Chapter 1. Linear Representations of Finite Groups

Representation theory of finite groups is originally concerned with the ways of writing a finite group G as a group of matrices, that is using group homomorphisms from G to the general linear group $\mathrm{GL}_n(K)$ of invertible $n \times n$-matrices with coefficients in a field K for some non-negative integer n.

Notation: throughout this chapter, unless otherwise specified, we let:

- G denote a finite group (in multiplicative notation);
- K denote a field of arbitrary characteristic; and
- V denote a K-vector space such that $\dim_K(V) < \infty$.

In general, unless otherwise stated, all groups considered are assumed to be finite and all K-vector spaces considered are assumed to be **finite-dimensional**.

1 Linear Representations

Definition 1.1 (*K-representation, matrix representation, faithfullness*)

(a) A **K-representation** of G (or a **(linear) representation of G (over K)**) is a group homomorphism

$$\rho : G \longrightarrow \mathrm{GL}(V)\,,$$

where V is a K-vector space of dimension $n \in \mathbb{Z}_{\geqslant 0}$.
(Here $\mathrm{GL}(V) := \mathrm{Aut}_K(V)$ is the group of K-automorphisms of V.)

(b) A **matrix representation** of G is a group homomorphism $R : G \longrightarrow \mathrm{GL}_n(K)$, where $n \in \mathbb{Z}_{\geqslant 0}$.

In both cases the integer n is called the **degree** of the representation. An injective (matrix) representation of G is called **faithful**.

Remark 1.2

We see at once that both concepts of a representation and of a matrix representation are closely connected.
Recall that every choice of an ordered basis B of V yields a group isomorphism

$$\begin{array}{cccc} \alpha_B: & \mathrm{GL}(V) & \longrightarrow & \mathrm{GL}_n(K) \\ & \varphi & \mapsto & (\varphi)_B \end{array}$$

where $(\varphi)_B$ denotes the matrix of φ in the basis B. Therefore, a K-representation $\rho : G \longrightarrow \mathrm{GL}(V)$ together with the choice of an ordered basis B of V gives rise to a matrix representation of G:

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[dr, dashed, "R_B:=\alpha_B\circ\rho"'] & \mathrm{GL}(V) \arrow[d, leftrightarrow, "\alpha_B", "\cong"'] \\ & \mathrm{GL}_n(K)\,. \end{tikzcd}$$

Explicitly, R_B sends an element $g \in G$ to the matrix $\big(\rho(g)\big)_B$ of $\rho(g)$ expressed in the basis B.

Another choice of a K-basis of V yields another matrix representation!!

It is also clear from the diagram that, conversely, any matrix representation $R : G \longrightarrow \mathrm{GL}_n(K)$ gives rise to a K-representation of G.

Throughout the lecture, we will favour the approach using representations rather than matrix representations in order to develop theoretical results. However, matrix representations are essential to carry out computations. Being able to pass back and forth from one approach to the other will be an essential feature.

Also note that Remark 1.2 allows us to transfer terminology/results from representations to matrix representations and conversely. Hence, from now on, in general we make new definitions for representations and use them for matrix representations as well.

Example 1

(a) If G is an arbitrary finite group and $V := K$, then

$$\begin{array}{cccc} \rho: & G & \longrightarrow & \mathrm{GL}(K) \cong K^\times \\ & g & \mapsto & \rho(g) := \mathrm{Id}_K \leftrightarrow 1_K \end{array}$$

is a K-representation of G, called **the trivial representation** of G.
Similarly $\rho : G \longrightarrow \mathrm{GL}(V), g \mapsto \mathrm{Id}_V$ with $\dim_K(V) =: n > 1$ is also a K-representation of G and is called **a trivial representation of G of degree n**.

(b) If G is a subgroup of $\mathrm{GL}(V)$, then the canonical inclusion

$$\begin{array}{ccc} G & \hookrightarrow & \mathrm{GL}(V) \\ g & \mapsto & g \end{array}$$

is a faithful representation of G, called the **tautological representation** of G.

(c) Let $G := S_n$ ($n \geqslant 1$) be the symmetric group on n letters. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $V := K^n$. Then

$$\begin{array}{cccc} \rho: & S_n & \longrightarrow & \mathrm{GL}(K^n) \\ & \sigma & \mapsto & \rho(\sigma) : K^n \longrightarrow K^n, e_i \mapsto e_{\sigma(i)} \end{array}$$

is a K-representation, called the **natural representation** of S_n.

