6, 11)$, $(2, 7, 13)$, $(2, 9, 17)$, $(2, 13, 19)$, or $(3, 7, 20)$. This settles case (a.2) of Theorem 1.2.

4.3. **Weight class as in Lemma 2.4(b), circuit (C) of Remark 2.2.** Let $F'_{s-4} = \{a, b, c, d\}$ and $F'_j = \{e, f\}$.

4.3.1. *Assume that m_e and m_f lie on different sides of the plane spanned by m_a, m_b, m_c and m_d.* Each of the two points m_e and m_f forms a pyramid with the triangle spanned by m_a, m_b, m_c and m_d as base. We assume that both of these pyramides contain no further lattice points.

Lemma 4.1
Let four lattice points m_a, m_b, m_c and m_d in an affine plane in $\mathbb{R}^3$ form a circuit of type (C) as in Remark 2.2. Let m_e be a fifth lattice point that forms a pyramid with this circuit as base and assume this pyramid contains no further lattice points. Then m_e has integral distance 1 or 3 from the plane spanned by the circuit.

Proof:
We can assume that the plane spanned by m_a, m_b, m_c and m_d is the $x = 0$-plane, and more precisely, we can assume that the (y, z)-coordinates of these four points are $(0, 0)$, $(1, 1)$, $(2, 1)$ and $(1, 2)$. Denote the triangle spanned by these points by T. Also we assume without restriction that the x-coordinate of m_e is positive. We have to show that it is then either 1 or 3. Consider a lattice point m with x-coordinate 1 and let C_m be the cone with vertex m and spanned by the rays m, $m - (0, 2, 1)$ and $m - (0, 1, 2)$. Intersect this cone with the plane $x = k$ for some choice of $k > 1$ (see Figure 10).

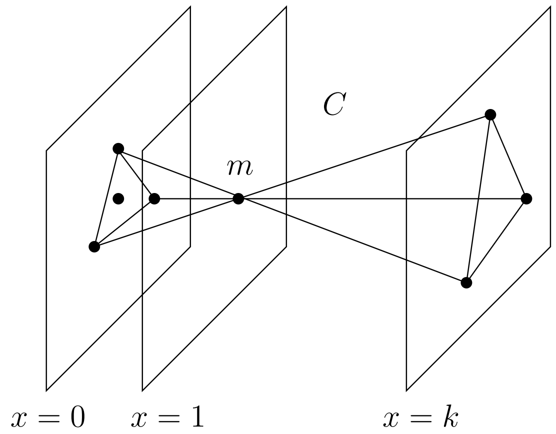

FIGURE 10. The cone C_m.

For any lattice point in $C_m \cap \{x = k\} \cap \mathbb{Z}^3$, consider the pyramid that this points forms with T as base. This pyramid will contain the point m. If we move m by a step of integer length 1, the triangle $C_m \cap \{x = k\}$ is shifted by k. Compared to T the triangle $C_m \cap \{x = k\}$ is grown by a factor of $k - 1$. Of course, we can also move m to a point with a different x-coordinate, this will add more triangles (smaller in size) such that for each point inside a triangle we know that the corresponding pyramid contains another lattice point. We show that for $k \neq 3$ the shifted triangles cover all lattice points with x-coordinate k. It follows that any pyramid with T as base and with a vertex with x-coordinate $k \neq 1, 3$ contains another lattice point. Figure 11 shows the plane $\{x = k\}$ with the k-shifts of the triangle $C_m \cap \{x = k\}$.

Let us compute the vertices of the right shaded region which is not yet covered by a triangle. Assume the left most vertex of the top right triangle has coordinates $(1, 1)$ in the plane, then it follows that the coordinates of the vertices of the shaded region are $(\frac{2k}{3} - 1, \frac{k}{3})$, $(\frac{2k}{3} + 1, \frac{k}{3} + 1)$ and $(\frac{2k}{3}, \frac{k}{3} - 1)$. Independently of k, this is a triangle of lattice area 3, with the point $(\frac{2k}{3}, \frac{k}{3})$ as midpoint from which we reach the three vertices by a lattice step to the left, down, and to the upper right. This triangle has an interior lattice point if and only if k is divisible by 3. In this case, the lattice point is $(\frac{2k}{3}, \frac{k}{3})$ (see Figure 12).

Analogously, we can compute the vertices of the left shaded region and see that it has an interior lattice point if and only if k is divisible by 3, and then this lattice point has coordinates $(\frac{k}{3}, \frac{2k}{3})$. It follows that for any k which is not divisible by 3

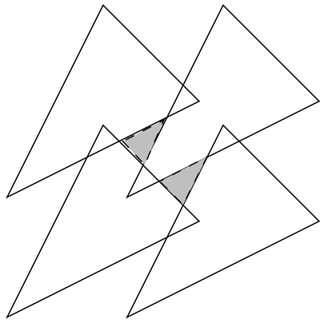

FIGURE 11. The shifts of the triangle $C_m \cap \{x = k\}$ on the plane $\{x = k\}$.

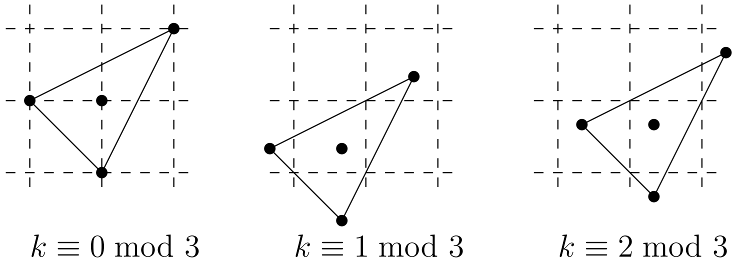

FIGURE 12. The non-covered region for different values of k.

the k-shifts of the triangle $C_m \cap \{x = k\} \cap \mathbb{Z}^3$ cover already all lattice points. That is, any pyramid with T as base and with a vertex with x-coordinate which is not divisible by 3 contains another lattice point with x-coordinate 1.

