# Chapter 6. Induction and Restriction of Characters

In this chapter we present important methods to construct / relate characters of a group, given characters of subgroups or overgroups. The main idea is that we would like to be able to use the character tables of groups we know already in order to compute the character tables of subgroups or overgroups of these groups.

**Notation**: throughout this chapter, unless otherwise specified, we let:

- $G$ denote a finite group, $H \leqslant G$ and $N \trianglelefteq G$.
- $K := \mathbb{C}$ be the field of complex numbers;
- $\mathrm{Irr}(G) := \{\chi_1, \ldots, \chi_r\}$ denote the set of pairwise distinct irreducible characters of $G$;
- $C_1 = [g_1], \ldots, C_r = [g_r]$ denote the conjugacy classes of $G$, where $g_1, \ldots, g_r$ is a fixed set of representatives; and
- we use the convention that $\chi_1 = \mathbf{1}_G$ and $g_1 = 1 \in G$.

In general, unless otherwise stated, all groups considered are assumed to be finite and all $\mathbb{C}$-vector spaces / modules over the group algebra considered are assumed to be finite-dimensional.

## 19 Induction and Restriction

We aim at *inducing* and *restricting* characters from subgroups, resp. overgroups. We start with the operation of induction, which is a subtle operation to construct a class function on $G$ from a given class function on a subgroup $H \leqslant G$. We will focus on characters in a second step.

**Definition 19.1 (*Induced class function*)**

Let $H \leqslant G$ and $\varphi \in \mathcal{C}l(H)$ be a class function on $H$. Then the **induction of $\varphi$ from $H$ to $G$** is

$$\begin{array}{rccl} \mathrm{Ind}_H^G(\varphi) =: \varphi\!\uparrow_H^G : & G & \longrightarrow & \mathbb{C} \\ & g & \mapsto & \varphi\!\uparrow_H^G(g) := \frac{1}{|H|}\sum_{x\in G} \varphi^\circ(x^{-1}gx)\,, \end{array}$$

where for $y \in G$, $\varphi^\circ(y) := \begin{cases} \varphi(y) & \text{if } y \in H, \\ 0 & \text{if } y \notin H. \end{cases}$

**Remark 19.2**

With the notation of Definition 19.1 the function $\varphi\uparrow_H^G$ is again a class function, because for every $g, y \in G$,

$$\varphi\uparrow_H^G(y^{-1}gy) = \frac{1}{|H|}\sum_{x\in G}\varphi^\circ(x^{-1}y^{-1}gyx) \overset{s:=yx}{=} \frac{1}{|H|}\sum_{s\in G}\varphi^\circ(s^{-1}gs) = \varphi\uparrow_H^G(g)\,.$$

Moreover, the map $\mathrm{Ind}_H^G : \mathcal{C}l(H) \longrightarrow \mathcal{C}l(G), \varphi \mapsto \varphi\uparrow_H^G$ is $\mathbb{C}$-linear [Exercise 23(d), Sheet 7].

In contrast, the operation of restriction is based on the more elementary idea that any map can be restricted to a subset of its domain. For class functions / representations / characters we are essentially interested in restricting these (seen as maps) to subgroups.

**Definition 19.3 (*Restricted class function*)**

Let $H \leqslant G$ and $\psi \in \mathcal{C}l(G)$ be a class function on $G$. Then the **restriction of $\psi$ from $G$ to $H$** is

$$\mathrm{Res}_H^G(\psi) := \psi\downarrow_H^G := \psi|_H\,.$$

This is obviously again a class function on $H$.

**Remark 19.4**

If $\psi$ is a character of $G$ afforded by the $\mathbb{C}$-representation $\rho : G \longrightarrow \mathrm{GL}(V)$, then clearly $\psi\downarrow_H^G$ is the character afforded by the $\mathbb{C}$-representation $\rho|_H : H \longrightarrow \mathrm{GL}(V)$.

**Exercise 19.5 (*Exercise 23, Sheet 7*)**

Let $H \leqslant J \leqslant G$. Prove that:

(a) $\varphi \in \mathcal{C}l(H) \implies (\varphi\uparrow_H^J)\uparrow_J^G = \varphi\uparrow_H^G$ (transitivity of induction);

(b) $\psi \in \mathcal{C}l(G) \implies (\psi\downarrow_J^G)\downarrow_H^J = \psi\downarrow_H^G$ (transitivity of restriction);

(c) $\varphi \in \mathcal{C}l(H)$ and $\psi \in \mathcal{C}l(G) \implies \psi\cdot\varphi\uparrow_H^G = \left(\psi\downarrow_H^G\cdot\varphi\right)\uparrow_H^G$ (Frobenius formula).

**Theorem 19.6 (*Frobenius reciprocity*)**

Let $H \leqslant G$, let $\varphi \in \mathcal{C}l(H)$ be a class function on $H$, and let $\psi \in \mathcal{C}l(G)$ be a class function on $G$. Then

$$\langle\varphi\uparrow_H^G, \psi\rangle_G = \langle\varphi, \psi\downarrow_H^G\rangle_H.$$

**Proof:** By the definitions of the scalar product and of the induction a direct computation yields:

$$\begin{aligned}\langle\varphi\uparrow_H^G,\psi\rangle_G = \frac{1}{|G|}\sum_{g\in G}\varphi\uparrow_H^G(g)\psi(g^{-1}) &= \frac{1}{|G|}\sum_{g\in G}\frac{1}{|H|}\sum_{x\in G}\varphi^\circ(x^{-1}gx)\psi(g^{-1})\\ &= \frac{1}{|G|\,|H|}\sum_{s\in G}\sum_{x\in G}\varphi^\circ(s)\psi(xs^{-1}x^{-1})\\ &= \frac{1}{|H|}\sum_{s\in G}\varphi^\circ(s)\psi(s^{-1}) = \frac{1}{|H|}\sum_{s\in H}\varphi(s)\psi(s^{-1}) = \langle\varphi,\psi\downarrow_H^G\rangle_H\,,\end{aligned}$$

where for the third equality we set $s := x^{-1}gx$ and the fourth equality comes from the fact that $\psi$ is a class function on $G$. ■

**Corollary 19.7**

Let $H \leqslant G$ and let $\chi$ be a character of $H$ of degree $n$. Then the induced class function $\chi\uparrow_H^G$ is a character of $G$ of degree $n|G:H|$.

