ZEROS OF BRAUER CHARACTERS AND THE DEFECT ZERO GRAPH

GUNTER MALLE

Abstract. We show that for each finite non-abelian simple group and each prime p either there exists an irreducible Brauer character which takes the value zero on some p-regular element, or $p = 2$ and all degrees of irreducible Brauer characters are powers of 2. This generalizes an old result of Burnside from ordinary to Brauer characters. For the proof we introduce the notion of the defect zero graph of a finite group, which may be of independent interest.

1. Introduction and statement of the main results

The main result of this paper answers in the affirmative a question on the zeros of Brauer characters of simple groups posed by Gabriel Navarro (a more precise formulation is contained in Theorem 1.2 below):

Theorem 1.1. *Let G be a finite non-abelian simple group and p a prime. Assume that no irreducible p-modular Brauer character of G vanishes on any p-regular element of G. Then $p = 2$ and the degrees of all irreducible p-Brauer characters of G are powers of 2.*

By a well-known observation of Burnside, any non-linear irreducible ordinary character of a finite group vanishes on some element, so the statement above is certainly true for primes p not dividing $|G|$.

We'll prove the theorem in a case by case analysis, using the classification of finite simple groups. More precisely, we'll show a statement which lends itself more easily to induction.

In order to formulate this, we'll say that a Brauer character φ of a finite group G has an *invariant zero*, if φ vanishes on a whole orbit of $\mathrm{Aut}(G)$ on the set of p-regular elements of G. For example, assume that φ vanishes on the p-regular class C of G. If one of φ, C is invariant under $\mathrm{Aut}(G)$, then φ has an invariant zero.

Theorem 1.2. *Let G be a finite non-abelian simple group and p a prime. Then there exists an irreducible p-Brauer character of G having an invariant zero on p-regular elements of G, unless $p = 2$ and*

(a) $G = \mathrm{L}_2(2^f)$ *with* $f \geq 2$,
(b) $G = \mathrm{L}_2(q)$ *with* $q = 2^m + 1 \geq 5$,
(c) $G = {}^2B_2(2^{2f+1})$ *with* $f \geq 1$, *or*
(d) $G = \mathrm{S}_4(2^f)$ *with* $f \geq 2$,

in which case all degrees of irreducible 2-Brauer characters of G are powers of 2.

Date: August 4, 2009.

The author thanks the MSRI (Berkeley) for its hospitality during the preparation of this work.

Clearly Theorem 1.1 follows from this.

As an immediate consequence of Theorem 1.2 we obtain the following extension to arbitrary non-solvable groups when p is odd.

Theorem 1.3. *Assume that G is not solvable and $p \neq 2$. Then there exists an irreducible p-Brauer character of G which vanishes on some p-regular element of G.*

Proof. Let G be a minimal counterexample, and $N \lhd G$ a minimal normal subgroup. Assume that N is solvable. Then G/N is not solvable, so by minimality of G there exists $gN \in G/N$ of order k prime to p and $\varphi \in \mathrm{IBr}_p(G/N)$ with $\varphi(gN) = 0$. Interpreting $\varphi \in \mathrm{IBr}_p(G)$ by inflation, we obtain that $\varphi(gn) = 0$ for all $n \in N$. Write $g = g_1g_2$ with g_1 the p-part, g_2 the p'-part of g. Then $g^k = g_1^k g_2^k \in N$, so $g_1^k, g_2^k \in N$. As $\gcd(k, p) = 1$ this implies that $g_1 \in N$, and thus $g_2 = gg_1^{-1}$ is p-regular and such that $\varphi(g_2) = 0$, a contradiction.

Hence N is not solvable. So it is a direct product of isomorphic simple groups $S_i \cong S$ of order divisible by p. By Theorem 1.2 there exists an irreducible p-Brauer character φ of S with an invariant zero, say on $g \in S$. Let $\mu := \varphi \times \ldots \times \varphi$, an irreducible Brauer character of N, which vanishes on $(g, \ldots, g)$. Let I be the inertia group of μ in G and $\tilde{\mu}$ an irreducible character of I lying above μ. Then $\tilde{\mu}|_N$ is a multiple of μ, so it has an invariant zero as well. The induced character $\tilde{\mu}^G$ is irreducible, and still vanishes on $(g, \ldots, g)$, which is a contradiction. □

It would seem interesting to understand what can be said in the corresponding situation for $p = 2$.

Example 1.4. We give two examples to show that there is no straightforward generalization of Theorem 1.3:

(a) The conclusion in Theorem 1.3 is definitely not true for solvable groups; for example take $p = 3$ and $G = \mathfrak{A}_4$.