If k is divisible by 3, then a pyramid with a vertex with coordinates $(k, \frac{2k}{3} + ik, \frac{k}{3} + jk)$ or $(k, \frac{k}{3} + ik, \frac{2k}{3} + jk)$ where $i, j \in \mathbb{Z}$ does not contain a lattice point with x-coordinate 1. Here, we take the effect of the k-shifts of $C_m \cap \{x = k\} \cap \mathbb{Z}^3$ on the shaded regions into account. Let us now assume that $k = 3^l \cdot h$, where $h \neq 1$ and $3 \nmid h$. Now move m to a point with x-coordinate 3^l. It follows using the same arguments as above that any pyramid with base T and a vertex with x-coordinate k contains a lattice point with x-coordinate 3^l. Next assume that $k = 3^l$, $l \geq 2$. Using m with x-coordinate 1 as before we see that the only possibilities to get a pyramid which does contain a lattice point with x-coordinate 1 are that the vertex has (y, z)-coordinates divisible by 3^{l-1} and not by 3^l. Using m with x-coordinate 3 we see that the only possibilities to get a pyramid which does not contain a lattice point with x-coordinate 3 are that the vertex has (y, z)-coordinates divisible by 3^{l-2} and not by 3^{l-1}. As there is no vertex which satisfies both it follows that any pyramid with a vertex with x-coordinate $k = 3^l$, $l \geq 2$, contains a lattice point with x-coordinate 1, or it contains a lattice point with x-coordinate 3. In any case, it contains another lattice point. It follows that the x-coordinate of the vertex m_e can only be 1 or 3. □

Remark 4.2
There are vertices m_e with integral distance 1 and 3 to the plane containing the circuit of type (C) of Remark 2.2 such that the pyramid formed by the circuit and m_e contains no further lattice points, e.g. the convex hull of the points $(0,1,0)$, $(0,0,1)$, $(0,2,2)$ and $(3,0,2)$, or the convex hull of the points $(0,0,0)$, $(0,1,2)$, $(0,2,1)$ and $(1,0,0)$.

Now solve the equations given by the tropical polynomial to get the positions of the two vertices corresponding to the two pyramides. The x-coordinates of m_e and m_f can either be the negative of each other, or one can be 3 and the other -1. Since m_a, m_b, m_c and m_d have biggest and equal height, it follows that the edge dual to the convex hull of m_a, m_b, m_c and m_d satisfies the equations $y = 0$ and $z = 0$. If $\lambda = u_{m_a}$ is the biggest weight (the weight of m_a, m_b, m_c and m_d), and $\mu = u_{m_e}$ is the weight of m_e and m_f, it follows that the vertex dual to the pyramid with vertex m_e is at $(\mu - \lambda, 0, 0)$ (resp. $(\frac{1}{3} \cdot (\mu - \lambda), 0, 0)$) and the vertex dual to the pyramid with vertex m_f is at $(\lambda - \mu, 0, 0)$ (resp. $(\frac{1}{3} \cdot (\lambda - \mu), 0, 0)$). It follows that the singular point $(0,0,0)$ is either exactly in the middle of the edge dual to the convex hull of m_a, m_b, m_c and m_d, or subdivides the edge into parts whose distances have ratio 1:3. This explains the first cases of (b.1) in Theorem 1.2.

4.3.2. *Assume that m_e and m_f lie on the same side of the plane spanned by m_a, m_b, m_c and m_d.* As before, assume that $m_a = (0,0,0)$, $m_b = (0,1,1)$, $m_c = (0,2,1)$ and $m_d = (0,1,2)$. Of course, due to Lemma 4.1 neither m_e nor m_f can have x-coordinate different from 1 or 3, since then we would get extra lattice points, contradicting our assumption that the plane is of maximal-dimensional type. If both m_e and m_f have x-coordinate 3, then we can conclude from the proof of Lemma 4.1 that their (y,z)-coordinates differ by a multiple of 3. Therefore we would again have extra lattice points, and thus this case is not possible.

It is possible that both m_e and m_f have x-coordinate 1. Then they form together with the circuit a "*triangular roof*". Thus, the corresponding subdivision does not correspond to a cone of codimension 1 of the secondary fan, and we therefore do not consider the situation.

It is also possible that m_e has x-coordinate 1 and m_f has x-coordinate 3, e.g. if the points are $(3,0,2)$ and $(1,1,0)$. Then the point m_f with x-coordinate 3 forms a pyramid P with the circuit as base, dual to a vertex V. The point m_e forms a pyramid with a face of P spanned by m_f and two vectors of the vectors m_a, m_c, m_d as base, and this pyramid is dual to a vertex V' adjacent to V. Then V is a vertex with four adjacent edges and six 2-dimensional polyhedra.

In this situation, we cannot describe the location of the singular point as some sort of midpoint as in the earlier cases, a description which does not change when we shift. When we solve for the position of V as before, and denote by $\lambda = u_{m_a}$ the highest weight, i.e. the coefficient of the four points $m_a, \ldots, m_d$, and by $\mu = u_{m_e}$ the coefficient of m_e and m_f, then as before we get $(\frac{1}{3} \cdot (\lambda - \mu), 0, 0)$ for the coordinates of V. The singular point is at $(0,0,0)$ which is a point of distance $\frac{\lambda-\mu}{3}$ from V. This distance will not change of course when we shift, however the coefficients λ and μ are going to be changed by adding a vector in the rowspace of A. Thus, for a given point $u \in \operatorname{Trop}(\operatorname{Sing}(\Delta))$ we have to write it as a sum of a vector in our weight class and a vector in the rowspace of A. Since the rowspace of A intersects the linear space spanned by the weight class transversely in this case, there is a unique way of writing u as such a sum, and we can in fact solve for the vector in the rowspace which we need. By our choice of coordinates for the point configuration, we can deduce that we need to add the vector of y-coordinates in the rowspace $(u_{m_b} - u_{m_c})$-times and the vector of z-coordinates $(u_{m_b} - u_{m_d})$-times. Then the four new coefficients of the circuit are equal, we have

$$\begin{aligned}\lambda &= u_{m_a} \\ &= u_{m_d} + (u_{m_b} - u_{m_c}) + 2 \cdot (u_{m_b} - u_{m_d}) \\ &= u_{m_d} + 2 \cdot (u_{m_b} - u_{m_c}) + (u_{m_b} - u_{m_d}) \\ &= u_{m_b} + (u_{m_b} - u_{m_c}) + (u_{m_b} - u_{m_d}).\end{aligned}$$

If M denotes the multiple of the x-vector that we add, then M has to satisfy the equality

$$\begin{aligned}\mu &= u_{m_f} + (u_{m_b} - u_{m_c}) \cdot m_{fy} + (u_{m_b} - u_{m_d}) \cdot m_{fz} + 3 \cdot M \\ &= u_{m_e} + (u_{m_b} - u_{m_c}) \cdot m_{ey} + (u_{m_b} - u_{m_d}) \cdot m_{ez} + M\end{aligned}$$

where m_{fy} is the second coordinate of m_f etc., so that then the new coefficients of m_e and m_f are also equal. So we can solve for M and then express the distance $\frac{\lambda-\mu}{3}$ of the singular point from V as

$$\begin{aligned}\frac{\lambda - \mu}{3} =& \frac{u_{m_a}}{3} - \left(\frac{u_{m_e}}{2} - \frac{u_{m_f}}{6}\right) - (u_{m_b} - u_{m_c}) \cdot \left(\frac{m_{ey}}{2} - \frac{m_{fy}}{6}\right) \\ &- (u_{m_b} - u_{m_d}) \cdot \left(\frac{m_{ez}}{2} - \frac{m_{fz}}{6}\right).\end{aligned}$$

This settles the case (b.1) of Theorem 1.2.

