Chapter 2. The Group Algebra and Its Modules

We now introduce the concept of a KG-module, and show that this more modern approach is equivalent to the concept of a K-representation of a given finite group G. Some of the material in the remainder of these notes will be presented in terms of KG-modules. As we will soon see with our second fundamental result – Schur's Lemma – there are several advantages to this approach to representation theory.

Notation: throughout this chapter, unless otherwise specified, we let:

- G denote a finite group;
- K denote a field of arbitrary characteristic; and
- V denote a K-vector space such that $\dim_K(V) < \infty$.

In general, unless otherwise stated, all groups considered are assumed to be finite and all K-vector spaces / modules over the group algebra considered are assumed to be finite-dimensional.

4 Modules over the Group Algebra

Lemma-Definition 4.1 (*Group algebra*)

The **group ring** KG is the ring whose elements are the K-linear combinations $\sum_{g\in G} \lambda_g g$ with $\lambda_g \in K$, and addition and multiplication are given by

$$\sum_{g\in G} \lambda_g g + \sum_{g\in G} \mu_g g = \sum_{g\in G} (\lambda_g + \mu_g) g \quad \text{and} \quad \Big(\sum_{g\in G} \lambda_g g\Big) \cdot \Big(\sum_{h\in G} \mu_h h\Big) = \sum_{g,h\in G} (\lambda_g \mu_h) gh$$

respectively. In fact KG is a K-vector space with basis G, hence a K-algebra. Thus we usually call KG the **group algebra of G over K** rather than simply *group ring*.

Note: In Definition 4.1, the field K can be replaced with a commutative ring R. E.g. if $R = \mathbb{Z}$, then $\mathbb{Z}G$ is called the *integral group ring* of G.

Proof: By definition KG is a K-vector space with basis G, and the multiplication in G is extended by K-bilinearity to the given multiplication $\cdot : KG \times KG \longrightarrow KG$. It is then straightforward to check that KG bears both the structures of a ring and of a K-vector space. Finally, axiom (A3) of K-algebras (see Appendix B) follows directly from the definition of the multiplication and the commutativity of K. ∎

Remark 4.2

Clearly $1_{KG} = 1_G$, $\dim_K(KG) = |G|$, and KG is commutative if and only if G is an abelian group.

Proposition 4.3

(a) Any K-representation $\rho : G \longrightarrow \mathrm{GL}(V)$ of G gives rise to a KG-module structure on V, where the external composition law is defined by the map

$$\begin{array}{rccl} \cdot : & KG \times V & \longrightarrow & V \\ & (\sum_{g\in G} \lambda_g g, v) & \mapsto & (\sum_{g\in G} \lambda_g g) \cdot v := \sum_{g\in G} \lambda_g \rho(g)(v) \ . \end{array}$$

(b) Conversely, every KG-module $(V, +, \cdot)$ defines a K-representation

$$\begin{array}{rccl} \rho_V : & G & \longrightarrow & \mathrm{GL}(V) \\ & g & \mapsto & \rho_V(g) : V \longrightarrow V, v \mapsto \rho_V(g)(v) := g \cdot v \end{array}$$

of the group G.

Proof: (a) Since V is a K-vectore space it is equipped with an internal addition $+$ such that $(V, +)$ is an abelian group. It is then straightforward to check that the given external composition law defined above verifies the KG-module axioms.

(b) A KG-module is in particular a K-vector space for the scalar multiplication defined for all $\lambda \in K$ and all $v \in V$ by

$$\lambda v := (\underbrace{\lambda 1_G}_{\in KG}) \cdot v \, .$$

Moreover, it follows from the KG-module axioms that $\rho_V(g) \in \mathrm{GL}(V)$ and also that

$$\rho_V(g_1 g_2) = \rho_V(g_1) \circ \rho_V(g_2)$$

for all $g_1, g_2 \in G$, hence ρ_V is a group homomorphism.

