Chapter 4. The Character Table

In Chapter 3 we have proved that for any finite group G the equality $|\operatorname{Irr}(G)| = |C(G)| =: r$ holds. Thus the values of the irreducible characters of G can be recorded in an $r \times r$-matrix, called the *character table* of G. The entries of this matrix are related to each other in subtle manners, many of which are encapsulated in the 1st Orthogonality Relations and their consequences, as for example the degree formula. Our aim in this chapter is to develop further tools and methods to compute character tables.

Notation: throughout this chapter, unless otherwise specified, we let:

- G denote a finite group;
- $K := \mathbb{C}$ be the field of complex numbers;
- $|\operatorname{Irr}(G)| = |C(G)| =: r$;
- $\operatorname{Irr}(G) = \{\chi_1, \dots, \chi_r\}$ denote the set of pairwise distinct irreducible characters of G;
- $C_1 = [g_1], \dots, C_r = [g_r]$ denote the conjugacy classes of G, where $g_1, \dots, g_r$ is a fixed set of representatives; and
- we use the convention that $\chi_1 = \mathbf{1}_G$ and $g_1 = 1 \in G$.

In general, unless otherwise stated, all groups considered are assumed to be finite and all $\mathbb{C}$-vector spaces / modules over the group algebra considered are assumed to be finite-dimensional.

11 The Character Table of a Finite Group

Definition 11.1 (*Character table*)

The **character table of** G is the matrix $X(G) := \Big(\chi_i(g_j)\Big)_{ij} \in M_r(\mathbb{C})$.

Example 4 (*The character table of a cyclic group*)

Let $G = \langle g \mid g^n = 1 \rangle$ be cyclic of order $n \in \mathbb{Z}_{>0}$. Since G is abelian,

$$\operatorname{Irr}(G) = \{\text{linear characters of } G\}$$

by Proposition 6.1 and $|\operatorname{Irr}(G)| = |G| = n$. Moreover, each conjugacy class is a singleton:

$$\forall\, 1 \leqslant j \leqslant r = n : \quad C_j = \{g_j\} \text{ and we set } g_j := g^{j-1}.$$

Let ζ be a primitive n-th root of unity in $\mathbb{C}$, so that $\{\zeta^i \mid 1 \leqslant i \leqslant n\}$ are all the n-th roots of unity. Now, each $\chi_i : G \to \mathbb{C}^\times$ is a group homomorphism and is determined by $\chi_i(g)$, which has to be an n-th root of $1_\mathbb{C}$. Therefore, we have n possibilities for $\chi_i(g)$. We set

$$\chi_i(g) := \zeta^{i-1} \quad \forall\, 1 \leqslant i \leqslant n \quad \Rightarrow \quad \chi_i(g^j) = \zeta^{(i-1)j} \quad \forall\, 1 \leqslant i \leqslant n, 0 \leqslant j \leqslant n-1$$

Thus the character table of G is

$$X(G) = \Big(\chi_i(g_j)\Big)_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}} = \Big(\chi_i(g^{j-1})\Big)_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}} = \Big(\zeta^{(i-1)(j-1)}\Big)_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}},$$

which we visualise as follows:

	1	g	g^2	$\dots$	g^{n-1}
$\chi_1 = 1_G$	1	1	1	$\dots$	1
χ_2	1	ζ	ζ^2	$\dots$	ζ^{n-1}
χ_3	1	ζ^2	ζ^4	$\dots$	$\zeta^{2(n-1)}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
χ_n	1	ζ^{n-1}	$\zeta^{2(n-1)}$	$\dots$	$\zeta^{(n-1)^2}$

Example 5 (*The character table of S_3*)