4.4. **Weight class as in Lemma 2.4(b), circuit (D) of Remark 2.2.** Let $F'_{s-4} = \{a, b, c, d\}$, $F'_j = \{e, f\}$, and assume first that m_e and m_f lie on different sides of the plane spanned by m_a, m_b, m_c and m_d. Since the two points m_e and m_f have the biggest heights of points outside the plane, it follows that both form a pyramid with m_a, m_b, m_c and m_d in the subdivision. By assumption both pyramides cannot have any lattice point besides the five vertices. It follows from Lemma 3.3 of [17] that the lattice distance of both points to the plane is one. Now solve the equations given by the tropical polytope to get the positions of the two vertices corresponding to the two pyramides. Without restriction, we can assume that m_a, m_b, m_c and m_d lie in the $x = 0$-plane, it follows that the x-coordinate of m_e is -1 and the x-coordinate of m_f is 1. Since m_a, m_b, m_c and m_d have biggest and equal height, it follows that the edge dual to the convex hull of m_a, m_b, m_c and m_d satisfies the equations $y = 0$ and $z = 0$. If $\lambda = u_{m_a}$ is the biggest weight (the weight of m_a, m_b, m_c and m_d), and $\mu = u_{m_e}$ is the weight of m_e and m_f, it follows that the vertex dual to the pyramid with vertex m_e is at $(\mu - \lambda, 0, 0)$ and the vertex dual to the pyramid with vertex m_f is at $(\lambda - \mu, 0, 0)$. The singular point $(0, 0, 0)$ is thus exactly in the middle of the edge dual to the convex hull of m_a, m_b, m_c and m_d.

Now assume m_e and m_f lie on the same side of the plane spanned by m_a, m_b, m_c and m_d. It follows from Lemma 3.3 of [17] again that none of these two points can have an integral distance larger than one to the plane, or it would form a pyramid with interior lattice points. Thus both m_e and m_f have integral distance one, and form a "*triangular roof*" with m_a, m_b, m_c and m_d. Again, then the dual subdivision does not correspond to a cone of codimension 1 of the secondary fan, and we do not consider the situation. This settles case (b.2) of Theorem 1.2.

4.5. **Weight class as in Lemma 2.4(c), circuit (E) of Remark 2.2.** With the notation from Lemma 2.4(c) let $F'_{s-4} = \{a, b, c\}$ and $F'_j = \{d, e, f\}$. We may assume that $m_a = (0, 0, 0)$, $m_b = (0, 0, 1)$, and $m_c = (0, 0, 2)$. We then distinguish two cases. Either there is no plane containing the z-axis such that m_d, m_e and m_f are all on one side of the plane, or there is such a plane.

4.5.1. *Assume there is no plane through the z-axis with m_d, m_e, and m_f all on the same side of the plane.* In a first step we want to classify the possible polytopes spanned by $m_a, \ldots, m_f$, and then we will see how the corresponding tropical surfaces locally at the singular point look like.

Lemma 4.3
Let $P = \operatorname{conv}\big((0, 0, 0), (0, 0, 2), m, m'\big)$ with $m, m' \in \mathbb{Z}^3$ be a 3-dimensional lattice polytope such that

$$P \cap \mathbb{Z}^3 = \{(0, 0, 0), (0, 0, 1), (0, 0, 2), m, m'\}. \tag{3}$$

Projecting P orthogonally onto the xy-plane we get a triangle T which contains no interior lattice point and where the edges with vertex $(0, 0)$ contain no relative interior point.

Proof:
We denote by $\pi : P \longrightarrow \mathbb{R}^2 : (x, y, z) \mapsto (x, y)$ the orthogonal projection onto the xy-plane, so that $T = \pi(P)$.

Applying a suitable coordinate change in $\mathrm{Gl}_3(\mathbb{Z})$ we may assume that $m' = (0, \beta', \gamma')$ and $m = (\alpha, \beta, \gamma)$ with $\beta' > 0$. If $\beta' > 1$ then $\pi^{-1}(0, 1)$ is a line segment of Euclidean length at least one and it thus contains a lattice point in contradiction to (3). Applying a coordinate change again we can assume $0 \leq \beta < \alpha$. Since $\beta' = 1$ the edge of T connecting the vertex $(0, 0)$ with $(0, \beta')$ has no relative interior point. If $\beta = 0$ or $\beta = 1$ the statement holds obviously, since then T is a triangle of lattice height one (see Figure 13). Note here that for $\beta = 0$ necessarily $\alpha = 1$ since otherwise above $\pi^{-1}(1, 0)$ would contain an interior lattice point.

FIGURE 13. Lattice triangles of lattice height one.

We may therefore assume

$$m' = (0, 1, \gamma') \text{ and } m = (\alpha, \beta, \gamma) \text{ with } 2 \leq \beta < \alpha. \tag{4}$$

Moreover, we must have $\gcd(\alpha, \beta) = 1$, since $\alpha = k \cdot d$ and $\beta = l \cdot d$ with $d \geq 2$ would imply that $\pi^{-1}(k, l)$ is a line segment of lattice length at least one and thus contains a lattice point in contradiction to (3), see Figure 14. Therefore, also the

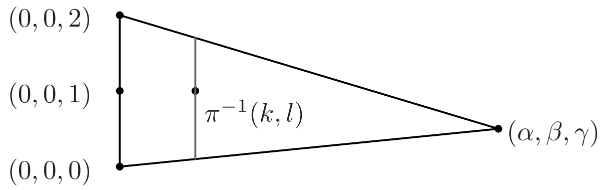

FIGURE 14. $\pi^{-1}(k,l)$ contains a lattice point.

edge of T connecting vertex $(0,0)$ with (α,β) has no relative interior point, and if we divide α by β with remainder we get

$$\alpha = q\cdot\beta + r \text{ with } 1 \le r \le \beta-1 \text{ and } q \ge 1. \tag{5}$$

The triangle T can be described by inequalities as follows

$$T = \left\{ (x,y) \;\Big|\; x \ge 0,\ y \ge \frac{\beta}{\alpha}\cdot x,\ y \le \frac{\beta-1}{\alpha}\cdot x + 1 \right\},$$

which ensures that

$$(q,1) \in T.$$

We now want to show that

$$\pi^{-1}(q,1) \cap \mathbb{Z}^3 \ne \emptyset,$$

which will be a contradiction to (3).