(d) More generally, if X is a finite G-set, i.e. a finite set endowed with a left action $\cdot : G \times X \longrightarrow X$, and V is a K-vector space with basis $\{e_x \mid x \in X\}$, then

$$\begin{array}{rccl} \rho_X : & G & \longrightarrow & \mathrm{GL}(V) \\ & g & \mapsto & \rho_X(g) : V \longrightarrow V, e_x \mapsto e_{g\cdot x} \end{array}$$

is a K-representation of G, called the **permutation representation** associated with X.

Notice that (c) is a special case of (d) with $G = S_n$ and $X = \{1, 2, \ldots, n\}$.
If $X = G$ and the left action $\cdot : G \times X \longrightarrow X$ is just the multiplication in G, then

$$\rho_X =: \rho_{\mathrm{reg}}$$

is called the **regular representation** of G.

We shall see later on in the lecture that K-representations are a special case of a certain *algebraic structure* (in the sense of the lecture *Algebraische Strukturen*). Thus, next, we define the notions that shall correspond to a *homomorphism* and an *isomorphism* of this algebraic structure.

Definition 1.3 (*Homomorphism of representations, equivalent representations*)

Let $\rho_1 : G \longrightarrow \mathrm{GL}(V_1)$ and $\rho_2 : G \longrightarrow \mathrm{GL}(V_2)$ be two K-representations of G, where V_1, V_2 are two finite-dimensional K-vector spaces.

(a) A K-homomorphism $\alpha : V_1 \longrightarrow V_2$ such that $\rho_2(g) \circ \alpha = \alpha \circ \rho_1(g)$ for each $g \in G$ is called a **homomorphism of representations** (or a G**-homomorphism**) between ρ_1 and ρ_2.

$$\begin{tikzcd} V_1 \arrow[r, "\rho_1(g)"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & V_1 \arrow[d, "\alpha"] \\ V_2 \arrow[r, "\rho_2(g)"'] & V_2 \end{tikzcd}$$

(b) If, moreover, α is a K-isomorphism, then it is called an **isomorphism of representations** (or a G**-isomorphism**), and the K-representations ρ_1 and ρ_2 are called **equivalent** (or **isomorphic**). In this case we write $\rho_1 \sim \rho_2$.

(c) Two matrix representations $R_1, R_2 : G \longrightarrow \mathrm{GL}_n(K)$ are called **equivalent** iff $\exists\ T \in \mathrm{GL}_n(K)$ such that

$$R_2(g) = TR_1(g)T^{-1} \qquad \forall\, g \in G\,.$$

In this case we write $R_1 \sim R_2$.

Remark 1.4

(a) Equivalent representations have the same degree.

(b) Clearly $\sim$ is an equivalence relation.

(c) Consequence: it essentially suffices to study representations up to equivalence (as it essentially suffices to study groups up to isomorphism).

Remark 1.5

If $\rho : G \longrightarrow \mathrm{GL}(V)$ is a K-representation of G and $E := (e_1, \dots, e_n)$, $F := (f_1, \dots, f_n)$ are two ordered bases of V, then by Remark 1.2, we have two matrix representations:

$$R_E : \begin{array}{ccc} G & \longrightarrow & \mathrm{GL}_n(K) \\ g & \mapsto & \big(\rho(g)\big)_E \end{array} \qquad \text{and} \qquad R_F : \begin{array}{ccc} G & \longrightarrow & \mathrm{GL}_n(K) \\ g & \mapsto & \big(\rho(g)\big)_F \end{array}$$

These matrix representations are equivalent since $R_F(g) = TR_E(g)T^{-1}$ $\forall\, g \in G$, where T is the change-of-basis matrix.

2 Subrepresentations and (Ir)reducibility

Subrepresentations allow us to introduce one of the main notions that will enable us to break representations in elementary pieces in order to simplify their study: the notion of (ir)reducibility.

Definition 2.1 (*G-invariant subspace, irreducibility*)

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a K-representation of G.

(a) A K-subspace $W \subseteq V$ is called **G-invariant** if

$$\rho(g)\big(W\big) \subseteq W \qquad \forall g \in G\,.$$

(In fact, in this case the reverse inclusion holds as well, since for each $w \in W$ we can write $w = \rho(gg^{-1})(w) = \rho(g)\big(\rho(g^{-1})(w)\big) \in \rho(g)\big(W\big)$, hence $\rho(g)\big(W\big) = W$.)