See [Exercise 7, Sheet 2] for the details (Hint: use the remark below!). ■

Remark 4.4

In fact in Proposition 4.3(a) checking the KG-module axioms is equivalent to checking that for all $g, h \in G$, $\lambda \in K$ and $u, v \in V$:

(1) $(gh) \cdot v = g \cdot (h \cdot v)$;

(2) $1_G \cdot v = v$;

(4) $g \cdot (u + v) = g \cdot u + g \cdot v$;

(3) $g \cdot (\lambda v) = \lambda(g \cdot v) = (\lambda g) \cdot v$,

or in other words, that the binary operation

$$\begin{array}{rccl} \cdot : & G \times V & \longrightarrow & V \\ & (g, v) & \mapsto & g \cdot v := \rho(g)(v) \end{array}$$

is a *K-linear action of the group G on V*. Indeed, the external multiplication of KG on V is just the extension by K-linearity of the latter map. For this reason, sometimes, KG-modules are also called *G-vector spaces*. See [Exercise 6, Sheet 2] for the details.

Lemma 4.5

Two representations $\rho_1 : G \longrightarrow \mathrm{GL}(V_1)$ and $\rho_2 : G \longrightarrow \mathrm{GL}(V_2)$ are equivalent if and only if $V_1 \cong V_2$ as KG-modules.

Proof: If $\rho_1 \sim \rho_2$ and $\alpha : V_1 \longrightarrow V_2$ is a K-isomorphism such that $\rho_2(g) = \alpha \circ \rho_1(g) \circ \alpha^{-1}$ for each $g \in G$, then by Proposition 4.3(a) for every $v \in V_1$ and every $g \in G$ we have

$$g \cdot \alpha(v) = \rho_2(g)(\alpha(v)) = \alpha(\rho_1(g)(v)) = \alpha(g \cdot v)\,.$$

Hence α is a KG-isomorphism.
Conversely, if $\alpha : V_1 \longrightarrow V_2$ is a KG-isomorphism, then certainly it is a K-homomorphism and for each $g \in G$ and by Proposition 4.3(b) for each $v \in V_2$ we have

$$\alpha \circ \rho_1(g) \circ \alpha^{-1}(v) = \alpha(\rho_1(g)(\alpha^{-1}(v))) = \alpha(g \cdot \alpha^{-1}(v)) = g \cdot \alpha(\alpha^{-1}(v)) = g \cdot v = \rho_2(g)(v)\,,$$

hence $\rho_2(g) = \alpha \circ \rho_1(g) \circ \alpha^{-1}$ for each $g \in G$. ■

Remark 4.6 (*Dictionary*)

More generally, through Proposition 4.3, we may transport terminology and properties from KG-modules to representations and conversely.

This lets us build the following **dictionary**:

REPRESENTATIONS		MODULES
K-representation of G	$\longleftrightarrow$	KG-module
degree	$\longleftrightarrow$	K-dimension
homomorphism of representations	$\longleftrightarrow$	homomorphism of KG-modules
subrepresentation / G-invariant subspace	$\longleftrightarrow$	KG-submodule
direct sum of representations $\rho_{V_1} \oplus \rho_{V_2}$	$\longleftrightarrow$	direct sum of KG-modules $V_1 \oplus V_2$
irreducible representation	$\longleftrightarrow$	simple (= irreducible) KG-module
the trivial representation	$\longleftrightarrow$	the trivial KG-module K
the regular representation of G	$\longleftrightarrow$	the regular KG-module KG
Corollary 3.6 to Maschke's Theorem:	$\longleftrightarrow$	Corollary 3.6 to Maschke's Theorem:
If $\mathrm{char}(K) \nmid \lvert G\rvert$, then every K-representation of G is completely reducible.		If $\mathrm{char}(K) \nmid \lvert G\rvert$, then every KG-module is semisimple.
...		...

Virtually, any result, we have seen in Chapter 1, can be reinterpreted using this translation table. E.g. Property 2.4(c) tells us that the image and the kernel of homomorphisms of KG-modules are KG-submodules, ...

In this lecture, we introduce the equivalence between representations and modules for the sake of completeness. In the sequel we keep on stating results in terms of representations as much as possible. However, we will use modules when we find them more fruitful. In contrast, the M.Sc. Lecture *Representation Theory* will consistently use the module approach to representation theory.