(b) There are plenty of groups with p-Brauer characters without zeros whose degree is not a power of p: $\mathrm{IBr}_p(\mathfrak{A}_n)$, for $p \in \{3, 5\}$ and $n \in \{7, 8\}$, contains an irreducible Brauer character of degree 13 without zeros, $\mathrm{IBr}_2(Co_2)$ contains an irreducible Brauer character of degree $83948 = 2^2.31.677$ without zeros.

In order to prove Theorem 1.2 in Section 2 we introduce a new combinatorial object associated to a finite group, the defect zero graph, which may be of independent interest, and study its properties in the case of finite simple groups. Section 3 is then devoted to the proof of Theorem 1.2.

2. A character degree graph

We introduce the following graph associated to a finite group G.

Definition 2.1. The *defect zero graph* $\Delta(G)$ of a finite group G has as vertices the primes dividing the order $|G|$, and two vertices $p \neq q$ are joined by an edge if there exists an irreducible complex character $\chi \in \mathrm{Irr}(G)$ which is of defect zero for both p and q. We let $n_\Delta(G)$ denote the number of connected components of $\Delta(G)$.

Closely related graphs have been studied extensively. Let $\Delta'(G)$ have same vertex set as $\Delta(G)$, with two primes being connected if there is some irreducible character whose degree is divisible by both. Clearly, $\Delta(G)$ is a subgraph of $\Delta'(G)$. Now remove those primes from $\Delta'(G)$ which divide no character degree. It has been shown by Manz–Staszewski–Willems [15] that this latter graph has at most three connected components. (In fact they consider some kind of dual graph $\Gamma(G)$, with vertices the degrees of non-linear characters, and edges given by common prime divisors, but which clearly has the same number of connected components.)

Clearly, if G has no character of p-defect zero, then p is an isolated vertex of $\Delta(G)$. But this criterion is only sufficient, not necessary, as can be seen for the defining prime of groups of Lie type. The pairs (G, p) of finite non-abelian simple groups G having no p-defect zero character are listed in [6, Cor. 2]. In particular it follows that there are simple groups for which $\Delta(G)$ has at least three components. Surprisingly enough, this gives already the correct upper bound for the number of components. More precisely we show:

Theorem 2.2. *Let G be a finite nonabelian simple group. Then $\Delta(G)$ has at most three components. Moreover, the sum of the diameters of the components is at most* $5 - n_\Delta(G)$.

The proof will be given in a case by case analysis, using the classification of finite simple groups. More detailed information on $\Delta(G)$ in the individual cases is given below.

From the ordinary character tables contained in the Atlas [4] one finds:

Proposition 2.3. *Let G be a sporadic simple group. Then*
(a) $n_\Delta(G) = 3$ *for* $G \in \{M_{12}, Suz, Co_3\}$,
(b) $n_\Delta(G) = 2$ *for* $G \in \{M_{11}, M_{22}, J_2, HS, M_{24}, He, Ru, Co_1, B\}$,
(c) $n_\Delta(G) = 1$ *otherwise.*

In particular, $n_\Delta(G) \leq 3$ *for sporadic groups. Furthermore, in each case the sum of the diameters of the connected components of* $\Delta(G)$ *is at most* $4 - n_\Delta(G)$.

For example, $\Delta(J_1)$ and $\Delta(M_{23})$ are connected of diameter 3, so the bound for the sum of diameters is best possible.

Our next aim is the determination of $n_\Delta(G)$ in the case of groups of Lie type. It is well-known that the Steinberg character of a simple group of Lie type G in characteristic p is the only character of p-defect zero, so the defining prime p is always an isolated vertex in the defect zero graph $\Delta(G)$.

2.1. **The setup for groups of Lie type.** Any simple group of Lie type can be obtained as $\mathbf{G}^F/Z(\mathbf{G}^F)$, where $\mathbf{G}$ is a simple algebraic group of simply-connected type, and $F : \mathbf{G} \to \mathbf{G}$ a Frobenius map with finite group of fixed points $G := \mathbf{G}^F$, with the sole exception of the Tits group ${}^2F_4(2)'$. We denote by r the defining characteristic of $\mathbf{G}$.