**Proof:** For $\psi \in \mathrm{Irr}(G)$ by Frobenius reciprocity we can set

$$m_\psi := \langle \chi\uparrow_H^G, \psi\rangle_G = \langle \chi, \psi\downarrow_H^G\rangle_H \in \mathbb{Z}_{\geqslant 0}\,,$$

which is an integer because both $\chi$ and $\psi\downarrow_H^G$ are characters of $H$. Therefore,

$$\chi\uparrow_H^G = \sum_{\psi\in\mathrm{Irr}(G)} m_\psi \psi$$

is a non-negative integral linear combination of irreducible characters of $G$, hence a character of $G$. Moreover,

$$\chi\uparrow_H^G(1) = \frac{1}{|H|}\sum_{x\in G}\chi^\circ(1) = \frac{1}{|H|}|G|\chi(1) = \chi(1)|G:H|\,.$$

■

**Example 11**

(a) The restriction of the trivial character of $G$ from $G$ to $H$ is obviously the trivial character of $H$.

(b) If $H = \{1\}$, then $\mathbf{1}_{\{1\}}\uparrow_{\{1\}}^G = \chi_{\mathrm{reg}}$. Indeed, if $g \in G$ then

$$\mathbf{1}_{\{1\}}\uparrow_{\{1\}}^G(g) = \frac{1}{|\{1\}|}\sum_{x\in G}\underbrace{\mathbf{1}_{\{1\}}^\circ(x^{-1}gx)}_{=0 \text{ unless } g=1} = \delta_{1g}|G| = \chi_{\mathrm{reg}}(g)$$

by Corollary 10.2.

(c) Let $G = S_3$, $H = \langle(1\ 2)\rangle$, and let $\varphi : H \to \mathbb{C}$ with $\varphi(\mathrm{Id}) = 1$, $\varphi((1\ 2)) = -1$ be the sign homomorphism on $H$. By the remark, it is enough to compute $\varphi\uparrow_H^G$ on representatives of the conjugacy classes of $S_3$, e.g. Id, $(1\ 2)$ and $(1\ 2\ 3)$:

$$\varphi\uparrow_H^G(\mathrm{Id}) = \frac{1}{2}\sum_{x\in S_3}\varphi^\circ(\mathrm{Id}) = \frac{1}{2}\cdot|S_3|\cdot 1 = 3\,,$$

$$\varphi\uparrow_H^G((1\ 2\ 3)) = \frac{1}{2}\sum_{x\in S_3}\varphi^\circ(x^{-1}(1\ 2\ 3)x) = \frac{1}{2}\sum_{x\in S_3} 0 = 0\,,$$

(as the conjugacy class of a 3-cycle contains only 3-cycles and $\varphi(\text{3-cycle}) = 0$)

$$\varphi\uparrow_H^G((1\ 2)) = \frac{1}{2}\sum_{x\in S_3}\varphi^\circ(x^{-1}(1\ 2)x) = \frac{1}{2}\left(2\varphi^\circ((1\ 2)) + 2\varphi^\circ((1\ 3)) + 2\varphi^\circ((2\ 3))\right) = -1\,.$$

Moreover we see from the character table of $S_3$ (Example 5) that $\varphi\uparrow_H^G = \chi_2 + \chi_3$. But we can also compute with Frobenius reciprocity, that

$$0 = \langle \varphi, \chi_1\downarrow_H^G\rangle_H = \langle \varphi\uparrow_H^G, \chi_1\rangle_G$$

and similarly

$$1 = \langle \varphi, \chi_2\downarrow_H^G\rangle_H = \langle \varphi\uparrow_H^G, \chi_2\rangle_G \quad \text{and} \quad 1 = \langle \varphi, \chi_3\downarrow_H^G\rangle_H = \langle \varphi\uparrow_H^G, \chi_3\rangle_G\,.$$

**Example 12** (***The character table of the alternating group $A_5$***)

The conjugacy classes of $G = A_5$ are

$$C_1 = \{\mathrm{Id}\}\ ,\ C_2 = [(1\ 2)(3\ 4)]\ ,\ C_3 = [(1\ 2\ 3)]\ ,\ C_4 \cup C_5 = \{\text{5-cycles}\}\,,$$

i.e. $g_1 = \mathrm{Id}, g_2 = (1\ 2)(3\ 4), g_3 = (1\ 2\ 3)$ and $g \in C_4 \Rightarrow o(g) = 5$ and $g^{-1} \in C_4$ but $g^2, g^3 \in C_5$ so that we can choose $g_4 := (1\ 2\ 3\ 4\ 5)$ and $g_5 := (1\ 3\ 5\ 2\ 4)$. This yields:

$$|\operatorname{Irr}(A_5)| = 5 \text{ and } |C_1| = 1,\ |C_2| = 15,\ |C_3| = 20,\ |C_4| = |C_5| = 12\,.$$

We obtain the character table of $A_5$ as follows:

- We know that the trivial character $\mathbf{1}_G = \chi_1$ is one of the irreducible characters, hence we need to determine $\operatorname{Irr}(A_5)\backslash\{\mathbf{1}_G\} = \{\chi_2, \chi_3, \chi_4, \chi_5\}$.

