

Übungen zu Lie-Algebren — Blatt 14

Prof. Dr. G. Malle

Abgabetermin: **6. März 2012, 11:30 Uhr**

PD Dr. M. Cuntz

WS 11/12

Exercise 51: Let $\lambda \in \Lambda^+$. Show that 0 occurs as a weight in $L(\lambda)$ if and only if λ is a sum of roots.

Exercise 52: Prove that the Casimir element $c(\kappa)$ lies in the center of the universal enveloping algebra $U(L)$.

Exercise 53: Let $\lambda, \mu \in \Lambda^+$, $V := L(\lambda)$, $W := L(\mu)$. Show that

$$\mathcal{P}(V \otimes W) = \{\nu + \nu' \mid \nu \in \mathcal{P}(V), \nu' \in \mathcal{P}(W)\}$$

and

$$\dim(V \otimes W)_{\nu+\nu'} = \sum_{\substack{\pi, \pi' \\ \pi+\pi'=\nu+\nu'}} \dim V_{\pi} \cdot \dim W_{\pi'}.$$

Conclude that $\text{ch } V \otimes W = \text{ch } V \cdot \text{ch } W$.

Exercise 54: Let $L = \mathfrak{sl}_2(\mathbb{C})$. Use the previous exercise to determine the dimensions of the weight spaces in $V_d \otimes V_e$ and prove the Clebsch-Gordan formula

$$V_d \otimes V_e = V_{d+e} \oplus V_{d+e-2} \oplus \dots \oplus V_{d-e} \quad (\text{for } d \geq e)$$

for the decomposition of the tensor product of two irreducible finite-dimensional \mathfrak{sl}_2 -modules.