MATHEMATIK

TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

Complements on Modules and Algebras

Character Theory of Finite Groups, SS 2022

Jun.-Prof. Dr. Caroline Lassueur
AG Algebra, Geometrie und Computeralgebra

Monday, the 2nd of May 2022

REFERENCE.

[Rot10] J. J. Rotman. Advanced modern algebra. 2nd ed. Providence, RI: American Mathematical Society (AMS), 2010.

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NOTATION.

Throughout we let $R = (R, +, \cdot)$ denote a unital associative ring, with multiplicative unit 1_R (or simply 1).

Definition. [Left R-module]

A **left R-module** is an ordered triple $(M, +, \cdot)$, where M is a set endowed with an **internal composition law**

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satisfying the following axioms:

(M1) $(M, +)$ is an abelian group;

(M2) $(r_1 + r_2) \cdot m = r_1 \cdot m + r_2 \cdot m$ for every $r_1, r_2 \in R$ and every $m \in M$;

(M3) $r \cdot (m_1 + m_2) = r \cdot m_1 + r \cdot m_2$ for every $r \in R$ and every $m_1, m_2 \in M$;

(M4) $(rs) \cdot m = r \cdot (s \cdot m)$ for every $r, s \in R$ and every $m \in M$.

(M5) $1_R \cdot m = m$ for every $m \in M$.

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(3) If the ring R is commutative, then any right module can be made into a left module via $r \cdot m := m \cdot r \; \forall\, r \in R, \forall\, m \in M$, and conversely.

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In this case we write $U_1 \oplus U_2$.

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- if $M = N$ and φ is invertible, then the set-theoretic *inverse map* φ^{-1} is also an R-homomorphism.

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  This is an abelian group for the pointwise addition of maps:

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- $\mathrm{End}_R(M) := \mathrm{Hom}_R(M,M)$.
  This is a ring for the pointwise addition of maps and the usual composition of maps.

Appendix A: Modules – Quotients

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Of course, the "usual" theorems on quotients hold:

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If U is an R-submodule of an R-module M, then there is a bijection:

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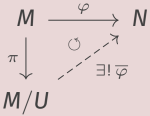
Universal Property of the Quotient

Let $\varphi : M \longrightarrow N$ be a homomorphism of R-modules. If U is an R-submodule of M such that $U \subseteq \ker(\varphi)$, then there exists a unique R-module homomorphism $\overline{\varphi} : M/U \longrightarrow N$ such that $\overline{\varphi} \circ \pi = \varphi$:



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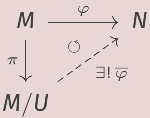

Concretely, $\overline{\varphi}(m + U) = \varphi(m) \;\; \forall\, m + U \in M/U$.

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(c) **3rd isomorphism theorem.** If $U_1 \subseteq U_2$ are R-submodules of M, then there is an isomorphism of R-modules

$$(M/U_1)\,/\,(U_2/U_1) \cong M/U_2\,.$$

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(b) A map $f : A \to B$ between two R-algebras is called an **algebra homomorphism** iff:

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[The internal composition law "$\cdot$" and the external composition law "$*$" coincide!]

(b) $M_n(R)$, with $n \in \mathbb{Z}_{\geq 1}$ and R a commutative ring, is an R-algebra for its usual R-module and ring structures.
[In particular R-algebras need not be commutative rings in general!]

(c) If K is a field and V a finite-dimensional K-vector space, then $\mathrm{End}_K(V)$ is a K-algebra.

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Definition. [Centre]

The **centre** of an R-algebra $(A, +, \cdot, *)$ is $Z(A) := \{a \in A \mid a \cdot b = b \cdot a \ \ \forall\, b \in A\}$.