5 Schur's Lemma and Schur's Relations

Schur's Lemma is a basic result concerning simple modules, or in other words irreducible representations. Though elementary to state and prove, it is fundamental to representation theory of finite groups.

Theorem 5.1 (Schur's Lemma)

(a) Let V, W be simple KG-modules. Then the following assertions hold.

(i) Any homomorphism of KG-modules $\varphi : V \longrightarrow V$ is either zero or invertible. In other words $\mathrm{End}_{KG}(V)$ is a skew-field.

(ii) If $V \ncong W$, then $\mathrm{Hom}_{KG}(V, W) = 0$.

(b) If K is an algebraically closed field and V is a simple KG-module, then

$$\mathrm{End}_{KG}(V) = \{\lambda\, \mathrm{Id}_V \mid \lambda \in K\} \cong K\,.$$

Notice that here we state Schur's Lemma in terms of modules, rather than in terms of representations, because part (a) holds in greater generality for arbitrary unital associative rings and part (b) holds for finite-dimensional algebras over an algebraically closed field.

Proof:

(a) First, we claim that every $\varphi \in \mathrm{Hom}_{KG}(V, W)\setminus\{0\}$ admits an inverse in $\mathrm{Hom}_{KG}(W, V)$.
Indeed, $\varphi \neq 0 \implies \ker\varphi \subsetneq V$ is a proper KG-submodule of V and $\{0\} \neq \mathrm{Im}\,\varphi$ is a non-zero KG-submodule of W. But then, on the one hand, $\ker\varphi = \{0\}$, because V is simple, hence φ is injective, and on the other hand, $\mathrm{Im}\,\varphi = W$ because W is simple. It follows that φ is also surjective, hence bijective. Therefore, by Properties A.7, φ is invertible with inverse $\varphi^{-1} \in \mathrm{Hom}_{KG}(W, V)$.
Now, (ii) is straightforward from the above. For (i), first recall that $\mathrm{End}_{KG}(V)$ is a ring (see Notation A.8), which is obviously non-zero as $\mathrm{End}_{KG}(V) \ni \mathrm{Id}_V$ and $\mathrm{Id}_V \neq 0$ because $V \neq 0$ since it is simple. Thus, as any $\varphi \in \mathrm{End}_{KG}(V)\setminus\{0\}$ is invertible, $\mathrm{End}_{KG}(V)$ is a skew-field.

(b) Let $\varphi \in \mathrm{End}_{KG}(V)$. Since $K = \overline{K}$, φ has an eigenvalue $\lambda \in K$. Let $v \in V\setminus\{0\}$ be an eigenvector of φ for λ. Then $(\varphi - \lambda\, \mathrm{Id}_V)(v) = 0$. Therefore, $\varphi - \lambda\, \mathrm{Id}_V$ is not invertible and

$$\varphi - \lambda\, \mathrm{Id}_V \in \mathrm{End}_{KG}(V) \quad \overset{(a)}{\Longrightarrow} \quad \varphi - \lambda\, \mathrm{Id}_V = 0 \quad \Longrightarrow \quad \varphi = \lambda\, \mathrm{Id}_V\,.$$

Hence $\mathrm{End}_{KG}(V) \subseteq \{\lambda\, \mathrm{Id}_V \mid \lambda \in K\}$, but the reverse inclusion also obviously holds, so that

$$\mathrm{End}_{KG}(V) = \{\lambda\, \mathrm{Id}_V\} \cong K\,.$$

■

Exercise 5.2 (*Exercise 8, Sheet 2*)

Prove that in terms of matrix representations the following statement holds:

Lemma 5.3 (*Schur's Lemma for matrix representations*)

Let $R : G \longrightarrow \mathrm{GL}_n(K)$ and $R' : G \longrightarrow \mathrm{GL}_{n'}(K)$ be two irreducible matrix representations. If there exists $A \in M_{n\times n'}(K)\setminus\{0\}$ such that $AR'(g) = R(g)A$ for every $g \in G$, then $n = n'$ and A is invertible (in particular $R \sim R'$).