Let now $G := S_3$ be the symmetric group on 3 letters. Recall from the AGS/*Einführung in die Algebra* that the conjugacy classes of S_3 are

$$C_1 = \{\mathrm{Id}\},\ C_2 = \{(1\ 2), (1\ 3), (2\ 3)\},\ C_3 = \{(1\ 2\ 3), (1\ 3\ 2)\}$$

$$\Rightarrow \quad r = 3, |C_1| = 1, |C_2| = 3, |C_3| = 2\,.$$

In Example 2(d) we have exhibited three non-equivalent irreducible matrix representations of S_3, which we denoted ρ_1, ρ_2, ρ_3. For each $1 \leqslant i \leqslant 3$ let χ_i be the character of ρ_i and n_i be its degree, so that $n_1 = n_2 = 1$ and $n_3 = 2$. Hence

$$n_1^2 + n_2^2 + n_3^2 = 6 = |G|\,.$$

Therefore, the degree formula tells us that ρ_1, ρ_2, ρ_3 are all the irreducible matrix representations of S_3, up to equivalence. We note that $n_1 = n_2 = 1$, $n_3 = 2$ is in fact the unique solution (up to relabelling) to the equation given by the degree formula! Taking traces of the matrices in Example 2(d) yields the character table of S_3.

	Id	(1 2)	(1 2 3)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

In the next sections we want to develop further techniques to compute character tables of finite groups, before we come back to further examples of such tables for larger groups.

Exercise 11.2 (*Exercise 14(d), Sheet 4*)

Compute the character table of the Klein-four group $C_2 \times C_2$ and of $C_2 \times C_2 \times C_2$.

12 The 2nd Orthogonality Relations

The 1st Orthogonality Relations provide us with orthogonality relations between the rows of the character table. They can be rewritten as follows in terms of matrices.

Exercise 12.1 (*Exercise 14(a), Sheet 4*)

Let G be a finite group. Set $X := X(G)$ and

$$C := \begin{bmatrix} |C_G(g_1)| & 0 & \cdots & 0 \\ 0 & |C_G(g_2)| & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & |C_G(g_r)| \end{bmatrix} \in M_r(\mathbb{C})\,.$$

Use the Orbit-Stabiliser Theorem in order to prove that the 1st Orthogonality Relations can be rewritten under the form

$$XC^{-1}\overline{X}^{\mathrm{Tr}} = I_r\,,$$

where $\overline{X}^{\mathrm{Tr}}$ denotes the transpose of the complex-conjugate $\overline{X}$ of the character table X of G. Deduce that the character table is invertible.

There are also some orthogonality relations between the columns of the character table. These can easily be deduced from the 1st Orthogonality Relations given above in terms of matrices.

Theorem 12.2 (*2nd Orthogonality Relations*)

With the notation of Exercise 12.1 we have

$$X^{\mathrm{Tr}}\,\overline{X} = C\,.$$

In other words,

$$\sum_{\chi\in\mathrm{Irr}(G)} \chi(g_i)\overline{\chi(g_j)} = \delta_{ij}\frac{|G|}{|[g_i]|} = \delta_{ij}|C_G(g_i)| \qquad \forall\, 1\leqslant i,j\leqslant r\,.$$

Proof: Taking complex conjugation of the formula given by the 1st Orthogonality Relations (Exercise 12.1) yields:

$$XC^{-1}\overline{X}^{\mathrm{Tr}} = I_r \qquad \Longrightarrow \qquad \overline{X}C^{-1}X^{\mathrm{Tr}} = I_r$$

Now, since X is invertible, so are all the matrices in the above equations and hence $X^{\mathrm{Tr}} = \left(\overline{X}C^{-1}\right)^{-1}$. It follows that

$$X^{\mathrm{Tr}}\,\overline{X} = \left(\overline{X}C^{-1}\right)^{-1}\overline{X} = C\overline{X}^{-1}\overline{X} = C\,.$$

The second formula is now obtained by considering the entry (i,j) in the above matrix equation for all $1\leqslant i,j\leqslant r$. ■

Exercise 12.3 (*Exercise 13(a), Sheet 4*)

Prove that the degree formula can be read off from the 2nd Orthogonality Relations.

13 Tensor Products of Representations and Characters

Tensor products of vector spaces and matrices are recalled/introduced in Appendix C. We are now going to use this construction to build *products* of characters.