- Now, $H := A_4 \leqslant A_5$ and we have already computed the character table of $A_4$ in Exercise Sheet 5. Therefore, inducing the trivial character of $A_4$ from $A_4$ to $A_5$ we obtain that

$$\begin{aligned}
&\mathbf{1}_H\!\uparrow_H^G(\mathrm{Id}) = 1 \cdot |G : H| = 5 \quad \text{(see Cor. 19.7)}\\
&\mathbf{1}_H\!\uparrow_H^G\big((1\ 2)(3\ 4)\big) = \tfrac{1}{12} \cdot 12 = 1\\
&\mathbf{1}_H\!\uparrow_H^G\big((1\ 2\ 3)\big) = \tfrac{1}{12} \cdot 24 = 2\\
&\mathbf{1}_H\!\uparrow_H^G(\text{5-cycle}) = \tfrac{1}{12} \cdot 0 = 0
\end{aligned}$$

  Now, by Frobenius reciprocity

$$\langle \mathbf{1}_H\!\uparrow_H^G, \chi_1\rangle_G = \langle \mathbf{1}_H, \underbrace{\chi_1\!\downarrow_H^G}_{=\mathbf{1}_H}\rangle_H = 1\,.$$

  It follows (check it) that $\langle \mathbf{1}_H\!\uparrow_H^G - \chi_1, \mathbf{1}_H\!\uparrow_H^G - \chi_1\rangle_G = 1$, so $\mathbf{1}_H\!\uparrow_H^G - \chi_1$ is an irreducible character, say $\chi_4 := \mathbf{1}_H\!\uparrow_H^G - \chi_1$. The values of $\chi_4$ are given by $(4, 0, 1, -1, -1)$ on $C_1, C_2, C_3, C_4, C_5$ respectively.

- Next, as $A_5$ is a non-abelian simple group, we have $A_5/[A_5, A_5] = 1$, and hence the trivial character is the unique linear character of $A_5$ and $\chi_2(1), \chi_3(1), \chi_5(1) \geqslant 3$. (You have also proved in Exercise 19, Sheet 6 that simple groups do not have irreducible characters of degree 2.) Then the degree formula yields

$$\chi_2(1)^2 + \chi_3(1)^2 + \chi_5(1)^2 = |A_5| - \chi_1(1)^2 - \chi_4(1)^2 = 20 - 1 - 16 = 43\,.$$

  As degrees of characters must divide the group order, it follows from this formula that $\chi_2(1), \chi_3(1), \chi_5(1) \in \{3, 4, 5, 6\}$, but then also that it is not possible to have an irreducible character of degree 6. From this we easily see that only possibility, up to relabelling, is $\chi_2(1) = \chi_3(1) = 3$ and $\chi_5(1) = 5$. Hence at this stage, we already have the following part of the character table:

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ |
|---|---|---|---|---|---|
| $\lvert C_k\rvert$ | 1 | 15 | 20 | 12 | 12 |
| $\lvert C_G(g_k)\rvert$ | 60 | 4 | 3 | 5 | 5 |
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 3 | . | . | . | . |
| $\chi_3$ | 3 | . | . | . | . |
| $\chi_4$ | 4 | 0 | 1 | $-1$ | $-1$ |
| $\chi_5$ | 5 | . | . | . | . |

· Next, we have that

$$\gcd(\chi_2(1), |C_3|) = \gcd(\chi_3(1), |C_3|) = \gcd(\chi_5(1), |C_4|) = \gcd(\chi_5(1), |C_5|) = 1\,,$$

so that the corresponding character values must all be zero by Corollary 17.7 and we get:

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ |
|---|---|---|---|---|---|
| $\lvert C_k\rvert$ | 1 | 15 | 20 | 12 | 12 |
| $\lvert C_G(g_k)\rvert$ | 60 | 4 | 3 | 5 | 5 |
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 3 | . | 0 | . | . |
| $\chi_3$ | 3 | . | 0 | . | . |
| $\chi_4$ | 4 | 0 | 1 | $-1$ | $-1$ |
| $\chi_5$ | 5 | . | . | 0 | 0 |

· Applying the Orthogonality Relations yields:
1st, 3rd column $\Rightarrow \chi_5(g_3) = -1$ and the scalar product $\langle \chi_1, \chi_5\rangle_G = 0 \Rightarrow \chi_5(g_2) = 1$.

· Finally, to fill out the remaining gaps, we can induce from the cyclic subgroup $Z_5 := \langle (1\ 2\ 3\ 4\ 5)\rangle \leqslant A_5$. Indeed, choosing the non-trivial irreducible character $\psi$ of $Z_5$ which was denoted "$\chi_3$" in Example 4 gives

$$\psi\uparrow_{Z_5}^{G} = (12, 0, 0, \zeta^2 + \zeta^3, \zeta + \zeta^4)$$

where $\zeta = \exp(2\pi\mathbf{i}/5)$ is a primitive 5-th root of unity. Then we compute that

$$\langle \psi\uparrow_{Z_5}^{G}, \chi_4\rangle_G = 1 = \langle \psi\uparrow_{Z_5}^{G}, \chi_5\rangle_G \quad\Longrightarrow\quad \psi\uparrow_{Z_5}^{G} -\chi_4 - \chi_5 = (3, -1, 0, -\zeta - \zeta^4, -\zeta^2 - \zeta^3)$$

and this character must be irreducible, because it is not the sum of 3 copies of the trivial character. Hence we set $\chi_2 := \psi\uparrow_{Z_5}^{G} -\chi_4 - \chi_5$ and the values of $\chi_3$ then easily follow from the 2nd Othogonality Relations:

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ |
|---|---|---|---|---|---|
| $\lvert C_k\rvert$ | 1 | 15 | 20 | 12 | 12 |
| $\lvert C_G(g_k)\rvert$ | 60 | 4 | 3 | 5 | 5 |
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 3 | $-1$ | 0 | $-\zeta - \zeta^4$ | $-\zeta^2 - \zeta^3$ |
| $\chi_3$ | 3 | $-1$ | 0 | $-\zeta^2 - \zeta^3$ | $-\zeta - \zeta^4$ |
| $\chi_4$ | 4 | 0 | 1 | $-1$ | $-1$ |
| $\chi_5$ | 5 | 1 | $-1$ | 0 | 0 |

**Remark 19.8 (*Induction of $KH$-modules*)**

If you have attended the lecture *Commutative Algebra* you have studied the *tensor product of modules*. In the M.Sc. lecture *Representation Theory* you will see that induction of modules is defined through a tensor product, extending the scalars from $\mathbb{C}H$ to $\mathbb{C}G$. More precisely, if $M$ is a $KH$-module, then the induction of $M$ from $H$ to $G$ is defined to be $KG \otimes_{KH} M$. Moreover, if $M$ affords the character $\chi$, then $KG \otimes_{KH} M$ affords the character $\chi\uparrow_H^G$.

## 20 Clifford Theory

Clifford theory is a generic term for a series of results relating the representation / character theory of a given group $G$ to that of a normal subgroup $N \trianglelefteq G$ through induction and restriction.

**Definition 20.1 (*Conjugate class function / inertia group*)**

Let $H \leqslant G$, let $\varphi \in \mathcal{C}l(H)$ and let $g \in G$.

(a) We define ${}^g\varphi \in \mathcal{C}l(gHg^{-1})$ to be the class function on $gHg^{-1}$ defined by

$$ {}^g\varphi : gHg^{-1} \longrightarrow \mathbb{C}, x \mapsto \varphi(g^{-1}xg)\,. $$

(b) The subgroup $\mathcal{I}_G(\varphi) := \{g \in G \mid {}^g\varphi = \varphi\} \leqslant G$ is called the **inertia group** of $\varphi$ in $G$.

**Exercise 20.2 (*Exercise 24, Sheet 7*)**

With the notation of Definition 20.1, prove that:

(a) ${}^g\varphi$ is indeed a class function on $gHg^{-1}$;

(b) $\mathcal{I}_G(\varphi) \leqslant G$ and $H \leqslant \mathcal{I}_G(\varphi) \leqslant N_G(H)$;

(c) given $g, h \in G$ we have: ${}^g\varphi = {}^h\varphi \;\Leftrightarrow\; h^{-1}g \in \mathcal{I}_G(\varphi) \;\Leftrightarrow\; g\mathcal{I}_G(\varphi) = h\mathcal{I}_G(\varphi)$;

(d) if $\rho : H \longrightarrow \mathrm{GL}(V)$ is a $\mathbb{C}$-representation of $H$ with character $\chi$, then

$$ {}^g\rho : gHg^{-1} \longrightarrow \mathrm{GL}(V), x \mapsto \rho(g^{-1}xg) $$

is $\mathbb{C}$-representation of $gHg^{-1}$ with character ${}^g\chi$ and ${}^g\chi(1) = \chi(1)$;

(e) if $J \leqslant H$ then ${}^g(\varphi\downarrow_J^H) = ({}^g\varphi)\downarrow_{gJg^{-1}}^{gHg^{-1}}$.

**Lemma 20.3**

(a) If $H \leqslant G$, $\varphi, \psi \in \mathcal{C}l(H)$ and $g \in G$, then $\langle {}^g\varphi, {}^g\psi\rangle_{gHg^{-1}} = \langle \varphi, \psi\rangle_H$.

(b) If $N \trianglelefteq G$ and $g \in G$, then we have $\psi \in \mathrm{Irr}(N) \;\Leftrightarrow\; {}^g\psi \in \mathrm{Irr}(N)$.

(c) If $N \trianglelefteq G$ and $\psi$ is a character of $N$, then $(\psi\uparrow_N^G)\downarrow_N^G = |\mathcal{I}_G(\psi) : N| \sum_{g \in [G/\mathcal{I}_G(\psi)]} {}^g\psi$.

**Proof:** (a) Clearly

$$ \begin{aligned} \langle {}^g\varphi, {}^g\psi\rangle_{gHg^{-1}} &= \frac{1}{|gHg^{-1}|} \sum_{x \in gHg^{-1}} {}^g\varphi(x)\overline{{}^g\psi(x)} \\ &= \frac{1}{|H|} \sum_{x \in gHg^{-1}} \varphi(g^{-1}xg)\overline{\psi(g^{-1}xg)} \\ &\overset{y := g^{-1}xg}{=} \frac{1}{|H|} \sum_{y \in H} \varphi(y)\overline{\psi(y)} = \langle \varphi, \psi\rangle_H\,. \end{aligned} $$

(b) As $N \trianglelefteq G$, $gNg^{-1} = N$. Thus, if $\psi \in \mathrm{Irr}(N)$, then on the one hand ${}^g\psi$ is also a character of $N$ by Exercise 20.2(d), and on the other hand it follows from (a) that $\langle {}^g\psi, {}^g\psi\rangle_N = \langle \psi, \psi\rangle_N = 1$. Hence ${}^g\psi$ is an irreducible character of $N$. Therefore, if ${}^g\psi \in \mathrm{Irr}(N)$, then $\psi = {}^{g^{-1}}({}^g\psi) \in \mathrm{Irr}(N)$, as required.