An easy computation shows that

$$\pi^{-1}(q,1) = \left\{ \left(q, 1, \frac{q\cdot\gamma + r\cdot\gamma' + z}{q\cdot\beta + r} \right) \;\Big|\; 0 \le z \le 2\cdot q\cdot(\beta-1) \right\},$$

and we have to show that there is a $0 \le z \le 2\cdot q\cdot(\beta-1)$ such that

$$q\cdot\beta + r \;\big|\; (q\cdot\gamma + r\cdot\gamma') + z. \tag{6}$$

We consider first the special case $\beta = 2$. Then necessarily $r = 1$ and there is of course a $0 \le z \le 2\cdot q$ such that $q\cdot\beta + r = 2\cdot q + 1$ divides $(q\cdot\gamma + \gamma') + z$.

Next we consider the special case $(q,r) = (1,\beta-1)$, and we have to check if $q\cdot\beta + r = 2\cdot\beta - 1$ divides $(\gamma + (\beta-1)\cdot\gamma') + z$ for some $0 \le z \le 2\cdot\beta - 2$, which is obviously the case.

For the general case we may now assume that $\beta \ge 3$ and $(q,r) \ne (1,\beta-1)$. Taking (4) and (5) into account it follows that

$$\beta \ge 2 + \frac{r}{q},$$

or equivalently

$$2\cdot q\cdot(\beta-1) \ge q\cdot\beta + r.$$

But then, there is definitely a $0 \le z \le 2\cdot q\cdot(\beta-1)$ such that (6) is satisfied.

So the case $2 \le \beta < \alpha$ cannot occur, and this finishes the proof. □

Proposition 4.4
Let P be a lattice polytope which is the convex hull of a circuit of type (E) and three additional lattice points m, m' and m'' such that any two of these together with the circuit span $\mathbb{R}^3$, P contains only the given six lattice points, and there is no plane through the z-axis such that m, m' and m'' are all on the same side of the plane, see Figure 15.

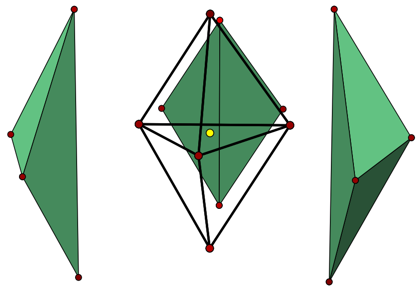

FIGURE 15. A lattice polytope P as in Proposition 4.4 with subdivision.

Then after a suitable integral unimodular affine transformation we may assume that the circuit is given by $(0,0,0)$, $(0,0,1)$, and $(0,0,2)$, and the lattice points m, m', and m'' satisfy the conditions in exactly one of the following cases:

(a) $m = (0,1,\gamma)$, $m' = (1,0,\gamma')$, *and* $m'' = (-1,-1,\gamma'')$ *with* $\gamma,\gamma',\gamma'' \in \mathbb{Z}$ *arbitrary.*
(b) $m = (0,1,\gamma)$, $m' = (2,1,\gamma')$, *and* $m'' = (-1,-1,\gamma'')$ *with* $\gamma,\gamma',\gamma'' \in \mathbb{Z}$ *such that* $\gamma \not\equiv \gamma' \pmod 2$.
(c) $m = (0,1,\gamma)$, $m' = (3,1,\gamma')$, *and* $m'' = (-1,-1,\gamma'')$ *with* $\gamma,\gamma',\gamma'' \in \mathbb{Z}$ *such that* $\gamma \not\equiv \gamma' \pmod 3$ *and* $\gamma' \not\equiv \gamma'' \pmod 2$.
(d) $m = (0,1,\gamma)$, $m' = (3,1,\gamma')$, *and* $m'' = (-3,-2,\gamma'')$ *with* $\gamma,\gamma',\gamma'' \in \mathbb{Z}$ *such that* $\gamma \not\equiv \gamma' \not\equiv \gamma'' \not\equiv \gamma \pmod 3$.

Proof:
It is clear that we can transform the circuit (E) by an integral unimodular affine transformation to $(0,0,0)$, $(0,0,1)$, and $(0,0,2)$. If we denote by $\pi : P \longrightarrow \mathbb{R}^2 : (x,y,z) \mapsto (x,y)$ the projection onto the xy-plane then $\pi(P)$ is a triangle which decomposes into three triangles $\pi(P) = T \cup T' \cup T''$ as in Lemma 4.3, see Figure 16. Lemma 4.3 therefore implies that $(0,0)$ is the only interior lattice point of $\pi(P)$. Lattice polygons with exactly one interior lattice lattice point have been classified up to integral unimodular affine transformations, see e.g. [15] or [14], and among them are exactly five triangles as shown in Figure 17, where the interior lattice point is $(0,0)$. Applying a $\mathbb{Z}$-linear coordinate change we may therefore assume that $\pi(P)$ is one of these five triangles. In each of the cases it remains to check whether there exist polytopes P that project to the triangle and what restrictions

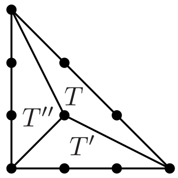

Figure 16. $\pi(P) = T \cup T' \cup T''$ decomposes as a union of three triangles.

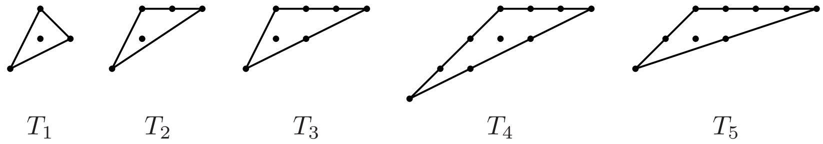

Figure 17. The five lattice triangles with one interior lattice point.

this poses on the third component of the lattice points m, m', and m''. Actually, the only obstruction is that above the relative interior lattice points on the edges of the triangles there should be no lattice point in P. If such an edge has k relative interior lattice points and the z-coordinates of the vertices of the edge differ by l, then some of the relative interior lattice points lifts to a lattice point if and only if $k+1$ and l are not coprime. Therefore, $T_1, \ldots, T_4$ lead to the four cases mentioned in the statement of the proposition. For T_5 we would need points $m = (0, 1, \gamma)$, $m' = (4, 1, \gamma')$, and $m'' = (-2, -1, \gamma'')$ such that each of the differences $\gamma - \gamma'$, $\gamma - \gamma''$ and $\gamma' - \gamma''$ is coprime to two. That is obviously not possible, so that T_5 cannot be the projection of any P. $\square$

In order to understand how the tropicalisation of the singular point locally looks like in the case we are considering, assume first that the subdivision contains a polytope as considered in Proposition 4.4, and it is subdivided into the three polytopes $\Delta_1 = \text{conv}(m_a, m_c, m_d, m_e)$, $\Delta_2 = \text{conv}(m_a, m_c, m_d, m_f)$ and $\Delta_3 = \text{conv}(m_a, m_c, m_e, m_f)$, see Figure 15. The circuit $\{m_a, m_b, m_c\}$ is then dual to a triangle in the tropical surface whose vertices are dual to Δ_1, Δ_2, and Δ_3, see Figure 18. We assume as before that $m_a = (0, 0, 0)$, $m_b = (0, 0, 1)$ and $m_c = (0, 0, 2)$.