Let $\mathbf{G}^*$ denote a group in duality with $\mathbf{G}$ and with corresponding Frobenius map $F^* : \mathbf{G}^* \to \mathbf{G}^*$ and fixed points $G^* := \mathbf{G}^{*F^*}$. Let $\mathbf{T} \leq \mathbf{G}^*$ be an F^*-stable maximal torus containing an F-stable regular semisimple element $s \in T := \mathbf{T}^{F^*}$. Then the corresponding Deligne-Lusztig character $R_{T,s}$ is, up to sign, an irreducible complex character of G of degree $R_{T,s}(1) = \pm|G : T|_{r'}$ (see [2, Cor. 7.3.5], for example). Moreover, if s lies in the derived subgroup of G^*, then $R_{T,s}$ has $Z(G)$ in its kernel, so defines a character of $S = G/Z(G)$, see [13].

In the subsequent statement, we consider ${}^2G_2(3)' = \mathrm{L}_2(8)$ and $G_2(2)' = \mathrm{U}_3(3)$ not as exceptional simple groups of Lie type.

Proposition 2.4. *Let S be an exceptional simple group of Lie type. Then $n_\Delta(S) = 2$. One component consists just of the defining prime, while the second component has diameter at most 2.*

Proof. The statement can be checked easily for ${}^2F_4(2)'$ from its known character table, so we exclude this group from now on. Let $(\mathbf{G}, F)$ be such that $S = G/Z(G)$ where $G := \mathbf{G}^F$, as above. We have already noted above that the defining characteristic always forms an isolated component of $\Delta(S)$, so we have to prove that all other prime divisors of $|S|$ lie in one connected component.

TABLE 1

Two tori for exceptional groups

G		$\|T_1\|$	$\|T_2\|$	d
${}^2B_2(q^2)$	$(q^2 \geq 8)$	Φ_8'	Φ_8''	1
${}^2G_2(q^2)$	$(q^2 \geq 27)$	Φ_{12}'	Φ_{12}''	1
$G_2(q)$	$(q \geq 3)$	Φ_3	Φ_6	1
${}^3D_4(q)$		Φ_3^2	Φ_{12}	1
${}^2F_4(q^2)$	$(q^2 \geq 8)$	Φ_{24}'	Φ_{24}''	1
$F_4(q)$		Φ_8	Φ_{12}	1
$E_6(q)$		$\Phi_3\Phi_{12}$	Φ_9	$(3, q-1)$
${}^2E_6(q)$		$\Phi_6\Phi_{12}$	Φ_{18}	$(3, q+1)$
$E_7(q)$		$\Phi_1\Phi_7$	$\Phi_2\Phi_{14}$	$(2, q-1)$
$E_8(q)$		Φ_{15}	Φ_{30}	1

(Here, Φ_n denotes the nth cyclotomic polynomial over $\mathbb{Q}$ evaluated at q, and Φ_n', Φ_n'' denote the two irreducible factors of Φ_n over $\mathbb{Q}(\sqrt{2})$ for $n = 8, 24$, respectively over $\mathbb{Q}(\sqrt{3})$ for $n = 12$.)

In Table 1 we produce two maximal tori T_1, T_2 of G^*, which are uniquely determined, up to conjugacy, by their order (see for example [14]). In each case except for type ${}^3D_4$, the largest cyclotomic polynomial Φ_{m_i} dividing $|T_i|$ has the following property: T_i is the unique maximal torus of G (up to conjugation) of order divisible by Φ_{m_i}. By Zsigmondy's theorem, there exists a primitive prime divisor p_i of $\Phi_{m_i}(q)$ in each case. Now let $s_i \in T_i$ be a semisimple element of order p_i. Assume that $C_{G^*}(s_i)$ is not a torus. Then its semisimple rank is at least 2, whence it contains two maximal tori of different order. Both of these must have order divisible by p_i, which is not the case by our previous observation. Thus s_i is a regular semisimple element. For ${}^3D_4$, it can be checked directly that there exist regular elements in T_1 of order a Zsigmondy prime for Φ_3. Since $|G^* : (G^*)'|$ is not divisible by these Zsigmondy primes, the s_i lie in the derived subgroup $(G^*)'$. So the corresponding Deligne-Lusztig characters $R_i := R_{T_i,s_i}$ are, up to sign, irreducible characters of S. By the above degree formula R_i is of defect zero for all prime divisors of $|G|$ different from the defining characteristic r and not dividing $|T_i|$.

In all cases, $|S|_{r'}$ is divisible by a third prime p_3 not dividing either of the $|T_i|$. Thus, all primes $p \neq r$ not dividing $d := \gcd(|T_1|, |T_2|)$ are connected to this third prime, and in particular lie in one connected component of $\Delta(G)$.

In the cases where $d > 1$, we have that $|S| = |G/Z(G)| = |G|/d$, so that

$$R_i(1) = \frac{|G|_{r'}}{|T_i|} = \frac{|S|_{r'}}{|T_i|/d} \quad \text{for } i = 1, 2.$$

As $\gcd(|T_1|/d, |T_2|/d) = 1$ we may in fact conclude as before.