The next lemma is a general principle, which we have already used in the proof of Maschke's Theorem, and which allows us to transform K-linear maps into KG-linear maps.

Lemma 5.4

Assume $\mathrm{char}(K) \nmid |G|$. Let V, W be two KG-modules and let $\rho_V : G \longrightarrow \mathrm{GL}(V)$, $\rho_W : G \longrightarrow \mathrm{GL}(W)$ be the associated K-representations. If $\psi : V \longrightarrow W$ is K-linear, then the map

$$\widetilde{\psi} := \frac{1}{|G|} \sum_{g \in G} \rho_W(g) \circ \psi \circ \rho_V(g^{-1})$$

from V to W is KG-linear.

Proof: Same argument as in (3) of the proof of Maschke's Theorem: replace π by ψ and apply the fact that a G-homomorphism between representations corresponds to a KG-hmomorphism between the corresponding KG-modules. ■

Proposition 5.5

Assume $\mathrm{char}(K) \nmid |G|$. Let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ and $\rho_W : G \longrightarrow \mathrm{GL}(W)$ be two irreducible K-representations.

(a) If $\rho_V \not\sim \rho_W$ and $\psi : V \longrightarrow W$ is a K-linear map, then

$$\widetilde{\psi} = \frac{1}{|G|} \sum_{g \in G} \rho_W(g) \circ \psi \circ \rho_V(g^{-1}) = 0\,.$$

(b) Assume moreover that $K = \overline{K}$ and $\mathrm{char}(K) \nmid n := \dim_K V$. If $\psi : V \longrightarrow V$ is a K-linear map, then

$$\widetilde{\psi} := \frac{1}{|G|} \sum_{g \in G} \rho_V(g) \circ \psi \circ \rho_V(g^{-1}) = \frac{\mathrm{Tr}(\psi)}{n} \cdot \mathrm{Id}_V\ .$$

Proof: Since ρ_V and ρ_W are irreducible, the associated KG-modules are simple. Moreover, by Lemma 5.4, both in (a) and (b) the map $\widetilde{\psi}$ is KG-linear. Therefore Schur's Lemma yields:

(a) $\widetilde{\psi} = 0$ since $V \not\cong W$.

(b) $\widetilde{\psi} = \lambda \cdot \mathrm{Id}_V$ for some scalar $\lambda \in K$. Therefore, on the one hand

$$\mathrm{Tr}(\widetilde{\psi}) = \frac{1}{|G|} \sum_{g \in G} \underbrace{\mathrm{Tr}\left(\rho_V(g) \circ \psi \circ \rho_V(g^{-1})\right)}_{=\mathrm{Tr}(\psi)} = \frac{1}{|G|} |G|\, \mathrm{Tr}(\psi) = \mathrm{Tr}(\psi)$$

and on the other hand

$$\mathrm{Tr}(\widetilde{\psi}) = \mathrm{Tr}(\lambda \cdot \mathrm{Id}_V) = \lambda\, \mathrm{Tr}(\mathrm{Id}_V) = n \cdot \lambda\,,$$

hence $\lambda = \frac{\mathrm{Tr}(\psi)}{n}$. ■

Next, we see that Schur's Lemma implies certain "orthogonality relations" for the entries of matrix representations.

Theorem 5.6 (Schur's Relations)

Assume $\mathrm{char}(K) \nmid |G|$. Let $Q : G \longrightarrow \mathrm{GL}_n(K)$ and $P : G \longrightarrow \mathrm{GL}_m(K)$ be irreducible matrix representations.

(a) If $P \not\sim Q$, then $\frac{1}{|G|}\sum_{g\in G} P(g)_{ri}Q(g^{-1})_{js} = 0$ for all $1 \leqslant r, i \leqslant m$ and all $1 \leqslant j, s \leqslant n$.

(b) If $\mathrm{char}(K) \nmid n$, then $\frac{1}{|G|}\sum_{g\in G} Q(g)_{ri}Q(g^{-1})_{js} = \frac{1}{n}\delta_{ij}\delta_{rs}$ for all $1 \leqslant r, i, j, s \leqslant n$.