(b) The representation ρ is called **reducible** if V admits a non-trivial proper G-invariant subspace $\{0\} \subsetneq W \subsetneq V$, whereas it ρ is called **irreducible** if it admits exactly two G-invariant subspaces: $\{0\}$ and V itself.

Notice that V itself and the zero subspace $\{0\}$ are always G-invariant subspaces. Moreover, ρ is irreducible if it is not reducible and $V \neq \{0\}$.

Definition 2.2 (*Subrepresentation*)

If $\rho : G \longrightarrow \mathrm{GL}(V)$ is a K-representation and $W \subseteq V$ is a G-invariant subspace, then

$$\rho_W : \begin{array}{ccl} G & \longrightarrow & \mathrm{GL}(W) \\ g & \mapsto & \rho_W(g) := \rho(g)|_W : W \longrightarrow W \end{array}$$

is called a **subrepresentation** of ρ. (This is clearly again a representation of G.)

Clearly, a representation $\rho : G \longrightarrow \mathrm{GL}(V)$ is *irreducible* if and only if ρ does not possess any proper subrepresentation and $V \neq \{0\}$.

Remark 2.3

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a K-representation and $0 \neq W \subseteq V$ be a G-invariant subspace. Now choose an ordered basis B' of W and complete it to an ordered basis B of V. Then for each $g \in G$

the corresponding matrix representation is of the form

$$\big(\rho(g)\big)_B = \begin{array}{c} \left[\begin{array}{c|c} \big(\rho_W(g)\big)_{B'} & * \\ \hline 0 & * \end{array}\right] \\ \begin{array}{cc} B' & B\setminus B' \end{array} \end{array}.$$

Example 2

(a) Any K-representation of degree 1 is irreducible.

(b) Let $\rho : S_n \longrightarrow \mathrm{GL}(K^n)$ be the natural representation of S_n ($n \geqslant 1$) and let $B := (e_1, \dots, e_n)$ be the standard basis of $V = K^n$. Then for each $g \in G$ we have

$$\rho(g)\Big(\sum_{i=1}^n e_i\Big) = \sum_{i=1}^n \rho(g)(e_i) = \sum_{i=1}^n e_i\,,$$

where the last equality holds because $\rho(g) : \{e_1, \dots, e_n\} \longrightarrow \{e_1, \dots, e_n\}, e_i \mapsto e_{g(i)}$ is a bijection. Thus

$$W := \big\langle \sum_{i=1}^n e_i \big\rangle_K$$

is an S_n-invariant subspace of K^n of dimension 1. It follows that ρ is reducible if $n > 1$.

(c) More generally, the trivial representation of a finite group G is a subrepresentation of any permutation representation of G. [Exercise 2, Sheet 1]

(d) The symmetric group $S_3 = \langle (1\ 2), (1\ 2\ 3) \rangle$ admits the following three non-equivalent irreducible matrix representations over $\mathbb{C}$:

$$\rho_1 : S_3 \longrightarrow \mathbb{C}^\times, \sigma \mapsto 1$$

i.e. the trivial representation,

$$\rho_2 : S_3 \longrightarrow \mathbb{C}^\times, \sigma \mapsto \mathrm{sign}(\sigma)$$

where $\mathrm{sign}(\sigma)$ denotes the sign of the permutation σ, and

$$\begin{array}{rccc} \rho_3 : & S_3 & \longrightarrow & \mathrm{GL}_2(\mathbb{C}) \\ & (1\ 2) & \mapsto & \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \\ & (1\ 2\ 3) & \mapsto & \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right). \end{array}$$

See [Exercise 1(a), Sheet 1].
We will prove later in the lecture that these are all the irreducible $\mathbb{C}$-representations of S_3 up to equivalence.

Properties 2.4

Let $\rho_1 : G \longrightarrow \mathrm{GL}(V_1)$ and $\rho_2 : G \longrightarrow \mathrm{GL}(V_2)$ be two K-representations of G and let $\alpha : V_1 \longrightarrow V_2$ be a G-homomorphism.

(a) If $W \subseteq V_1$ is a G-invariant subspace of V_1, then $\alpha(W) \subseteq V_2$ is G-invariant.

(b) If $W \subseteq V_2$ is a G-invariant subspace of V_2, then $\alpha^{-1}(W) \subseteq V_1$ is G-invariant.