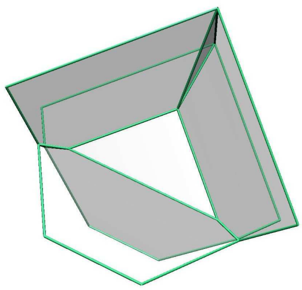

Figure 18. The triangle in the tropical surface dual to the circuit.

Recall that we can project P to the (x, y)-plane and obtain three triangles T, T' and T'' as in Figure 16. The midpoint is $(0, 0)$. Denote the coordinates of the three vertices by (r_1, s_1), (r_2, s_2) and (r_3, s_3). Let us use the tropical polynomial to solve for the coordinates (x, y, z) of the three vertices dual to Δ_1, Δ_2, and Δ_3. By assumption the heights associated to the lattice points satisfy $u_{m_a} = u_{m_b} = u_{m_c}$ and $u_{m_d} = u_{m_e} = u_{m_f}$, and we set $u = u_{m_a} - u_{m_d}$. For any $i = 1, 2, 3$, the equation $u + z = u$ has to be satisfied, so any of the three vertices has z-coordinate 0. In fact, the whole triangle dual to the circuit satisfies $z = 0$. So we only have to solve for the (x, y)-coordinates of the vertices. For any choice of $(i, j) = (1, 2), (2, 3)$ or $(3, 1)$, the vertex dual to the polytope which projects to the triangle spanned by $(0, 0)$, (r_i, s_i) and (r_j, s_j) has to satisfy the equations $u = r_i x + s_i y$ and $u = r_j x + s_j y$, which are solved by $(x, y) = \frac{1}{r_i s_j - s_i r_j} \cdot (s_j u - s_i u, r_i u - r_j u)$. Now assign to each of the vertices the area of the projection of the dual polytope, i.e. $(r_i s_j - s_i r_j)$, as weight. Then it follows that the weighted sum of the three vertices is $(0, 0, 0)$, i.e. the singular point. Thus, the singular point tropicalises precisely to the *weighted barycenter* of the triangle dual to the circuit. Figure 19 depicts this situation for the case that the projection is the triangle T_3 of Figure 17.

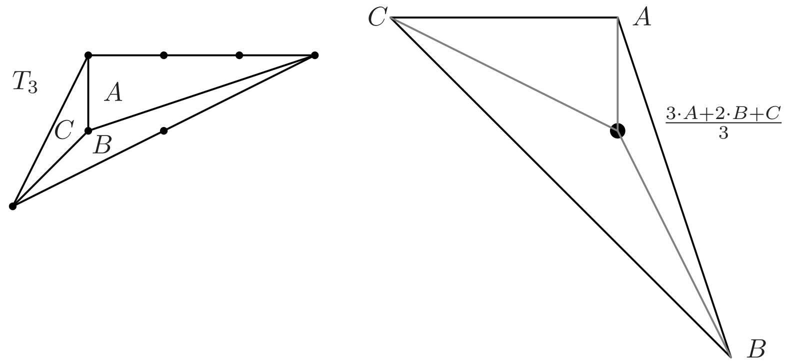

FIGURE 19. $T_3 = \pi(\Delta)$ and the dual triangle in the tropical surface showing $(0, 0, 0)$ as the weighted barycenter $\frac{3 \cdot A + 2 \cdot B + C}{3}$.

If the subdivision locally around the circuit contains further lattice points, the local picture may look more complicated. However, the circuit $\{m_a, m_b, m_c\}$ is still dual to a polygon Q in the $\{z = 0\}$-plane. Moreover, in the subdivision there will still be polytopes which contain $\operatorname{conv}(m_a, m_c, m_d)$ respectively $\operatorname{conv}(m_a, m_c, m_e)$ respectively $\operatorname{conv}(m_a, m_c, m_f)$ as a facet. Therefore, the polygon Q will have three edges dual to these facets. If one computes the intersection points of the lines through these edges, one gets three points A, B, and C which would be dual to the polytopes Δ_i, and the tropicalisation of the singular point is still the weighted sum of these three points, see Figure 20.

Example 4.5
A concrete example for this behaviour is the singular point $H = (-1, -1, 0)$ on the tropical surface in Example 1.4. We have seen in Remark 3.7 which weight class corresponds to the point $H = (-1, -1, 0)$. We have

$$\begin{aligned} m_a &= (0,0,0), & m_b &= (0,0,1), & m_c &= (0,0,2), \\ m_d &= (-1,-1,0), & m_e &= (0,1,0), & m_f &= (1,0,0), \end{aligned}$$

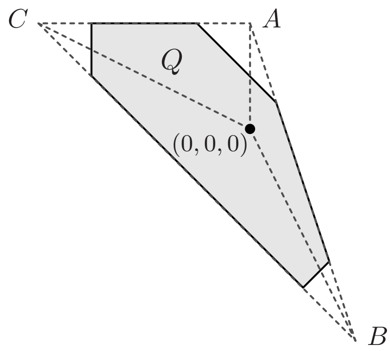

FIGURE 20. The origin as a generalised weighted barycenter.

and one further point $m_g = (1,2,1)$. The circuit m_a, m_b, m_c corresponds then to quadrangle $ABCD$ (see Figure 21, where the vertices $C = (5,-13,0)$ and

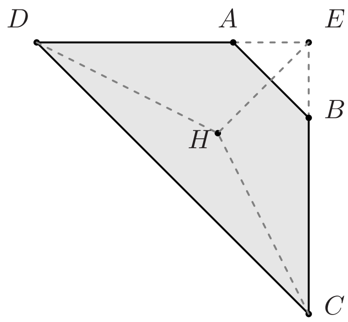

FIGURE 21. The singular point as barycenter.