Proof: Set $V := K^n$, $W := K^m$ and let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ and $\rho_W : G \longrightarrow \mathrm{GL}(W)$ be the K-representations induced by Q and P, respectively, as defined in Remark 1.2. Furthermore, consider the K-linear map $\psi : V \longrightarrow W$ whose matrix with respect to the standard bases of $V = K^n$ and $W = K^m$ is the elementary matrix

$$\begin{array}{cc} i & \begin{bmatrix} & \vdots & \\ \cdots\cdots & 1 & \cdots\cdots \\ & \vdots & \end{bmatrix} \\ & j \end{array} =: E_{ij} \in M_{m\times n}(K)$$

(i.e. the unique nonzero entry of E_{ij} is its (i, j)-entry).

(a) By Proposition 5.5(a),

$$\widetilde{\psi} = \frac{1}{|G|}\sum_{g\in G} \rho_W(g) \circ \psi \circ \rho_V(g^{-1}) = 0$$

because $P \not\sim Q$, and hence $\rho_V \not\sim \rho_W$. In particular the (r, s)-entry of the matrix of $\widetilde{\psi}$ with respect to the standard bases of $V = K^n$ and $W = K^m$ is zero. Thus,

$$0 = \frac{1}{|G|}\sum_{g\in G} \left[P(g)E_{ij}Q(g^{-1})\right]_{rs} = \frac{1}{|G|}\sum_{g\in G} P(g)_{ri} \cdot 1 \cdot Q(g^{-1})_{js}$$

because the unique nonzero entry of the matrix E_{ij} is its (i, j)-entry.

(b) Now we assume that $P = Q$, and hence $n = m$, $V = W$, $\rho_V = \rho_W$. Then by Proposition 5.5(b),

$$\widetilde{\psi} := \frac{1}{|G|}\sum_{g\in G} \rho_V(g) \circ \psi \circ \rho_V(g^{-1}) = \frac{\mathrm{Tr}(\psi)}{n} \cdot \mathrm{Id}_V = \begin{cases} \frac{1}{n} \cdot \mathrm{Id}_V & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Therefore the (r, s)-entry of the matrix of $\widetilde{\psi}$ with respect to the standard basis of $V = K^n$ is

$$\frac{1}{|G|}\sum_{g\in G} \left[Q(g)E_{ij}Q(g^{-1})\right]_{rs} = \begin{cases} \left(\frac{1}{n} \cdot \mathrm{Id}_V\right)_{rs} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Again, because the unique nonzero entry of the matrix E_{ij} is its (i, j)-entry, it follows that

$$\frac{1}{|G|}\sum_{g\in G} Q(g)_{ri}Q(g^{-1})_{js} = \frac{1}{n}\delta_{ij}\delta_{rs}\,.$$

■

6 $\mathbb{C}$-Representations of Finite Abelian Groups

In this section we give an immediate application of Schur's Lemma encoding the representation theory of finite abelian groups over the field $\mathbb{C}$ of complex numbers.

Proposition 6.1

If G is a finite abelian group, then any simple $\mathbb{C}G$-module has dimension 1. (Equivalently, any irreducible $\mathbb{C}$-representation of G has degree 1.)

Proof: Let V be a simple $\mathbb{C}G$-module, and let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ be the associated $\mathbb{C}$-representation (i.e. as given by Proposition 4.3).

Claim: any $\mathbb{C}$-subspace of V is in fact a $\mathbb{C}G$-submodule.