(c) If $n \in N$ then so does $g^{-1}ng\ \forall\ g \in G$, hence

$$\psi\!\uparrow_N^G\downarrow_N^G(n) = \psi\!\uparrow_N^G(n) = \frac{1}{|N|}\sum_{g\in G}\psi(g^{-1}ng) = \frac{1}{|N|}\sum_{g\in G}{}^g\psi(n) = \frac{|\mathcal{I}_G(\psi)|}{|N|}\sum_{g\in[G/\mathcal{I}_G(\psi)]}{}^g\psi(n).$$

■

**Notation 20.4**

Given $N \trianglelefteq G$ and $\psi \in \mathrm{Irr}(N)$, we set $\mathrm{Irr}(G \mid \psi) := \{\chi \in \mathrm{Irr}(G) \mid \langle \chi\downarrow_N^G, \psi\rangle_N \neq 0\}$.

**Theorem 20.5 (Clifford Theory)**

Let $N \trianglelefteq G$. Let $\chi \in \mathrm{Irr}(G)$, $\psi \in \mathrm{Irr}(N)$ and set $\mathcal{I} := \mathcal{I}_G(\psi)$. Then the following assertions hold.

(a) If $\psi$ is a constituent of $\chi\downarrow_N^G$, then

$$\chi\downarrow_N^G = e\sum_{g\in[G/\mathcal{I}_G(\psi)]}{}^g\psi\,,$$

where $e = \langle \chi\downarrow_N^G, \psi\rangle_N = \langle \chi, \psi\uparrow_N^G\rangle_G \in \mathbb{Z}_{>0}$ is called the **ramification index** of $\chi$ in $N$ (or of $\psi$ in $G$). In particular, all the constituents of $\chi\downarrow_N^G$ have the same degree.

(b) Induction from $\mathcal{I} = \mathcal{I}_G(\psi)$ to $G$ induces a bijection

$$\begin{array}{rccc}\mathrm{Ind}_{\mathcal{I}}^G: & \mathrm{Irr}(\mathcal{I} \mid \psi) & \longrightarrow & \mathrm{Irr}(G \mid \psi)\\ & \eta & \mapsto & \eta\uparrow_{\mathcal{I}}^G\end{array}$$

preserving ramification indices, i.e. $\langle \eta\downarrow_N^{\mathcal{I}}, \psi\rangle_N = \langle \eta\uparrow_{\mathcal{I}}^G\downarrow_N^{\mathcal{I}}, \psi\rangle_N = e$.

**Proof:**

(a) By Frobenius reciprocity, $\langle \chi, \psi\uparrow_N^G\rangle_G = \langle \chi\downarrow_N^G, \psi\rangle_N \neq 0$. Thus $\chi$ is a constituent of $\psi\uparrow_N^G$ and therefore $\chi\downarrow_N^G$ is a constituent of $\psi\uparrow_N^G\downarrow_N^G$.
Now, if $\eta \in \mathrm{Irr}(N)$ is an arbitrary constituent of $\chi\downarrow_N^G$ (i.e. $\langle \chi\downarrow_N^G, \eta\rangle_N \neq 0$) then by the above, we have

$$\langle \psi\uparrow_N^G\downarrow_N^G, \eta\rangle_N \geqslant \langle \chi\downarrow_N^G, \eta\rangle_N > 0\,.$$

Moroever, by Lemma 20.3(c) the constituents of $\psi\uparrow_N^G\downarrow_N^G$ are preciely $\{{}^g\psi \mid g \in [G/\mathcal{I}_G(\psi)]\}$. Hence $\eta$ is $G$-conjugate to $\psi$. Furthermore, for every $g \in G$ we have

$$\begin{aligned}\langle \chi\downarrow_N^G, {}^g\psi\rangle_N = \frac{1}{|N|}\sum_{h\in N}\chi(h){}^g\psi(h^{-1}) &= \frac{1}{|N|}\sum_{h\in N}\chi(h)\psi(g^{-1}h^{-1}g)\\ &\overset{\chi\in\mathcal{C}l(G)}{=} \frac{1}{|N|}\sum_{h\in N}\chi(g^{-1}hg)\psi(g^{-1}h^{-1}g)\\ &\overset{s:=g^{-1}hg\in N}{=} \frac{1}{|N|}\sum_{s\in N}\chi(s)\psi(s^{-1}) = \langle \chi\downarrow_N^G, \psi\rangle_N = e\,.\end{aligned}$$

Therefore, every $G$-conjugate ${}^g\psi$ ($g \in [G/\mathcal{I}_G(\psi)]$) of $\psi$ occurs as a constituent of $\chi\downarrow_N^G$ with the same multiplicity $e$. The claim about the degrees is then clear as ${}^g\psi(1) = \psi(1)\ \forall g \in G$.

(b) **Claim 1:** $\eta \in \mathrm{Irr}(\mathcal{I} \mid \psi) \;\Rightarrow\; \eta\!\uparrow_{\mathcal{I}}^{G} \in \mathrm{Irr}(G|\psi)$.