Since all primes $p \neq r$ are connected to p_3, the diameter of this component is at most 2. □

We now turn to the groups of classical type.

Proposition 2.5. *Let S be a classical simple group of Lie type. Then $n_\Delta(S) \leq 3$. Furthermore, $n_\Delta(S) = 2$ unless one of*

(a) $S = \mathrm{L}_2(q)$, $q \geq 4$,
(b) $S = \mathrm{L}_3(q)$, $q = 2^{6m+1} - 1$ *with* $6m + 1$ *prime, or* $q = 3$,
(c) $S = \mathrm{U}_3(q)$, $q = 2^{2^{2m}} + 1$ *with* $m \geq 0$, *or* $q = 9$,
(d) $S = \mathrm{S}_4(q)$ *with* $q \in \{3, 5\}$.

In all cases, the sum of the diameters of the components is at most 3.

Proof. We keep the setup $(\mathbf{G}, F, G = \mathbf{G}^F)$ as above and proceed as in the previous proof by producing two maximal tori T_1, T_2 of G^* with suitable properties in Table 2, see [14, Table 3.5].

TABLE 2

Two tori for classical groups.

G		$\lvert T_1\rvert$	$\lvert T_2\rvert$	d
A_n		$(q^{n+1}-1)/(q-1)$	$q^n - 1$	$(n+1, q-1)$
${}^2A_n$	$(n \geq 3$ odd$)$	$(q^{n+1}-1)/(q+1)$	$q^n + 1$	$(n+1, q+1)$
	$(n \geq 2$ even$)$	$(q^{n+1}+1)/(q+1)$	$q^n - 1$	$(n+1, q+1)$
B_n, C_n	$(n \geq 3$ odd$)$	$q^n + 1$	$q^n - 1$	$(2, q-1)$
	$(n \geq 2$ even$)$	$q^n + 1$	$(q^{n-1}+1)(q+1)$	$(2, q-1)$
D_n	$(n \geq 5$ odd$)$	$(q^{n-1}+1)(q+1)$	$q^n - 1$	$(4, q^n - 1)$
	$(n \geq 4$ even$)$	$(q^{n-1}+1)(q+1)$	$(q^{n-1}-1)(q-1)$	$(4, q^n - 1)$
${}^2D_n$	$(n \geq 4)$	$q^n + 1$	$(q^{n-1}+1)(q-1)$	$(4, q^n + 1)$

Again, in all cases there exists a cyclotomic polynomial dividing $|T_i|$ which divides $|G|$ exactly once, see [14, Prop. 3.4]. Furthermore, $d := \gcd(|T_1|, |T_2|) = |Z(G)|$, and there is a third prime divisor of $|G|$ not dividing $|T_1|\,|T_2|$ and different from the defining characteristic of G, so whenever both Zsigmondy primes exist, we may conclude as in the proof of Proposition 2.4.

The cases where not both Zsigmondy primes exist are the groups of types A_1, A_2, ${}^2A_2$, B_2 and the groups $\mathrm{L}_6(2)$, $\mathrm{L}_7(2)$, $\mathrm{U}_4(2)$, $\mathrm{S}_6(2)$, $\mathrm{S}_8(2)$, $\mathrm{O}_8^+(2)$ and $\mathrm{O}_8^-(2)$. The character tables of the latter seven groups are contained in [4], and the assertion can easily be checked. Now assume that $S = \mathrm{L}_2(q)$, so $|S| = q(q^2-1)/\gcd(2, q-1)$. Here, the character degrees are $q-1, q, q+1$, and $(q\pm 1)/2$ if q is odd, where the sign depends on the congruence of q modulo 4. Clearly, $\Delta(S)$ has three components, consisting of the prime divisors of $q-\epsilon$, respectively those of $(q+\epsilon)/2$, respectively the defining prime (where $q \equiv \epsilon \pmod 4$), each being a complete graph.