Proposition 13.1

Let G and H be finite groups, and let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ and $\rho_W : H \longrightarrow \mathrm{GL}(W)$ be $\mathbb{C}$-representations with characters χ_V and χ_W respectively. Then

$$\begin{array}{rccl} \rho_V \otimes \rho_W : & G \times H & \longrightarrow & \mathrm{GL}(V \otimes_{\mathbb{C}} W) \\ & (g,h) & \mapsto & (\rho_V \otimes \rho_W)(g,h) := \rho_V(g) \otimes \rho_W(h) \end{array}$$

(where $\rho_V(g) \otimes \rho_W(h)$ is the tensor product of the $\mathbb{C}$-endomorphisms $\rho_V(g) : V \longrightarrow V$ and $\rho_W(h) : W \longrightarrow W$ as defined in Lemma-Definition C.4) is a $\mathbb{C}$-representation of $G \times H$, called the **tensor product** of ρ_V and ρ_W, and the corresponding character, which we denote by $\chi_{V \otimes_{\mathbb{C}} W}$, is

$$\chi_{V \otimes_{\mathbb{C}} W} = \chi_V \cdot \chi_W ,$$

where $\chi_V \cdot \chi_W(g,h) := \chi_V(g) \cdot \chi_W(h) \ \forall \ (g,h) \in G \times H$.

Proof: First note that $\rho_V \otimes \rho_W$ is well-defined by Lemma-Definition C.4 and it is a group homomorphism because

$$\begin{aligned} (\rho_V \otimes \rho_W)(g_1g_2, h_1h_2)[v \otimes w] &= \big(\rho_V(g_1g_2) \otimes \rho_W(h_1h_2)\big)[v \otimes w] \\ &= \rho_V(g_1g_2)[v] \otimes \rho_W(h_1h_2)[w] \\ &= \rho_V(g_1) \circ \rho_V(g_2)[v] \otimes \rho_W(h_1) \circ \rho_W(h_2)[w] \\ &= \rho_V(g_1) \otimes \rho_W(h_1)\big[\rho_V(g_2)[v] \otimes \rho_W(h_2)[w]\big] \\ &= \big(\rho_V(g_1) \otimes \rho_W(h_1)\big) \circ \big(\rho_V(g_2) \otimes \rho_W(h_2)\big)[v \otimes w] \\ &= \big(\rho_V \otimes \rho_W\big)(g_1,h_1) \circ \big(\rho_V \otimes \rho_W\big)(g_2,h_2)[v \otimes w] \end{aligned}$$

$\forall\ g_1, g_2 \in G$, $h_1, h_2 \in H$, $v \in V$, $w \in W$. Furthermore, for each $g \in G$ and each $h \in H$,

$$\chi_{V \otimes_{\mathbb{C}} W}(g,h) = \mathrm{Tr}\left((\rho_V \otimes \rho_W)(g,h)\right) = \mathrm{Tr}\left(\rho_V(g) \otimes \rho_W(h)\right) = \mathrm{Tr}\left(\rho_V(g)\right) \cdot \mathrm{Tr}\left(\rho_W(h)\right) = \chi_V(g) \cdot \chi_W(h)$$

by Lemma-Definition C.4, hence $\chi_{V \otimes_{\mathbb{C}} W} = \chi_V \cdot \chi_W$. ■

Remark 13.2

The diagonal inclusion $\iota : G \longrightarrow G \times G, g \mapsto (g,g)$ of G in the product $G \times G$ is a group homomorphism with $\iota(G) \cong G$. Therefore, if $G = H$, then

$$G \xrightarrow{\iota} G \times G \xrightarrow{\chi_V \cdot \chi_W} \mathbb{C}, g \mapsto (g,g) \mapsto \chi_V(g) \cdot \chi_W(g)$$

becomes a character of G, which we also denote by $\chi_V \cdot \chi_W$.

Corollary 13.3

If G and H are finite groups, then $\mathrm{Irr}(G \times H) = \{\chi \cdot \psi \mid \chi \in \mathrm{Irr}(G), \psi \in \mathrm{Irr}(H)\}$.