Since $\mathcal{I} = \mathcal{I}_{\mathcal{I}}(\psi)$, (a) implies that $\eta\!\downarrow_N^{\mathcal{I}} = e'\psi$ with $e' = \langle \eta\!\downarrow_N^{\mathcal{I}}, \psi\rangle_N = \frac{\eta(1)}{\psi(1)} > 0$. Now, let $\chi \in \mathrm{Irr}(G)$ be a constituent of $\eta\!\uparrow_{\mathcal{I}}^{G}$. By Frobenius Reciprocity we have

$$0 \neq \langle \chi, \eta\!\uparrow_{\mathcal{I}}^{G}\rangle_G = \langle \chi\!\downarrow_{\mathcal{I}}^{G}, \eta\rangle_{\mathcal{I}}\,.$$

It follows that $\eta\!\downarrow_N^{\mathcal{I}}$ is a constituent of $\chi\!\downarrow_{\mathcal{I}}^{G}\!\downarrow_N^{\mathcal{I}}$ and

$$e := \langle \chi\!\downarrow_N^G, \psi\rangle_N = \langle \chi\!\downarrow_{\mathcal{I}}^{G}\!\downarrow_N^{\mathcal{I}}, \psi\rangle_N \geqslant \langle \eta\!\downarrow_N^{\mathcal{I}}, \psi\rangle_N = e' > 0\,,$$

hence $\chi \in \mathrm{Irr}(G|\psi)$. Moreover, by (a) we have $e = \langle \chi\!\downarrow_N^G, {}^g\psi\rangle_N \geqslant e'$ for each $g \in G$. Therefore,

$$\chi(1) = e \sum_{g \in [G/\mathcal{I}]} {}^g\psi(1) \overset{(a)}{=} e|G:\mathcal{I}|\psi(1) \geqslant e'|G:\mathcal{I}|\psi(1) = |G:\mathcal{I}|\eta(1) = \eta\!\uparrow_{\mathcal{I}}^{G}(1) \geqslant \chi(1)\,.$$

Thus $e = e'$, $\eta\!\uparrow_{\mathcal{I}}^{G} = \chi \in \mathrm{Irr}(G)$, and therefore $\eta\!\uparrow_{\mathcal{I}}^{G} \in \mathrm{Irr}(G|\psi)$.

**Claim 2:** $\chi \in \mathrm{Irr}(G \mid \psi) \;\Rightarrow\; \exists!\, \eta \in \mathrm{Irr}(\mathcal{I} \mid \psi)$ such that $\langle \chi\!\downarrow_{\mathcal{I}}^{G}, \eta\rangle_{\mathcal{I}} \neq 0$.

Again by (a), as $\chi \in \mathrm{Irr}(G \mid \psi)$, we have $\chi\!\downarrow_N^G = e\sum_{g\in[G/\mathcal{I}]} {}^g\psi$, where $e = \langle \chi\!\downarrow_N^G, \psi\rangle_N \in \mathbb{Z}_{>0}$. Therefore, there exists $\eta \in \mathrm{Irr}(\mathcal{I})$ such that

$$\langle \chi\!\downarrow_{\mathcal{I}}^{G}, \eta\rangle_{\mathcal{I}} \neq 0 \neq \langle \eta\!\downarrow_N^{\mathcal{I}}, \psi\rangle_N$$

because $\chi\!\downarrow_N^G = \chi\!\downarrow_{\mathcal{I}}^{G}\!\downarrow_N^{\mathcal{I}}$, so in particular $\eta \in \mathrm{Irr}(\mathcal{I} \mid \psi)$. Hence existence holds and it remains to see that uniqueness holds. Again by Frobenius reciprocity we have $0 \neq \langle \chi, \eta\!\uparrow_{\mathcal{I}}^{G}\rangle_G$. By Claim 1 this forces $\chi = \eta\!\uparrow_{\mathcal{I}}^{G}$ and $\eta\!\downarrow_N^{\mathcal{I}} = e\psi$, so $e$ is also the ramification index of $\psi$ in $\mathcal{I}$.
Now, write $\chi\!\downarrow_{\mathcal{I}}^{G} = \sum_{\lambda\in\mathrm{Irr}(\mathcal{I})} a_\lambda \lambda = \sum_{\lambda\neq\eta} a_\lambda\lambda + a_\eta\eta$ with $a_\lambda \geqslant 0$ for each $\lambda \in \mathrm{Irr}(\mathcal{I})$ and $a_\eta > 0$. It follows that

$$(a_\eta - 1)\eta\!\downarrow_N^{\mathcal{I}} + \sum_{\lambda\neq\eta} a_\lambda\lambda\!\downarrow_N^{\mathcal{I}} = \underbrace{\chi\!\downarrow_N^G}_{=e\sum_{g\in[G/\mathcal{I}]} {}^g\psi} - \underbrace{\eta\!\downarrow_N^{\mathcal{I}}}_{=e\psi} = e \sum_{g\in[G/\mathcal{I}]\setminus[1]} {}^g\psi.$$

Since $\psi$ does not occur in this sum, but occurs in $\eta\!\downarrow_N^{\mathcal{I}}$, the only possibility is $a_\eta = 1$ and $\lambda \notin \mathrm{Irr}(\mathcal{I}|\psi)$ for $\lambda \neq \eta$. Thus $\eta$ is uniquely determined as the only constituent of $\chi\!\downarrow_{\mathcal{I}}^{G}$ in $\mathrm{Irr}(\mathcal{I} \mid \psi)$.

Finally, Claims 1 and 2 prove that $\mathrm{Ind}_{\mathcal{I}}^G : \mathrm{Irr}(\mathcal{I} \mid \psi) \longrightarrow \mathrm{Irr}(G \mid \psi), \eta \mapsto \eta\!\uparrow_{\mathcal{I}}^{G}$ is well-defined and bijective, and the proof of Claim 2 shows that the ramification indices are preserved. ∎

**Example 13 (*Normal subgroups of index 2*)**

Let $N < G$ be a subgroup of index $|G:N| = 2$ ($\Rightarrow N \lhd G$) and let $\chi \in \mathrm{Irr}(G)$, then either

(1) $\chi\!\downarrow_N^G \in \mathrm{Irr}(N)$, or

(2) $\chi\!\downarrow_N^G = \psi + {}^g\psi$ for a $\psi \in \mathrm{Irr}(N)$ and a $g \in G\setminus N$.