For $L_3(q)$, $U_3(q)$ and $S_4(q)$ the Zsigmondy prime p_1 exists. (Note here that $U_3(2)$ is solvable.) Thus all prime divisors of $|S|$ not dividing $q|T_1|/d$ form a complete subgraph in $\Delta(S)$. For $S = L_3(q)$ the Deligne-Lusztig character corresponding to a regular element of a totally split torus of order $(q-1)^2$ connects the prime divisors of $|T_1|/d$ to those which are prime to $|T_1|/d$, unless $q+1$ is a power of 2. Such regular elements exist whenever $q \geq 5$. The cases $L_3(3)$ and $L_3(4)$ can be checked directly. If $q+1$ is a power of 2 then $q \equiv 0, 1 \pmod 3$. Since $q \geq 5$ we actually have $q \equiv 1 \pmod 3$. The Deligne-Lusztig character $R_2 = R_{T_2,s}$ for some regular element $s \in T_2$ is of defect zero for all prime divisors of T_1 different from 3. If $q \not\equiv 1 \pmod 9$, then R_2 is also of 3-defect zero, as is R_1, so that $\Delta(S)$ has two components. On the other hand, if $q \equiv 1 \pmod 9$ then it can be checked from the explicitly known character degrees of $L_3(q)$ that the prime divisors of $|T_1|/3$ are not connected to any other primes in $\Delta(S)$, so that we obtain three components, each a complete graph.

For $S = U_3(q)$, if $q-1$ is not a power of 2 then the Deligne-Lusztig character R_3 corresponding to a regular semisimple element of a maximal torus of order $(q+1)^2$ connects the divisors of $|T_1|$ to the other non-defining primes, and we obtain two components. So assume that $q = 2^m + 1$, hence $q \equiv 0, 2 \pmod 3$. If $q \equiv 0 \pmod 3$ then $q \in \{3, 9\}$, and the claim can be checked directly. Thus $q \equiv 2 \pmod 3$. If $q \not\equiv 8 \pmod 9$ then R_3 is also of 3-defect zero, and all non-defining primes form a complete graph. Otherwise, the prime divisors of $|T_1|/3$ are not connected to any other primes in $\Delta(S)$, so that we obtain three components, each a complete graph. Note that $2^{6k+4}+1$ is a prime power if and only if it is a Fermat prime.

Finally, consider $S = S_4(q)$. If a maximally split torus T_3 of order $(q-1)^2$ contains regular elements, and $q+1$ is not a power of 2, then the corresponding Deligne-Lusztig character joins the prime divisors of $|T_1|/d$ to the other non-defining primes. Similarly, the same holds if a torus T_4 of order $(q+1)^2$ contains regular elements and $q-1$ is not a power of 2. Such regular elements exist for all $q > 5$. Furthermore, at least one of $q+1, q-1$ is not a power of 2 if $q > 3$. The remaining cases $S_4(3), S_4(4), S_4(5)$ can be checked easily. Note that $S_4(2)' \cong L_2(9)$. □

For the treatment of alternating groups we need the following observation:

Lemma 2.6. *Let $\lambda \vdash n$ be a partition of n which is a p-core. Then the largest hook length in λ is smaller than*

$$\min\{2\sqrt{2n(p-1)} - 1, n/s + p(s-1) + 1 \mid s \in \mathbb{N}\}.$$

Proof. Let $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k)$, with $\lambda_1 \geq 1$, be a p-core. Then at most $p-1$ parts of λ can be equal. Writing $k = m(p-1)+r$, with $0 \leq r < p-1$, we obtain

$$\begin{aligned} n = \sum_{i=1}^{k} \lambda_i \geq& (p-1)\lambda_1 + \ldots + (p-1)(\lambda_1+m-1) + r(\lambda_1+m) \\ =& m(p-1)\lambda_1 + r\lambda_1 + (p-1)\frac{m(m-1)}{2} + rm \\ =& k\lambda_1 + \frac{m}{2}(2k-(m+1)(p-1)) \\ \geq& k + \frac{m}{2}(k-p+1) \geq k + \frac{mk}{2} - \frac{k}{2} = \frac{1}{2}k(m+1), \end{aligned}$$

since $m(p-1) \leq k$. Using that $k < (m+1)(p-1)$ this gives

$$\frac{1}{2}k\frac{k}{p-1} < \frac{1}{2}k(m+1) \leq n,$$

so $k < \sqrt{2n(p-1)}$. Applying the same argument to the transposed partition λ' we obtain that also $\lambda_k < \sqrt{2n(p-1)}$, so λ has largest hook length $k + \lambda_k - 1 < 2\sqrt{2n(p-1)} - 1$.

For the second bound, we consider the lengths of the (disjoint) hooks at the diagonal entries of λ. If h denotes the length of the longest hook, the next one has length at least $h-2(p-1)-2$, the third at least $h-4(p-1)-4$, and so on. Adding these up gives the claim. □

The proof of the following statement on the defect zero graph of alternating groups makes essential use of the existence of p-cores for all primes $p \geq 5$.

