Tuesday, the 1st of May is a public holiday: *please, exceptionally hand in your Exercises in Patrick Wegener's mailbox by Monday 6pm, instead Tuesday 10am.*

Unless otherwise stated, throughout these exercises R is a ring, all rings are unital and associative and modules are left modules.

Exercise 1
Let A, B be R-modules. Prove that:

(a) $\mathrm{End}_R(A)$, endowed with the usual composition and the usual sum, is a ring.

(b) If R is commutative then the abelian group $\mathrm{Hom}_R(A, B)$ is a left R-module.

Exercise 2
Prove:

(a) the universal property of the direct product (Proposition 4.1 of the lecture).

(b) the universal property of the direct sum (Proposition 4.2 of the lecture).

Exercise 3
Prove that if $\varphi : M \longrightarrow N$ is an R-module homomorphism, then there is always an exact sequence

$$0 \longrightarrow \ker(\varphi) \longrightarrow M \xrightarrow{\varphi} N \longrightarrow \mathrm{Coker}(\varphi) \longrightarrow 0\,.$$

Compute the sequences associated with the following $\mathbb{Z}$-homomorphisms:
(i) $\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z}$, multiplication by a prime p in $\mathbb{Z}$;
(ii) $\mathbb{Z}/15\mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z}/15\mathbb{Z}$, multiplication by 3.

Exercise 4
Let K be a field and n a positive integer.

(a) Prove that in $K[X]$ the multiples of X^{2018} form a submodule, isomorphic to $K[X]$ as a $K[X]$-module.

(b) Prove that the data of a $K[X]/{<}X^n{>}$-module is equivalent to the data of a pair (V, φ) where V is a K-vector space and $\varphi : V \longrightarrow V$ is a K-endomorphism such that $\varphi^n = 0$.

(c) Give a comprehensive list of the submodules of $K[X]/{<}X^n{>}$, up to isomorphism.

(d) For $1 \leq k \leq n$ denote by V_k the $K[X]/{<}X^n{>}$-module $K[X]/{<}X^k{>}$. Compute the matrix of V_k in a "well-chosen" K-basis and show that for every $1 \leq r < k$ there exists an exact sequence

$$0 \longrightarrow V_r \longrightarrow V_k \longrightarrow V_{k-r} \longrightarrow 0\,.$$

Exercise 5
Let Q be an R-module and let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a short exact sequence of R-modules.

(a) Find a counterexample in which the functor $\mathrm{Hom}_R(Q, -)$ does not preserve surjectivity, i.e find a surjective R-linear map $g : B \longrightarrow C$ such that the induced homomorphism $g_* : \mathrm{Hom}_R(Q, B) \longrightarrow \mathrm{Hom}_R(Q, C)$ is not surjective.

(b) Prove that the induced sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}_R(C, Q) \xrightarrow{g^*} \mathrm{Hom}_R(B, Q) \xrightarrow{f^*} \mathrm{Hom}_R(A, Q)$$

is exact.

Find a counterexample of an injective R-linear map $f : A \longrightarrow B$ such that the induced homomorphism $f^* : \mathrm{Hom}_R(B, Q) \longrightarrow \mathrm{Hom}_R(A, Q)$ is not surjective.