(c) In particular, $\ker(\alpha)$ and $\mathrm{Im}(\alpha)$ are G-invariant subspaces of V_1 and V_2 respectively.

Proof: [Exercise 3, Sheet 1]. ■

3 Maschke's Theorem

We now come to our first major result in the representation theory of finite groups, namely Maschke's Theorem, which provides us with a criterion for representations to decompose into direct sums of irreducible subrepresentations.

Definition 3.1 (*Direct sum of subrepresentations*)

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a K-representation. If $W_1, W_2 \subseteq V$ are two G-invariant subspaces such that $V = W_1 \oplus W_2$, then we say that ρ is the **direct sum** of the subrepresentations ρ_{W_1} and ρ_{W_2} and we write $\rho = \rho_{W_1} \oplus \rho_{W_2}$.

Remark 3.2

With the notation of Definition 3.1, if we choose an ordered basis B_i of W_i ($i = 1, 2$) and consider the ordered K-basis $B := B_1 \sqcup B_2$ of V, then the corresponding matrix representation is of the form

$$\big(\rho(g)\big)_B = \begin{array}{cc} \left[\begin{array}{c|c} \big(\rho_{W_1}(g)\big)_{B_1} & 0 \\ \hline 0 & \big(\rho_{W_2}(g)\big)_{B_2} \end{array}\right] \\ \begin{array}{cc} B_1 & \quad\quad B_2 \end{array} \end{array} \qquad \forall\, g \in G\,.$$

The following exercise shows that it is not always possible to decompose representations into direct sums of irreducible subrepresentations.

Exercise 3.3 (*Exercise 4, Sheet 1*)

Let p be an odd prime number, let $G := C_p = \langle g \mid g^p = 1\rangle$, let $K := \mathbb{F}_p$, and let $V := \mathbb{F}_p^2$ with its canonical basis $B = (e_1, e_2)$. Consider the matrix representation

$$\begin{array}{rccc} R : & G & \longrightarrow & \mathrm{GL}_2(K) \\ & g^b & \mapsto & \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \,. \end{array}$$

(a) Prove that Ke_1 is G-invariant and deduce that R is reducible.

(b) Prove that there is no direct sum decomposition of V into irreducible G-invariant subspaces.

Theorem 3.4 (Maschke)

Let G be a finite group and let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a K-representation of G. If $\mathrm{char}(K) \nmid |G|$, then every G-invariant subspace W of V admits a G-invariant complement in V, i.e. a G-invariant subspace $U \subseteq V$ such that $V = W \oplus U$.

Proof: To begin with, choose an arbitrary complement U_0 to W in V, i.e. $V = W \oplus U_0$ as K-vector spaces. (Note that, however, U_0 is possibly not G-invariant!) Next, consider the projection onto W along U_0, that is the K-linear map

$$\pi : V = W \oplus U_0 \longrightarrow W$$

which maps an element $v = w + u$ with $w \in W, u \in U_0$ to w, and define a new K-linear map

$$\begin{array}{rccc} \widetilde{\pi}: & V & \longrightarrow & V \\ & v & \mapsto & \frac{1}{|G|}\sum_{g\in G}\rho(g)\pi\rho(g^{-1})(v)\,. \end{array}$$

Notice that it is allowed to divide by $|G|$ because the hypothesis that $\mathrm{char}(K) \nmid |G|$ implies that $|G|\cdot 1_K$ is invertible in the field K.

We prove the following assertions:

(1) $\mathrm{Im}\,\widetilde{\pi} \subseteq W$: indeed, if $v \in V$, then

$$\widetilde{\pi}(v) = \frac{1}{|G|}\sum_{g\in G}\underbrace{\rho(g)\,\underbrace{\pi\rho(g^{-1})(v)}_{\in W}}_{\in W\ (G\text{-invariance})} \quad \in W\,.$$

(2) $\widetilde{\pi}|_W = \mathrm{Id}_W$: indeed, if $w \in W$, then

$$\widetilde{\pi}(w) = \frac{1}{|G|}\sum_{g\in G}\rho(g)\,\underbrace{\pi\ \underbrace{\rho(g^{-1})(w)}_{\substack{\in W \\ \text{(by } G\text{-invariance)}}}}_{\substack{=\rho(g^{-1})(w) \\ \text{(by def. of } \pi)}} = \frac{1}{|G|}\sum_{g\in G}\underbrace{\rho(g)\rho(g^{-1})}_{\substack{=\rho(gg^{-1}) \\ =\rho(1_G) \\ =\mathrm{Id}_V}}(w) = \frac{1}{|G|}\sum_{g\in G} w = w\,.$$