Indeed, let $\psi \in \mathrm{Irr}(N)$ be a constituent of $\chi\!\downarrow_N^G$. Since $|G:N| = 2$, we have $\mathcal{I}_G(\psi) \in \{N, G\}$. Theorem 20.5 yields the following:

- If $\mathcal{I}_G(\psi) = N$ then $\mathrm{Irr}(\mathcal{I}_G(\psi) \mid \psi) = \{\psi\}$ and $\psi\!\uparrow_N^G = \chi$, so that $e = 1$ and we get $\chi\!\downarrow_N^G = \psi + {}^g\psi$ for any $g \in G\setminus N$.
- If $\mathcal{I}_G(\psi) = G$ then $G/\mathcal{I}_G(\psi) = \{1\}$, so that

$$\chi\!\downarrow_N^G = e\psi \qquad \text{with } e = \langle \chi\!\downarrow_N^G, \psi\rangle_N = \langle \chi, \psi\!\uparrow_N^G\rangle_G\,.$$

Moroever, by Lemma 20.3(c),

$$\psi\uparrow_N^G\downarrow_N^G = |\mathcal{I}_G(\psi):N| \sum_{g\in G/\mathcal{I}_G(\psi)} {}^g\psi = 2\psi\,.$$

Hence

$$2\psi(1) = \psi\uparrow_N^G\downarrow_N^G(1) \geqslant \chi\downarrow_N^G(1) = \chi(1) = e\psi(1) \quad \Rightarrow \quad e \leqslant 2\,.$$

Were $e = 2$ then we would have $2\psi(1) = \psi\uparrow_N^G(1)$, hence $\chi = \psi\uparrow_N^G$ and thus

$$1 = \langle \chi, \psi\uparrow_N^G\rangle_G = \langle \chi\downarrow_N^G, \psi\rangle_N = e = 2$$

a contradiction. Whence $e = 1$, which implies that $\chi\downarrow_N^G \in \mathrm{Irr}(N)$. Moreover, $\psi\uparrow_N^G = \chi + \chi'$ for some $\chi' \in \mathrm{Irr}(G)$ such that $\chi' \neq \chi$.

Remember that we have proved (Exercise 24, Sheet 6) that the degree of an irreducible character of a finite group $G$ divides the index of the centre $|G:Z(G)|$. The following consequence of Clifford's theorem due to N. Itô provides us with yet a stronger divisibility criterion.

**Theorem 20.6 (Itô)**

Let $A \leqslant G$ be an abelian subgroup of $G$ and let $\chi \in \mathrm{Irr}(G)$. Then the following assertions hold:

(a) $\chi(1) \leqslant |G:A|$; and

(b) if $A \trianglelefteq G$, then $\chi(1) \,\big|\, |G:A|$.

**Proof:**

(a) Exercise 27, Sheet 7.

(b) Let $\psi \in \mathrm{Irr}(A)$ be a constituent of $\chi\downarrow_A^G$, so that in other words $\chi \in \mathrm{Irr}(G \mid \psi)$. By Theorem 20.5(b) there exists $\eta \in \mathrm{Irr}(\mathcal{I}_G(\psi) \mid \psi)$ such that $\chi = \eta\uparrow_{\mathcal{I}_G(\psi)}^G$ and $\eta\downarrow_A^{\mathcal{I}_G(\psi)} = e\psi$ (proof of Claim 2). Now, as $A$ is abelian, all the irreducible characters of $A$ have degree 1 and for each $x \in A$, $\psi(x)$ is an $o(x)$-th root of unity. Hence $\forall\, x \in A$ we have

$$|\eta(x)| = |\eta\downarrow_A^{\mathcal{I}_G(\psi)}(x)| = |e\psi(x)| = e|\psi(x)| = e\cdot 1 = e = \eta(1) \quad \Rightarrow \quad A \subseteq Z(\eta)\,.$$

Therefore, by Remark 17.5, we have

$$\eta(1) \,\Big|\, |\mathcal{I}_G(\psi):Z(\eta)| \,\Big|\, |\mathcal{I}_G(\psi):A|$$

and since $\chi = \eta\uparrow_{\mathcal{I}_G(\psi)}^G$ it follows that

$$\chi(1) = |G:\mathcal{I}_G(\psi)|\eta(1) \,\Big|\, |G:\mathcal{I}_G(\psi)|\cdot|\mathcal{I}_G(\psi):A| = |G:A|\,.$$

■

## 21 The Theorem of Gallagher

In the context of Clifford theory (Theorem 20.5) we understand that irreducibility of characters is preserved by induction from $\mathcal{I}_G(\psi)$ to $G$. Thus we need to understand induction of characters from $N$ to

$\mathcal{I}_G(\psi)$, in particular what if $G = \mathcal{I}_G(\psi)$. What can be said about $\mathrm{Irr}(G \mid \psi)$?