$D = (-13,5,0)$ correspond to the polytopes $\Delta_C = \text{conv}\{m_a, m_c, m_d, m_e\}$ respectively $\Delta_D = \text{conv}\{m_a, m_c, m_d, m_f\}$ in the subdivision. The polytope $\Delta_E = \text{conv}\{m_a, m_c, m_e, m_f\}$, however, is not part of the subdivision due to the presence of m_g with an appropriate height. However, Δ_E defines a virtual point $E = (5,5,0)$, which is the intersection of the two lines determined by the facets $\text{conv}\{m_a, m_c, m_e\}$ and $\text{conv}\{m_a, m_c, m_f\}$ of Δ_C respectively Δ_D, and

$$H = \frac{1}{3} \cdot (C + D + E)$$

is the barycenter of this virtual triangle in the tropical surface.

4.5.2. *Assume there is a plane through the z-axis with m_d, m_e, and m_f all on the same side of the plane.* Again we first want to classify the possible polytopes spanned by $m_a, \ldots, m_f$, and then we will see how the corresponding tropical surfaces locally at the singular point look like.

Proposition 4.6
Let P be a lattice polytope which is the convex hull of a circuit of type (E) and three additional lattice points m, m', and m'' such that any two of these together with the circuit span $\mathbb{R}^3$, P contains only the given six lattice points, and there is a plane

through the z-axis such that m, m' and m'' are all on the same side of the plane, see Figure 26.

Then after a suitable integral unimodular affine transformation we may assume that the circuit is given by $(0,0,0)$, $(0,0,1)$, and $(0,0,2)$, and the lattice points m, m', and m'' (up to reordering) satisfy the conditions in exactly one of the following cases:

(a) $m = (-1,0,\gamma)$, $m' = (0,1,\gamma')$, *and* $m'' = (\alpha'',1,\gamma'')$ *with* $\alpha'' \geq 1$, $\gamma \in \mathbb{Z}$ *arbitrary and* $\gcd(\gamma''-\gamma',\alpha'') = 1$.
(b) $m = (\alpha,1,\gamma)$, $m' = (\alpha+l,1,\gamma+k)$, *and* $m'' = (\alpha+2\cdot l,1,\gamma+2\cdot k)$ *with* $\alpha,\gamma \in \mathbb{Z}$ *arbitrary and* $\gcd(l,k) = 1$.
(c) $m = (\alpha,1,\gamma)$, $m' = (\alpha',1,\gamma')$, *and* $m'' = (\alpha'',1,\gamma'')$ *with*

$$\det\begin{pmatrix}\alpha'-\alpha & \alpha''-\alpha\\ \gamma'-\gamma & \gamma''-\gamma\end{pmatrix} = \pm 1.$$

Proof:
Applying an integral unimodular affine transformation we may assume that the circuit is $(0,0,0)$, $(0,0,1)$, and $(0,0,2)$. Projecting Δ to the xy-plane the points $\pi(m)$, $\pi(m')$, and $\pi(m'')$ lie in one half plane. Due to the assumptions on Δ no two of these points lie on the same line through the origin, and ordering these lines by their angle clockwise we may assume up to reordering that the points $\pi(m)$, $\pi(m')$, and $\pi(m'')$ come in this order, see Figure 22 for possible configurations.

FIGURE 22. $\pi(\Delta)$ with the separating hyperplane.

We should note here first that in $\pi(\Delta)$ the point $\pi(m')$ cannot be an interior point of the triangle $\operatorname{conv}\big((0,0),\pi(m),\pi(m'')\big)$, since otherwise $\operatorname{conv}\big((0,0,0),(0,0,2),m,m''\big)$ will be a polytope in the subdivision of Δ which therefore satisfies the assumptions on Lemma 4.3, but $\pi(m')$ would violate these assumptions. It is then natural to distinguish the two cases that either $\pi(m')$ is on the line segment connecting $\pi(m)$ and $\pi(m'')$, i.e. $\pi(\Delta)$ is a triangle as shown on the right hand side of Figure 22, or $\pi(\Delta)$ is a quadrangle as shown on the left hand side of Figure 22. In any case, applying Lemma 4.3 to the convex hull of the circuit and two of the further lattice points m, m', and m'', we see that each of the points $\pi(m)$, $\pi(m')$, and $\pi(m'')$ has lattice distance one from the origin.

Let us first consider the case that $\pi(\Delta)$ is a triangle. Applying an integral unimodular linear transformation we may assume that the line through $\pi(m)$, $\pi(m')$, and $\pi(m'')$ is parallel to the x-axis, i.e. $\pi(m) = (\alpha,\beta)$, $\pi(m') = (\alpha',\beta)$, and $\pi(m'') = (\alpha'',\beta)$ with $\alpha < \alpha' < \alpha''$. By Lemma 4.3 the triangle $\operatorname{conv}\big((0,0),\pi(m),\pi(m'')\big)$ has no interior lattice point and the number of lattice points on the boundary is $\alpha''-\alpha+2$, so that Pick's Formula implies $\beta = 1$.

This case now subdivides into two subcases, namely, that the points m, m', and m'' lie on a line, respectively that they form a triangle. If the three points lie on

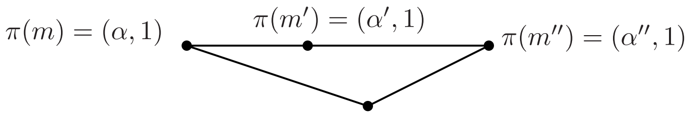

FIGURE 23. The normal form of $\pi(\Delta)$ when it is a triangle.

a line, then m' must be the midpoint of the line segment from m to m'' and the line segment contains no further lattice point. Thus, $\gcd\big(\alpha'' - \alpha, \gamma'' - \gamma\big) = 2$ is the only obstruction that has to be satisfied, and we are thus in Case (b) of the proposition with $l = \frac{\alpha''-\alpha}{2}$ and $k = \frac{\gamma''-\gamma}{2}$. If the three points m, m', and m'' form a triangle, then the only obstruction to the condition that Δ contains no further lattice points is that this triangle should have lattice area one. This is precisely the condition of Case (c) in the proposition.

It remains to consider the case that $\pi(\Delta)$ is a quadrangle. As in the proof of Lemma 4.3 we may apply an integral unimodular linear transformation such that $m' = (0, 1, \gamma')$ and $m'' = (\alpha'', \beta'', \gamma'')$ with $0 \leq \beta'' < \alpha''$. Moreover, since the triangle $T = \operatorname{conv}\big((0,0), \pi(m'), \pi(m'')\big)$ contains no interior lattice point due to Lemma 4.3 Pick's Formula implies that $\beta'' \in \{0, 1\}$, and if $\beta'' = 1$ then necessarily $\alpha'' = 1$, since the lattice distance from $\pi(m'')$ to the origin is one. See Figure 24.