Proof: [Exercise 15(c), Sheet 5]. Hint: Use Corollary 9.8(d) and the degree formula. ■

Exercise 13.4 (*Exercise 15(a)+(b), Sheet 5*)

(a) If $\lambda, \chi \in \mathrm{Irr}(G)$ and $\lambda(1) = 1$, then $\lambda \cdot \chi \in \mathrm{Irr}(G)$.

(b) The set $\{\chi \in \mathrm{Irr}(G) \mid \chi(1) = 1\}$ of linear characters of a finite group G forms a group for the product of characters.

14 Normal Subgroups and Inflation

Whenever a group homomorphism $G \longrightarrow H$ and a representation of H are given, we obtain a representation of G by composition. In particular, we want to apply this principle to normal subgroups $N \trianglelefteq G$ and the corresponding quotient homomorphism, which we always denote by $\pi : G \longrightarrow G/N, g \mapsto gN$.

We will see that by this means, copies of the character tables of quotient groups of G all appear in the character table of G. This observation, although straightforward, will allow us to fill out the character table of a group very rapidly, provided it possesses normal subgroups.

Definition 14.1 (*Inflation*)

Let $N \trianglelefteq G$ and let $\pi : G \longrightarrow G/N, g \mapsto gN$ be the quotient homomorphism. Given a $\mathbb{C}$-representation $\rho : G/N \longrightarrow \mathrm{GL}(V)$, we set

$$\mathrm{Inf}_{G/N}^{G}(\rho) := \rho \circ \pi : G \longrightarrow \mathrm{GL}(V)\,.$$

This is a $\mathbb{C}$-representation of G, called the **inflation of ρ from G/N to G**. If the character of ρ is χ, then we denote by $\mathrm{Inf}_{G/N}^{G}(\chi)$ the character of $\mathrm{Inf}_{G/N}^{G}(\rho)$ and call it the **inflation of χ from G/N to G**.

Note that some texts also call $\mathrm{Inf}_{G/N}^{G}(\rho)$ (resp. $\mathrm{Inf}_{G/N}^{G}(\chi)$) the *lift* of ρ (resp. χ) along π.

Remark 14.2

The values of the character $\mathrm{Inf}_{G/N}^{G}(\chi)$ of G are obtained from those of χ as follows. If $g \in G$, then

$$\mathrm{Inf}_{G/N}^{G}(\chi)(g) = \mathrm{Tr}\left((\rho \circ \pi)(g)\right) = \mathrm{Tr}\left(\rho(gN)\right) = \chi(gN)\,.$$

Exercise 14.3 (*Exercise 16(a), Sheet 5*)

Let $N \trianglelefteq G$ and let $\rho : G/N \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation of G/N with character χ.

(a) Prove that if ρ is irreducible, then so is $\mathrm{Inf}_{G/N}^{G}(\rho)$.

(b) Compute the kernel of $\mathrm{Inf}_{G/N}^{G}(\rho)$ provided that ρ is faithful.

Definition 14.4 (*Kernel of a character*)

The **kernel of a character** χ of G is $\ker(\chi) := \{g \in G \mid \chi(g) = \chi(1)\}$.

Example 6

(a) $\chi = \mathbf{1}_G$ the trivial character $\Rightarrow \ker(\chi) = G$.

(b) $G = \mathfrak{S}_3$, $\chi = \chi_2$ the sign character $\Rightarrow \ker(\chi) = C_1 \cup C_3 = \langle(123)\rangle$; whereas $\ker(\chi_3) = \{1\}$. (See Example 5.)

Lemma 14.5

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation of G with character ψ. Then $\ker(\psi) = \ker(\rho)$, thus is a normal subgroup of G.

Proof: [Exercise 16(b), Sheet 5] ■

Theorem 14.6

Let $N \trianglelefteq G$. Then

$$\begin{array}{rccc} \mathrm{Inf}_{G/N}^{G} : & \{\text{characters of } G/N\} & \longrightarrow & \{\text{characters } \psi \text{ of } G \mid N \leqslant \ker(\psi)\} \\ & \chi & \mapsto & \mathrm{Inf}_{G/N}^{G}(\chi) \end{array}$$

is a bijection and so is its restriction to the irreducible characters

$$\begin{array}{rccc} \mathrm{Inf}_{G/N}^{G} : & \mathrm{Irr}(G/N) & \longrightarrow & \{\psi \in \mathrm{Irr}(G) \mid N \leqslant \ker(\psi)\} \\ & \chi & \mapsto & \mathrm{Inf}_{G/N}^{G}(\chi)\,. \end{array}$$

Proof: First we prove that the first map is well-defined and bijective.