Proposition 2.7. *Let $G = \mathfrak{A}_n$ with $n \geq 5$. Then $n_\Delta(G) \leq 3$. More precisely, all primes lie in a single component, of diameter at most 2, except that*

(a) *$p = 3$ is isolated if n is of the form $3m+1$ and has a prime divisor congruent 2* (mod 3) *to an odd power, and*

(b) *$p = 2$ is isolated if n is not of the form $m(m+1)/2$ or $m(m+1)/2+2$ for some $m \geq 2$, or if $n \in \{5, 6, 8, 10\}$.*

Proof. The complex irreducible characters of the symmetric group are naturally parameterized by partitions λ of n, and the degree of χ_λ equals $n!$ divided by all hook length of the Young diagram of λ. This allows to check the assertion by computer for all $n \leq 16$.

From now on, let $n \geq 17$. If p is odd then clearly the restriction to $\mathfrak{A}_n$ of any p-defect zero character of $\mathfrak{S}_n$ contains (at least) one defect zero character of $\mathfrak{A}_n$, thus for odd primes it suffices to consider character degrees in $\mathfrak{S}_n$. Let $k := \lceil\sqrt{n}\rceil$. Then clearly there exist partitions λ of n whose Young diagram has at most k rows and columns. Thus, the largest hook length of λ is at most $2k-1$. In particular, χ_λ is of p-defect zero for all primes $n \geq p \geq 2\lceil\sqrt{n}\rceil$. Thus, these primes form a complete subgraph of $\Delta(\mathfrak{A}_n)$. Note that, since $n \geq 17$, there always exists at least one such prime by Bertrand's postulate.

So now assume that $p < 2\lceil\sqrt{n}\rceil$. Then we show that a p-core, that is, a partition λ without p-hooks, is also a core for some other, large prime. Indeed, if λ is a p-core, then the largest hook of λ has length at most $2\sqrt{2n(p-1)} - 1 \leq 4n^{3/4} - 1$ by Lemma 2.6 and our assumption on p. Now for all $n \geq 5$ there exists a prime q between $n/2$ and n. For

$n \geq 4088$ our bound $4n^{3/4} - 1$ is smaller than $n/2$, so for those n, any p-core is also a q-core for some large prime $q > n/2$. Using a computer program it can be checked that in fact the largest prime $q \leq n$ is larger than our bound for all $n \geq 269$. For $29 \leq n \leq 268$, the second bound $n/2 + (2\lceil\sqrt{n}\rceil - 1) + 1$ from Lemma 2.6 is smaller than the biggest prime below n. Moreover, the only cases with $n \geq 17$ for which the inequality fails occur for $n = 26, 27, 28$ and $p = 11$. But in these cases, $(6)^4(n - 24)$ is an 11-core as well as a 23-core.

Thus, whenever there exists a partition of n which is a p-core, p is joined to the largest prime less or equal to n. Now the result of Granville and Ono [6, Thm. 1] guarantees the existence of p-cores for all n and all primes $p \geq 5$. In particular, all primes $p \geq 5$ lie in a single component of $\Delta(G)$, of diameter at most 2.

By [6, Cor. 2] there is no 3-core partition of n if and only if n is congruent to $1 \pmod 3$ and divisible by some prime congruent $2 \pmod 3$ to an odd power. Clearly, in these cases 3 is an isolated vertex.

Now let $p = 2$. By [6, Cor. 2] the alternating group $\mathfrak{A}_n$ has an irreducible character of 2-defect zero if n is of the form $m(m+1)/2$ or $m(m+1)/2+2$ for some integer $m \geq 2$. In the first case, the partition $(m)(m-1)\ldots(1)$ is a 2-core, hence connected to the largest prime below n by our above argument. In the second case, there is no 2-core of n. But the character indexed by the partition $(m+2)(m-1)(m-2)\ldots(1)$ (obtained from the 2-core for $n-2$ by adding a single 2-hook) has the full 2-part of $|\mathfrak{A}_n|$ in its degree and remains irreducible upon restriction to $\mathfrak{A}_n$. It is of defect zero for all primes $p > 2m+1$, hence connects 2 to the large primes when $n \geq 17$. □

Example 2.8. For small values of n we have $n_\Delta(\mathfrak{A}_n) = 3$ for $n \in \{5, 6, 7, 9, 11, 13\}$, $n_\Delta(\mathfrak{A}_n) = 2$ for $n \in \{8, 10, 14, 15, 16\}$ and $n_\Delta(\mathfrak{A}_{12}) = 1$.

It would be interesting to know whether the sum of diameters is bounded above by $4 - n_\Delta(G)$ for alternating groups as well. For this to hold, if both 2 and 3 are isolated, the other primes would have to form a complete graph.