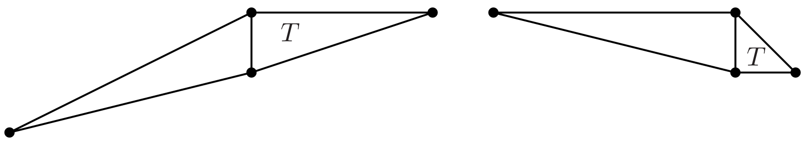

FIGURE 24. Possible configurations for the triangle $T = \operatorname{conv}\big((0,0), \pi(m'), \pi(m'')\big)$.

Let us now consider the case $\beta'' = 1$ in more detail. The point $\pi(m) = (\alpha, \beta)$ has to lie below the line $\{y = 1\}$ and above the line $\{\alpha \cdot y = x\}$. Thus $0 \geq \beta > \alpha$, and applying Pick's Formula once again we find $\beta = 0$, and then necessarily $\alpha = -1$. Analogously, we get in the case $\beta'' = 0$ that $\beta'' = 1$ and $\alpha \geq 1$. That is, $\pi(\Delta)$ is one of the quadrangles shown in Figure 25.

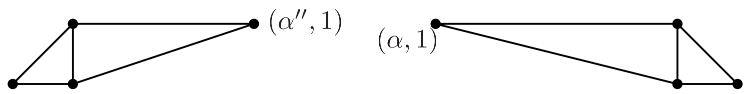

FIGURE 25. The normal forms of $\pi(\Delta)$ when it is a quadrangle.

Obviously, reflecting at the plane $\{x = 0\}$ and exchanging m and m'' the two possible configuration types are equivalent, so that we may assume that $\beta = 1$. We thus have $m = (-1, 0, \gamma)$, $m' = (0, 1, \gamma')$, and $m'' = (\alpha'', 1, \gamma'')$. Only above the line segment joining $\pi(m')$ and $\pi(m'')$ there could be an additional lattice point in Δ if the coordinates γ' and γ'' are chosen inappropriately, and the condition to avoid this is $\gcd(\gamma'' - \gamma', \alpha'') = 1$. We are thus in Case (a) of the proposition, and this finishes the proof. $\square$

Remark 4.7
If Δ was the polytope P in Proposition 4.6 (b), then Δ would be defective (see e.g. Example 2.12 in [4]), and we would not consider Δ at all. If Δ, however, contains further lattice points besides those in P, then Δ need not be defective. But the weight class C in Trop(Ker(A)) corresponding to this situation will still be defective, as we will see further down when considering this case.

We now have to see how the tropical surface looks locally at the tropicalisation of the singular point, i.e. locally at $(0,0,0)$. As in Subsection 4.5.1 we want to restrict first to the case where the Newton polytope Δ is just the convex hull of $m_a, \ldots, m_f$, and in the notation of Proposition 4.6 we may assume that $m_d = m$, $m_e = m'$, and $m_f = m''$. Moreover, we will consider the Case (a) in Proposition 4.6 first. In the subdivision of Δ there will be exactly two polytopes which contain the circuit m_a, m_b, and m_c, namely $\Delta_A = \operatorname{conv}(m_a, m_b, m_c, m_d, m_e)$ and $\Delta_B = \operatorname{conv}(m_a, m_b, m_c, m_e, m_f)$, see Figure 26. The subdivision may contain a third polytope $\operatorname{conv}(m_a, m_d, m_e, m_f)$ respectively $\operatorname{conv}(m_c, m_d, m_e, m_f)$ which does not contain the circuit, and which consequently will not matter for the singular point.

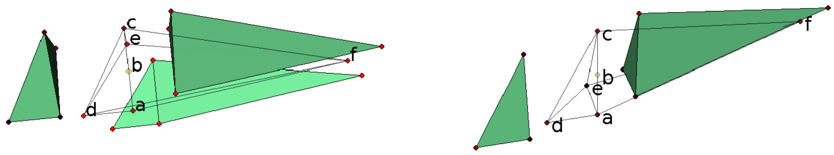

FIGURE 26. Possible subdivisions of Δ.

The tropicalisation of the singular point will then be contained in the plane segment dual to the circuit. This segment will be unbounded, but it has two vertices A and B which are dual the polytopes Δ_A and Δ_B. Moreover, if we consider the lines through the line segments which are dual to $\operatorname{conv}(m_a, m_b, m_c, m_d)$ and $\operatorname{conv}(m_a, m_b, m_c, m_f)$ respectively, then these will intersect in a point C which is dual to the polytope $\operatorname{conv}(m_a, m_b, m_c, m_d, m_f)$ which is not part of the subdivision. Anyway, if we assign to the points A, B, and C as weights the lattice area of the corresponding triangle in $\pi(\Delta)$, e.g. B gets as weight the lattice area α'' of $\operatorname{conv}\big((0,0),(0,1),(\alpha'',1)\big)$, and if we moreover consider the weight of C negatively, since C lies outside the plane segment, then the tropicalisation of the singular point is the weighted sum of A, B, and C. In the normal form a simple computation gives $A = (-u, u, 0)$, $B = (0, u, 0)$ and $C = (-u, (1+\alpha'')\cdot u, 0)$, and $\frac{A+\alpha''\cdot B-C}{3} = (0,0,0)$. We could thus interpret the tropicalisation of the singular point as a *virtual weighted barycenter* of the virtual triangle ABC.

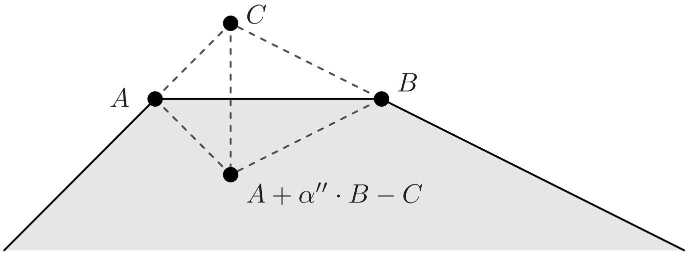

FIGURE 27. The singular point at the virtual barycenter.

In our classification we need not consider the Case (b) in 4.6, since there the weight class C in Trop(Ker(A)) corresponding to this situtation is defective because span(C) intersects the lineality space in the vector corresponding to the y-coordinates of the point configuration (see also Remark 4.7).

The Case (c) in Proposition 4.6 differs from Case (a) by the fact that the points A, B, and C all coincide, and that the plane segment corresponding to the circuit has only one vertex. However, it remains true that the tropicalisation of the singular point is the weighted sum of A, B, and C.

Finally, if the Newton polytope contains further points the situation becomes more complicated. The polytopes Δ_A and Δ_B might be subdivided further, and consequently the vertices A and B might be cut off, similar to the situation described in Figure 20. As in Subsection 4.5.1 we can still identify the virtual points A, B, and C and their weighted sum is the tropicalisation of the singular point.