- Let χ be a character of G/N afforded by the $\mathbb{C}$-representation $\rho : G/N \longrightarrow \mathrm{GL}(V)$. By Remark 14.2, N is in the kernel of $\mathrm{Inf}_{G/N}^{G}(\chi)$, hence the first map is well-defined.
- Now let ψ be a character of G with $N \leqslant \ker(\psi)$ and assume ψ is afforded by the $\mathbb{C}$-representation $\rho : G \longrightarrow \mathrm{GL}(V)$.

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[d, two heads, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{GL}(V) \\ G/N \arrow[ur, dashed, "\exists!\,\widetilde{\rho}"'] & \end{tikzcd}$$

By Lemma 14.5 we have $\ker(\psi) = \ker(\rho) \geqslant N$. Therefore, by the universal property of the quotient, ρ induces a unique $\mathbb{C}$-representation $\widetilde{\rho} : G/N \longrightarrow \mathrm{GL}(V)$ with the property that $\widetilde{\rho} \circ \pi = \rho$.

It follows that $\rho = \mathrm{Inf}_{G/N}^{G}(\widetilde{\rho})$ and $\psi = \mathrm{Inf}_{G/N}^{G}(\chi)$. Thus the 1st map is surjective. Its injectivity is clear.

The second map is well-defined by the above and Exercise 14.3(a). It is injective because it is just the restriction of the 1st map to the $\mathrm{Irr}(G/N)$, whereas it is surjective by the same argument as above as the constructed representation $\widetilde{\rho}$ is clearly irreducible if ρ is because $\widetilde{\rho} \circ \pi = \rho$. ■

Exercise 14.7 (*Exercise 16(c), Sheet 5*)

Let G be a finite group. Prove that if $N \trianglelefteq G$, then

$$N = \bigcap_{\substack{\chi \in \mathrm{Irr}(G) \\ N \subseteq \ker(\chi)}} \ker(\chi)\,.$$

It follows immediately from the above exercise that the lattice of normal subgroups of G can be read off from its character table. The theorem also implies that it can be read off from the character table, whether the group is abelian or simple.

Corollary 14.8

(a) Inflation from the abelianisation induces a bijection

$$\mathrm{Inf}_{G/G'}^{G} : \mathrm{Irr}(G/G') \xrightarrow{\sim} \{\psi \in \mathrm{Irr}(G) \mid \psi(1) = 1\} \ ;$$

in particular, G has precisely $|G : G'|$ linear characters.

(b) The group G is abelian if and only if all its irreducible characters are linear.

Proof: (a) First, we claim that if $\psi \in \mathrm{Irr}(G)$ is linear, then G' is in its kernel. Indeed, if $\psi(1) = 1$, then $\psi : G \longrightarrow \mathbb{C}^\times$ is a group homomorphism. Therefore, as $\mathbb{C}^\times$ is abelian,

$$\psi([g,h]) = \psi(ghg^{-1}h^{-1}) = \psi(g)\psi(h)\psi(g)^{-1}\psi(h)^{-1} = \psi(g)\psi(g)^{-1}\psi(h)\psi(h)^{-1} = 1$$

for all $g, h \in G$, and hence $G' = \langle [g,h] \mid g, h \in G \rangle \leqslant \ker(\chi)$. In addition, any irreducible character of G/G' is linear by Proposition 6.1 because G/G' is abelian. Thus Theorem 14.6 yields a bijection

$$\mathrm{Irr}(G/G') \xrightarrow[\mathrm{Inf}_{G/G'}^{G}]{\sim} \{\psi \in \mathrm{Irr}(G) \mid G' \leqslant \ker(\psi)\} = \{\psi \in \mathrm{Irr}(G) \mid \psi(1) = 1\} \, ,$$

as required.