Proof: Fix $g \in G$ and consider $\rho_V(g)$. By definition $\rho_V(g) \in \mathrm{GL}(V)$, hence it is a $\mathbb{C}$-linear endomorphism of V. We claim that it is in fact $\mathbb{C}G$-linear. Indeed, as G is abelian, $\forall\, h \in G$, $\forall\, v \in V$ we have

$$\begin{aligned}\rho_V(g)(h \cdot v) = \rho_V(g)\big(\rho_V(h)(v)\big) &= \big[\rho_V(g)\rho_V(h)\big](v)\\ &= \big[\rho_V(gh)\big](v)\\ &= \big[\rho_V(hg)\big](v)\\ &= \big[\rho_V(h)\rho_V(g)\big](v)\\ &= \rho_V(h)\big(\rho_V(g)(v)\big)\\ &= h \cdot \big(\rho_V(g)(v)\big)\end{aligned}$$

and it follows from Remark 4.4 that $\rho_V(g)$ is $\mathbb{C}G$-linear. Now, because $\mathbb{C}$ is algebraically closed, by part (b) of Schur's Lemma, there exists $\lambda_g \in \mathbb{C}$ (depending on g) such that

$$\rho_V(g) = \lambda_g \cdot \mathrm{Id}_V \, .$$

As this holds for every $g \in G$, it follows that any $\mathbb{C}$-subspace of V is G-invariant, which in terms of $\mathbb{C}G$-modules means that any $\mathbb{C}$-subspace of V is a $\mathbb{C}G$-submodule of V.

To conclude, as V is simple, we deduce from the Claim that the $\mathbb{C}$-dimension of V must be equal to 1. ■

Theorem 6.2 (*Diagonalisation Theorem*)

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation of an arbitrary finite group G. Fix $g \in G$. Then, there exists an ordered $\mathbb{C}$-basis B of V with respect to which

$$\big(\rho(g)\big)_B = \begin{bmatrix} \varepsilon_1 & 0 & \cdots & \cdots & 0 \\ 0 & \varepsilon_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \varepsilon_n \end{bmatrix},$$

where $n := \dim_{\mathbb{C}}(V)$ and each ε_i ($1 \leqslant i \leqslant n$) is an $o(g)$-th root of unity in $\mathbb{C}$.

Proof: Consider the restriction of ρ to the cyclic subgroup generated by g, that is the representation

$$\rho|_{\langle g \rangle} : \langle g \rangle \longrightarrow \mathrm{GL}(V)\, .$$

By Corollary 3.6 to Maschke's Theorem, we can decompose the representation $\rho|_{\langle g \rangle}$ into a direct sum of irreducible $\mathbb{C}$-representations, say

$$\rho|_{\langle g \rangle} = \rho_{V_1} \oplus \cdots \oplus \rho_{V_n}\, ,$$

where $V_1, \dots, V_n \subseteq V$ are $\langle g \rangle$-invariant. Since $\langle g \rangle$ is abelian $\dim_{\mathbb{C}}(V_i) = 1$ for each $1 \leqslant i \leqslant n$ by Proposition 6.1. Now, if for each $1 \leqslant i \leqslant n$ we choose a $\mathbb{C}$-basis $\{x_i\}$ of V_i, then there exist $\varepsilon_i \in \mathbb{C}$

($1 \leqslant i \leqslant n$) such that $\rho_{V_i}(g) = \varepsilon_i$ and $B := (x_1, \dots, x_n)$ is an ordered $\mathbb{C}$-basis of V such that

$$\big(\rho(g)\big)_B = \begin{bmatrix} \varepsilon_1 & 0 & \cdots & 0 \\ 0 & \varepsilon_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \varepsilon_n \end{bmatrix}.$$

Finally, as $g^{o(g)} = 1_G$, it follows that for each $1 \leqslant i \leqslant n$,

$$\varepsilon_i^{o(g)} = \rho_{V_i}(g)^{o(g)} = \rho_{V_i}(g^{o(g)}) = \rho_{V_i}(1_G) = 1_{\mathbb{C}}$$

and hence ε_i is an $o(g)$-th root of unity. ■

Scholium 6.3

If $\rho : G \longrightarrow \mathrm{GL}(V)$ is a $\mathbb{C}$-representation of a finite abelian group, then the $\mathbb{C}$-endomorphisms $\rho(g) : V \longrightarrow V$ with g running through G are simultaneously diagonalisable.

Proof: Same argument as in the previous proof, where we may replace "$\langle g \rangle$" with the whole of G. ■