Example 4.8
A concrete example for this behaviour is the singular point $G = (0, 0, 0)$ on the tropical surface in Example 1.4. Here

$$m_a = (0,0,0), \quad m_b = (0,0,1), \quad m_c = (0,0,2),$$
$$m_d = (1,2,1), \quad m_e = (0,1,0), \quad m_f = (1,0,0),$$

and one further point $m_g = (-1,-1,0)$. Note that the points m_d, m_e, m_f are all on the same side of the plane $x + y = 0$ through the circuit. The circuit m_a, m_b, m_c corresponds then to quadrangle $ABCD$ (see Figure 28, where the vertices $A =$

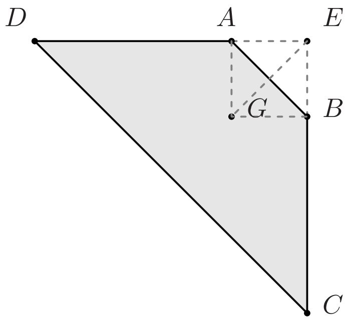

FIGURE 28. The singular point as barycenter.

$(0, 5, 0)$ and $B = (5, 0, 0)$ correspond to the polytopes $\Delta_A = \operatorname{conv}\{m_a, m_c, m_d, m_e\}$

respectively $\Delta_B = \mathrm{conv}\{m_a, m_c, m_d, m_f\}$ in the subdivision. The polytope $\Delta_E = \mathrm{conv}\{m_a, m_c, m_e, m_f\}$, however, is not part of the subdivision and defines only a virtual point $E = (5, 5, 0)$, which is the intersection of the two lines determined by the facets $\mathrm{conv}\{m_a, m_c, m_e\}$ and $\mathrm{conv}\{m_a, m_c, m_f\}$ of Δ_A respectively Δ_B. In this situation and

$$G = \frac{1}{3} \cdot (C + D - E)$$

is the virtual weighted barycenter of this virtual triangle in the tropical surface. Note here, that the virtual vertex E comes with a negative weight since it lies outside the plane segment dual to the circuit even if we only consider the points $m_a, \ldots, m_f$. Note also, that the plane segment dual to the circuit is bounded due to the presence of the additional point m_g.

4.6. **Weight class as in Lemma 2.4(d), circuit (E) of Remark 2.2.** Let $F'_{s-4} = \{a, b, c\}$, $F'_j = \{d, e\}$ and $F'_i = \{f, g\}$. We assume without restriction that $m_a = (0, 0, 0)$, $m_b = (0, 0, 1)$ and $m_c = (0, 0, 2)$. Dual to this circuit is then as before a 2-dimensional polyhedron satisfying $z = 0$. We know that in this situation, the points m_d and m_e lie in a plane with the line $\{x = y = 0\}$, we can assume that this plane satisfies $y = 0$. Let us first assume that m_d and m_e lie on different sides of the line, i.e. we assume that m_d has positive x-coordinate and m_e has negative x-coordinate. Then the triangle with vertices m_a, m_c and m_d (resp. m_e) will be a face of a polytope in the subdivision. If m_d or m_e had integral distance bigger one from the circuit, this face would contain extra lattice points, contradicting our assumption that the surface is of maximal-dimensional type. It follows that m_d has x-coordinate 1 and m_e has x-coordinate -1. Also, the triangle spanned by m_a, m_c and m_f (resp. m_g) are faces of the subdivision and thus m_f and m_g must have integral distance one to the plane $\{y = 0\}$. Let us first assume m_f has y-coordinate 1 and m_g has y-coordinate -1. Assume first that the subdivision locally contains only the polytopes $\mathrm{conv}(m_a, m_c, m_d, m_f)$, $\mathrm{conv}(m_a, m_c, m_e, m_f)$, $\mathrm{conv}(m_a, m_c, m_d, m_g)$ and $\mathrm{conv}(m_a, m_c, m_e, m_g)$. Then corresponding to this part of the subdivision we have a quadrangle on the surface. Let us solve for the (x, y)-coordinates of the four vertices. Assume $m_d = (1, 0, \gamma)$, $m_e = (-1, 0, \gamma')$, $m_f = (\alpha, 1, \gamma'')$ and $m_g = (\alpha', -1, \gamma''')$. Let us denote by $u = u_{m_a} - u_{m_d}$ the difference of the weights of m_a and m_d and by $w = u_{m_d} - u_{m_f}$ the difference of the weights of m_d and m_f. Then the coordinates of the four vertices are $A = (u, w+(1-\alpha)u)$, $B = (-u, w+(1+\alpha)u)$, $C = (u, -w + (\alpha' - 1)u)$ and $D = (-u, -w - (1 + \alpha')u)$. That is, the quadrangle is a trapeze with the singular point $(0, 0, 0) = \frac{A+B+C+D}{4}$ as its midpoint, as depicted in Figure 29 on the left.

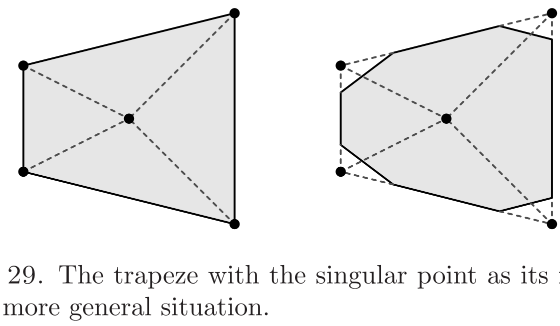

FIGURE 29. The trapeze with the singular point as its midpoint, and the more general situation.

If the subdivision contains more polytopes than just these four locally around the circuit, then we get a polygon with more sides. The four edges of the trapeze are still present, and the singular point is still the midpoint. This more general situation is depicted in Figure 29 on the right.

If m_d and m_e are on the same side of the circuit in the plane $\{y = 0\}$, then they must both be of integral distance one, and they form a quadrangle with the circuit which is a face of the subdivision. Thus the dual subdivision does not correspond to a cone of the secondary fan of codimension 1, and we do not consider the situation. Analogously, if m_f and m_g are on the same side of the plane $\{y = 0\}$, they must both have integral distance one to $\{y = 0\}$. However, since the edge connecting m_f and m_g and the circuit do not need to lie in a plane, it may be that only one of the points m_f or m_g forms a facet of the subdivision with the circuit. In this case, the dual subdivision corresponds to a cone of codimension 1. However, since the span of the corresponding weight class intersects the rowspace of A non-trivially (both contain the vector of x-coordinates of the points $m \in \mathcal{A}$), this weight class is defective and we do not consider the situation.

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