(b) If G is abelian, then $G/G' = G$. Hence the claim follows from (a). ■

Corollary 14.9

A finite group G is simple $\iff$ $\chi(g) \neq \chi(1)\ \ \forall\, g \in G \backslash \{1\}$ and $\forall\, \chi \in \mathrm{Irr}(G) \backslash \{\mathbf{1}_G\}$.

Proof: [Exercise 16(d), Sheet 5] ■

Exercise 14.10 (*Exercise 18, Sheet 5*)

Compute the complex character table of the alternating group A_4 through the following steps:

1. Determine the conjugacy classes of A_4 (there are 4 of them) and the corresponding centraliser orders.
2. Determine the degrees of the 4 irreducible characters of A_4.
3. Determine the linear characters of A_4.
4. Determine the non-linear character of A_4 using the 2nd Orthogonality Relations.

To finish this section we show how to compute the character table of the symmetric group S_4 combining several of the techniques we have developed in this chapter.

Example 7 (*The character table of S_4*)

Again the conjugacy classes of S_4 are given by the cycle types. We fix

$$C_1 = \{\mathrm{Id}\},\ C_2 = [(1\ 2)],\ C_3 = [(1\ 2\ 3)],\ C_4 = [(1\ 2)(3\ 4)],\ C_5 = [(1234)]$$

$$\Rightarrow \quad r = 5,\ |C_1| = 1,\ |C_2| = 6,\ |C_3| = 8,\ |C_4| = 3,\ |C_5| = 6\,.$$

Hence $|\operatorname{Irr}(G)| = |C(G)| = 5$ and as always we may assume that $\chi_1 = \mathbf{1}_G$ is the trivial character.

Recall that $V_4 = \{\mathrm{Id}, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \trianglelefteq S_4$ with $S_4/V_4 \cong S_3$ (AGS or Einführung in die Algebra!). Therefore, by Theorem 14.6 we can "inflate" the character table of $S_4/V_4 \cong S_3$ to S_4 (see Example 5 for the character table of S_3). This provides us with three irreducible characters χ_1, χ_2 and χ_3 of S_4:

	Id	(1 2)	(1 2 3)	(1 2)(3 4)	(1 2 3 4)
$\lvert C_G(g_i)\rvert$	24	4	3	8	4
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	·	·	·	·	·
χ_5	·	·	·	·	·

Here we have computed the values of χ_2 and χ_3 using Remark 14.2 as follows:

- Inflation preserves degrees, hence it follows from Example 5 that $\chi_2(\mathrm{Id}) = 1$ and $\chi_3(\mathrm{Id}) = 2$. (Up to relabelling!)
- As $C_4 = [(1\ 2)(3\ 4)] \subseteq V_4$, $(1\ 2)(3\ 4) \in \ker(\chi_i)$ for $i = 2, 3$ and hence $\chi_2((1\ 2)(3\ 4)) = 1$ and $\chi_3((1\ 2)(3\ 4)) = 2$.
- By Remark 14.2 the values of χ_2 and χ_3 at $(1\ 2)$ and $(1\ 2\ 3)$ are given by the corresponding values in the character table of S_3. (Here it is enough to argue that the isomorphism between S_4/V_4 and S_3 must preserve orders of elements, hence also the cycle type in this case.)
- Finally, we compute that $\overline{(1\ 2\ 3\ 4)} = \overline{(1\ 2)} \in S_4/V_4$, hence $\chi_i((1\ 2\ 3\ 4)) = \chi_i((1\ 2))$ for $i = 2, 3$.