Thus (1)+(2) imply that $\widetilde{\pi}$ is a projection onto W so that as a K-vector space

$$V = W \oplus \ker(\widetilde{\pi})\,.$$

(3) $\ker\widetilde{\pi}$ is G-invariant: indeed, for each $h \in G$ we have

$$\begin{aligned} \rho(h)\circ\widetilde{\pi} &= \frac{1}{|G|}\sum_{g\in G}\underbrace{\rho(h)\rho(g)}_{=\rho(hg)}\pi\rho(g^{-1}) \\ &= \frac{1}{|G|}\sum_{g\in G}\rho(hg)\pi\rho((hg)^{-1}h) \\ &\overset{s:=hg}{=} \frac{1}{|G|}\sum_{s\in G}\rho(s)\pi\rho(s^{-1}h) \\ &= \Big(\frac{1}{|G|}\sum_{s\in G}\rho(s)\pi\rho(s^{-1})\Big)\rho(h) = \widetilde{\pi}\circ\rho(h)\,. \end{aligned}$$

Hence $\widetilde{\pi}$ is a G-homomorphism and it follows from Property 2.4(c) that its kernel is G-invariant. Therefore we may set $U := \ker(\widetilde{\pi})$ and the claim follows. ∎

Definition 3.5 (*Completely reducible/semisimple representation / constituent*)

A K-representation which can be decomposed into a direct sum of irreducible subrepresentations is called **completely reducible** or **semisimple**. In this case, an irreducible subrepresentation occuring in such a decomposition is called a **constituent** of the representation.

Corollary 3.6

If G is a finite group and K is a field such that $\mathrm{char}(K) \nmid |G|$, then every K-representation of G is completely reducible.

Proof: Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a K-representation of G. W.l.o.g. we may assume $V \neq \{0\}$.

- Case 1: ρ is irreducible $\Rightarrow$ nothing to do ✓.
- Case 2: ρ is reducible. Thus $\dim_K(V) \geqslant 2$ and there exists an irreducible G-invariant subspace $0 \neq V_1 \subseteq V$. Now, by Maschke's Theorem, there exists a G-invariant complement $U \subseteq V$, i.e. such that $V = V_1 \oplus U$. As $\dim_K(V_1) \geqslant 1$, we have $\dim_K(U) < \dim_K(V)$. Therefore, an induction argument yields the existence of a decomposition

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_r \qquad (r \geqslant 2)$$

of V, where $V_1, \ldots, V_r$ are irreducible G-invariant subspaces. ■

Remark 3.7

(a) The hypothesis of Maschke's Theorem requiring that $\mathrm{char}(K) \nmid |G|$ is always verified if K is a field of characteristic zero. E.g. if $K = \mathbb{C}, \mathbb{R}, \mathbb{Q}, \ldots$

(b) The converse of Maschke's Theorem holds as well. It will be proved in the M.Sc. lecture *Representation Theory*.

(c) In the literature, a representation is called an **ordinary** representation if K is a field of characteristic zero (or more generally of characteristic not dividing $|G|$), and it is called a **modular** representation if $\mathrm{char}(K) \mid |G|$.

In this lecture we are going to reduce our attention to *ordinary representation theory* and, most of the time, even assume that K is the field $\mathbb{C}$ of complex numbers.

Exercise 3.8 (*Alternative proof of Maschke's Theorem over the field* $\mathbb{C}$. *Exercise 5, Sheet 2.*)

Assume $K = \mathbb{C}$ and let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation of G.

(a) Prove that there exists a G-invariant scalar product $\langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{C}$, i.e. such that

$$\langle g.u, g.v \rangle = \langle u, v \rangle \quad \forall\, g \in G, \forall\, u, v \in V\,.$$

[Hint: consider an arbitrary scalar product on V, say $(\ ,\ ) : V \times V \longrightarrow \mathbb{C}$, which is not necessarily G-invariant. Use a sum on the elements of G, weighted by the group order $|G|$, in order to produce a new G-invariant scalar product on V.]

(b) Deduce that every G-invariant subspace W of V admits a G-invariant complement.
[Hint: consider the orthogonal complement of W.]