3. Proof of the main theorem

In this section we prove Theorem 1.2 on zeros of Brauer characters of finite non-abelian simple groups. For groups of Lie type in non-defining characteristic, we use properties of their defect zero graphs, see Section 3.2. The case of defining characteristic is treated in Section 3.3. It is not amenable to this method and thus requires a different approach.

We start with the following elementary observations for arbitrary finite groups G which establish the connection between zeros of Brauer characters and the defect zero graph:

Lemma 3.1. *Assume that G has an ordinary irreducible character χ of p-defect zero which takes value zero on some p-regular element. Then G has an irreducible p-Brauer character taking value zero.*

Indeed, being of p-defect zero, the restriction of χ to the p-regular classes of G is an irreducible p-Brauer character. In particular we have:

Lemma 3.2. *Assume that G has an ordinary irreducible character χ of defect zero for two distinct primes p, q dividing $|G|$. Then G has an irreducible p-Brauer character with an invariant zero.*

Indeed, χ is irreducible modulo p and vanishes on all q-elements, which is an $\mathrm{Aut}(G)$-invariant set. We now consider the various classes of finite simple groups in turn.

3.1. **Alternating groups.** The alternating groups $\mathfrak{A}_5$ and $\mathfrak{A}_6$ are isomorphic to projective special linear groups, which will be treated later, so we may assume that $n \geq 7$ here.

Proposition 3.3. *Theorem 1.2 holds for $G = \mathfrak{A}_n$ with $n \geq 7$.*

Proof. For $n \leq 8$ the assertion can be checked from the known modular character tables [9], so assume now that $n \geq 9$. Let χ denote the non-trivial irreducible constituent of the natural permutation character of $\mathfrak{S}_n$ over $\mathbb{Q}$.

First assume that $p \not| n$. Then χ remains irreducible modulo p, so its restriction to p-regular classes is an irreducible p-Brauer character. It vanishes on all elements of $\mathfrak{S}_n$ with exactly one fixed point. If n is even (so $p \neq 2$), then permutations of at least one of the cycle shapes $(n-1)(1)$ or $(n-5)(2)^2(1)$ have order prime to p, and both lie in $\mathfrak{A}_n$. If n is odd, consider (even) permutations of cycle shapes $(n-3)(2)(1)$ and $(n-4)(3)(1)$. Since $p \not| n$, at least one of them has order prime to p. Thus in both cases we have found p-regular elements on which χ vanishes. Since χ extends to $\mathrm{Aut}(\mathfrak{A}_n) = \mathfrak{S}_n$, the claim follows.

Now assume that $p|n$. Then the p-modular reduction of χ contains one trivial constituent and one further irreducible Brauer character φ. Clearly, the latter vanishes on all permutations with exactly two fixed points, so in particular on those of cycle shapes $(n-2)(1)^2$ and $(n-4)(2)(1)^2$. One of them is even, hence contained in $\mathfrak{A}_n$. If $p \neq 2$, then both have order prime to p, and we are done as before. So finally assume that $p = 2$ and n is even. Then φ vanishes on permutations of cycle shape $(n-5)(3)(1)^2$, which have odd order. This completes the proof. ∎

Alternatively, we could have made use of our results on the defect 0 graph in Proposition 2.7, but the above proof has the advantage that it does not depend on the non-trivial existence result for p-cores.

3.2. **Groups of Lie type in non-defining characteristic.** Here the proof of the main theorem relies heavily on our above study of the defect zero graph, in view of Lemma 3.2.

Proposition 3.4. *Theorem 1.2 holds for simple groups of Lie type if p is different from the defining characteristic.*

Proof. Let G be a simple group of Lie type. Assume that p is not an isolated vertex of the defect zero graph of G. Then there exists another prime divisor q of $|G|$ and an irreducible complex character χ of G such that χ is of p-defect zero and of q-defect zero. Thus the claim holds for G and p by virtue of Lemma 3.2.

According to Propositions 2.4 and 2.5, all non-defining primes for G lie in a single component (which has at least two vertices since G is non-solvable), unless G is one of $\mathrm{L}_2(q)$ with $q \geq 4$, $\mathrm{L}_3(q)$, $q = 2^{6m+1} - 1$ with $6m+1$ prime, $\mathrm{U}_3(q)$, $q = 2^{2^{2m}} + 1$ with $m \geq 0$, or

$$G \in \{\mathrm{L}_3(3), \mathrm{U}_3(9), \mathrm{S}_4(3), \mathrm{S}_4(5)\}.$$

The assertion for the latter four groups can be checked directly from the known Brauer character tables [9].