Therefore, it remains to compute χ_4 and χ_5. To begin with the degree formula yields

$$\sum_{i=1}^{5} \chi_i(\mathrm{Id})^2 = 24 \quad \Longrightarrow \quad \chi_4(\mathrm{Id})^2 + \chi_5(\mathrm{Id})^2 = 18 \quad \Longrightarrow \quad \chi_4(\mathrm{Id}) = \chi_5(\mathrm{Id}) = 3\,.$$

Next, the 2nd Orthogonality Relations applied to the 3rd column with itself read

$$\sum_{i=1}^{5} \chi_i((1\ 2\ 3))\overline{\chi_i((1\ 2\ 3))} = \sum_{i=1}^{5} \chi_i((1\ 2\ 3))\chi_i((1\ 2\ 3)^{-1}) = |C_G((1\ 2\ 3))| = 3\,,$$

hence $1 + 1 + 1 + \chi_4((1\ 2\ 3))^2 + \chi_5(1\ 2\ 3))^2 = 3$ and it follows that $\chi_4((1\ 2\ 3)) = \chi_5((1\ 2\ 3)) = 0$. Similarly, the 2nd Orthogonality Relations applied to the 2nd column with itself / the 4th column with itself and the 5th column with itself yield that all other entries squared are equal to 1, hence

all other entries are ± 1.
The 2nd Orthogonality Relations applied to the 1st and 2nd columns give the 2nd column, i.e. $\chi_4((1\ 2)) = 1$ and $\chi_5((1\ 2)) = -1$ (up to swapping χ_4 and χ_5).
Then the 1st Orthogonality Relations applied to the 3rd and the 4th row yield

$$0 = \sum_{k=1}^{5} \frac{1}{|C_G(g_k)|}\chi_3(g_k)\overline{\chi_4(g_k)} = \frac{6}{24} + \frac{1}{4}\chi_4((1\ 2)(3\ 4)) \ \Rightarrow\ \chi_4((1\ 2)(3\ 4)) = -1\,.$$

Similar with the 3rd row and the 5th row: $\chi_5((1\ 2)(3\ 4)) = -1$. Finally the 1st Orthogonality Relations applied to the 1st and the 4th (resp. 5th) row yield $\chi_4((1\ 2\ 3\ 4)) = -1$ (resp. $\chi_5((1\ 2\ 3\ 4)) = 1$). Thus the character table of S_4 is:

	Id	(1 2)	(1 2 3)	(1 2)(3 4)	(1 2 3 4)
$\lvert C_G(g_i)\rvert$	24	4	3	8	4
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	-1	1

Remark 14.11

Two non-isomorphic groups can have the same character table. E.g.: Q_8 and D_8, but $Q_8 \not\cong D_8$. So the character table does not determine:

- the group up to isomorphism;
- the full lattice of subgroups;
- the orders of elements.

Exercise 14.12 (*Exercise 17(a), Sheet 5*)

Compute the character tables of D_8 and Q_8.
[Hint: In each case, determine the commutator subgroup and deduce that there are 4 linear characters.]

Exercise 14.13 (*The determinant of a representation / Exercise 17(b), Sheet 5*)

If $\rho : G \longrightarrow \mathrm{GL}(V)$ is a $\mathbb{C}$-representation of G and $\det : \mathrm{GL}(V) \longrightarrow \mathbb{C}^*$ denotes the determinant homomorphism, then we define a linear character of G via

$$\det_\rho := \det \circ \rho : G \longrightarrow \mathbb{C}^*\,,$$

called the **determinant** of ρ. Prove that, although the finite groups D_8 and Q_8 have the same character table, they can be distinguished by considering the determinants of their irreducible $\mathbb{C}$-representations.

Exercise 14.14 (*Exercise 19, Sheet 6*)

Prove the follwing assertions:

(a) If G is a non-abelian simple group (or more generally if G is perfect, i.e. $G = [G, G]$), then the image $\rho(G)$ of any $\mathbb{C}$-representation $\rho : G \longrightarrow \mathrm{GL}(V)$ is a subgroup of $\mathrm{SL}(V)$.

(b) No simple group G has an irreducible character of degree 2.
Assume that G is simple and $\rho : G \longrightarrow \mathrm{GL}_2(\mathbb{C})$ is an irreducible matrix representation of G with character χ and proceed as follows:

1. Prove that ρ is faithful and G is non-abelian.
3. Determine the determinant $\det_\rho$ of ρ.
4. Prove that $|G|$ is even and G admits an element x of order 2.
5. Prove that $\langle x \rangle \lhd G$ and conclude that assertion (b) holds.