**Lemma 21.1**

Let $N \trianglelefteq G$ and let $\psi \in \mathrm{Irr}(N)$ such that $\mathcal{I}_G(\psi) = G$. Then

$$\psi\uparrow_N^G = \sum_{\chi \in \mathrm{Irr}(G)} e_\chi \chi$$

where $e_\chi := \langle \chi\downarrow_N^G, \psi \rangle_N$ is the ramification index of $\chi$ in $N$; in particular

$$\sum_{\chi \in \mathrm{Irr}(G)} e_\chi^2 = |G : N|\,.$$

**Proof:** Write $\psi\uparrow_N^G = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi$ with suitable $a_\chi = \langle \chi, \psi\uparrow_N^G \rangle_G$. By Frobenius reciprocity, $a_\chi \neq 0$ if and only if $\chi \in \mathrm{Irr}(G \mid \psi)$. But by Theorem 20.5: if $\chi \in \mathrm{Irr}(G|\psi)$, then $\chi\downarrow_N^G = e_\chi \psi$, so that

$$e_\chi = \langle \chi\downarrow_N^G, \psi \rangle_N = \langle \chi, \psi\uparrow_N^G \rangle_G = a_\chi\,.$$

Therefore,

$$|G : N|\psi(1) = \psi\uparrow_N^G(1) = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi(1) = \sum_{\chi \in \mathrm{Irr}(G)} e_\chi \chi(1) = \sum_{\chi \in \mathrm{Irr}(G)} e_\chi^2 \psi(1) = \psi(1) \sum_{\chi \in \mathrm{Irr}(G)} e_\chi^2$$

and it follows that $|G : N| = \sum_{\chi \in \mathrm{Irr}(G)} e_\chi^2$. ■

Therefore the multiplicities $\{e_\chi\}_{\chi \in \mathrm{Irr}(G)}$ behave like the irreducible character degrees of the factor group $G/N$. This is not a coincidence in many cases.

**Definition 21.2 (*Extension of a character*)**

Let $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$ such that $\psi := \chi\downarrow_N^G$ is irreducible. Then we say that $\psi$ **extends to** $G$, and $\chi$ is an **extension of** $\psi$.

**Exercise 21.3 (*Exercise 26, Sheet 7*)**

Let $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$. Prove that

$$\chi\downarrow_N^G\uparrow_N^G = \mathrm{Inf}_{G/N}^G(\chi_{\mathrm{reg}}) \cdot \chi\,,$$

where $\chi_{\mathrm{reg}}$ is the regular character of $G/N$.

**Theorem 21.4 (Gallagher)**

Let $N \trianglelefteq G$ and let $\chi \in \mathrm{Irr}(G)$ such that $\psi := \chi\downarrow_N^G \in \mathrm{Irr}(N)$. Then

$$\psi\uparrow_N^G = \sum_{\lambda \in \mathrm{Irr}(G/N)} \lambda(1)\ \mathrm{Inf}_{G/N}^G(\lambda) \cdot \chi,$$

where the characters $\{\mathrm{Inf}_{G/N}^G(\lambda) \cdot \chi \mid \lambda \in \mathrm{Irr}(G/N)\}$ of $G$ are pairwise distinct and irreducible.

**Proof:** By Exercise 21.3 we have $\psi\uparrow_N^G = \mathrm{Inf}_{G/N}^G(\chi_{\mathrm{reg}})\cdot\chi$, where $\chi_{\mathrm{reg}}$ denotes the regular character of $G/N$. Recall that by Theorem 10.3, $\chi_{\mathrm{reg}} = \sum_{\lambda\in\mathrm{Irr}(G/N)}\lambda(1)\,\lambda$, so that we have

$$\psi\uparrow_N^G = \sum_{\lambda\in\mathrm{Irr}(G/N)}\lambda(1)\,\mathrm{Inf}_{G/N}^G(\lambda)\cdot\chi\,.$$

Now, by Lemma 21.1, we have

$$\begin{aligned}|G:N| = \sum_{\chi\in\mathrm{Irr}(G)} e_\chi^2 = \langle\psi\uparrow_N^G,\psi\uparrow_N^G\rangle_G &= \sum_{\lambda,\mu\in\mathrm{Irr}(G/N)}\lambda(1)\mu(1)\langle\mathrm{Inf}_{G/N}^G(\lambda)\cdot\chi,\mathrm{Inf}_{G/N}^G(\mu)\cdot\chi\rangle_G\\ &\geqslant \sum_{\lambda\in\mathrm{Irr}(G/N)}\lambda(1)^2 = |G:N|\,.\end{aligned}$$

Hence equality holds throughout. This proves that

$$\langle\mathrm{Inf}_{G/N}^G(\lambda)\cdot\chi,\mathrm{Inf}_{G/N}^G(\mu)\cdot\chi\rangle = \delta_{\lambda\mu}.$$

By Erercise 13.4, $\mathrm{Inf}_{G/N}^G(\lambda)\cdot\chi$ are characters of $G$ and hence the characters $\{\mathrm{Inf}_{G/N}^G(\lambda)\cdot\chi \mid \lambda\in\mathrm{Irr}(G/N)\}$ are irreducible and pairwise distinct, as claimed. ∎

Therefore, given $\psi\in\mathrm{Irr}(N)$ which extends to $\chi\in\mathrm{Irr}(G)$, we get $\mathrm{Inf}_{G/N}^G(\lambda)\cdot\chi$ ($\lambda\in\mathrm{Irr}(G/N)$) as further irreducible characters.

**Example 14**

Let $N < G$ with $|G:N| = 2$ ($\Rightarrow N\trianglelefteq G$) and let $\psi\in\mathrm{Irr}(N)$. We saw:

- if $\mathcal{I}_G(\psi) = N$ then $\psi\uparrow_N^G\in\mathrm{Irr}(G)$;
- if $\mathcal{I}_G(\psi) = G$ then $\psi$ extends to some $\chi\in\mathrm{Irr}(G)$ and $\psi^G = \chi+\chi'$ with $\chi'\in\mathrm{Irr}(G)\backslash\{\chi\}$. It follows that $\chi' = \chi\cdot\mathrm{sign}$, where sign is the inflation of the sign character of $G/N\cong S_2$ to $G$.