For $G = \mathrm{L}_2(q)$, $q = r^f$, we use the information on decomposition numbers obtained by Burkhardt [1]. First assume that $p \neq 2$. If $p|(q-1)$, then the Steinberg character of G of degree q remains irreducible modulo p by [1]. Since it vanishes on the class of r-elements, and remains invariant under all automorphisms of G, we are done in this case.

If $p|(q+1)$ then the p-modular reduction of the Steinberg character contains an irreducible constituent of degree $q-1$, which vanishes on all elements of order dividing $q-1$, and again remains invariant under all automorphisms.

It remains to consider the case that $p = 2$ and hence q is odd. If $q \equiv 1 \pmod 3$, then the 2-modular irreducible Brauer characters of G of degree $(q-1)/2$ vanish on the unique conjugacy class of elements of order 3. If $q \equiv -1 \pmod 3$ there exists an irreducible 2-Brauer character of G of degree $q-1$, invariant under automorphisms, which vanishes on all elements of odd order dividing $q-1$. Such elements exist unless $q-1$ is a power of 2. In the latter case, again by [1, VIII(a)], the degrees of Brauer characters are $1, (q-1)/2$ and $q-1$, which leads to exception (b) in Theorem 1.2. Finally, if $q = 3^f$ the irreducible 2-modular characters of degree $(q-1)/2$ vanish on all elements of odd order dividing $q-1$. Such elements exist unless $q = 9 = 2^3 + 1$.

Next let $G = \mathrm{L}_3(q)$, $q = 2^{6m+1} - 1$ with $6m+1$ prime. In the proof of Proposition 2.5 we showed that the non-defining primes not dividing $\Phi_3/3$ form a complete subgraph of the defect zero graph of G. Since there are at least two such primes, the result follows for them. For the primes dividing $\Phi_3/3$, the Deligne-Lusztig characters for regular elements in a maximally split torus are of defect zero and vanish on all elements of order $q+1$, providing the required zero.

Finally, for $\mathrm{U}_3(q)$, $q = 2^{2^{2m}} + 1$ with $m \geq 0$, we may argue similarly with $q+1$ and $q-1$ interchanged. □

3.3. **Groups of Lie type in defining characteristic.** In contrast to the case of non-defining characteristic, the defect zero graph is of no use here, since the defining prime always forms an isolated vertex. Instead, we produce zeros of Brauer characters by embedding small groups with known character values into the groups of Lie type.

Let $(\mathbf{G}, F)$ as in Section 2.1 above and $G := \mathbf{G}^F$, with p the defining characteristic. We'll need two important facts from the representation theory of groups of Lie type in their defining characteristic. First of all, any irreducible such representation of G is obtained as the restriction of some irreducible representation of $\mathbf{G}$. In particular, it extends to all $\mathbf{G}^{F^k}$ for any $k \geq 1$, see [11, Thm. 5.4.1] for example. Secondly, by Steinberg's tensor product theorem any irreducible representation of G is obtained as a tensor product of suitable Frobenius twists of the so-called basic representations, see [11, Thm. 5.4.5].

We start with the exceptional groups of Lie type.

Proposition 3.5. *Theorem 1.2 holds for exceptional groups of Lie type and the defining prime.*

Proof. According to Steinberg's tensor product theorem, the irreducible 2-modular representations of ${}^2B_2(2^{2f+1})$ are tensor products of Frobenius twists of the natural 4-dimensional representation of G inside $\mathrm{Sp}_4(2^{2f+1})$, so all have 2-power degree. This is exception (c) in Theorem 1.2.

The natural 7-dimensional representation of ${}^2G_2(3^{2f+1})$ in characteristic 3 restricts irreducibly to ${}^2G_2(3)' = \mathrm{L}_2(8)$, and for the latter it vanishes on all elements of order 7.

Similarly, the natural 26-dimensional representation of ${}^2F_4(2^{2f+1})$ in characteristic 2 restricts irreducibly to ${}^2F_4(2)'$, and thus vanishes on all elements of order 13 in that subgroup.

The natural 7-dimensional irreducible representation of $G_2(q)$, q odd, vanishes on all regular elements of order 7; the adjoint 14-dimensional representation of $G_2(7)$ vanishes on regular elements of order 8; the natural 6-dimensional representation of $G_2(2^f)$ restricts irreducibly to $G_2(2)' = \mathrm{U}_3(3)$ and vanishes there on the class of regular 3-elements.

The Steinberg triality group ${}^3D_4(q)$ contains a natural subgroup $G_2(q)$ (the centralizer of the graph-field automorphism), which in turn always contains a subgroup $G_2(2)$ by [10, Th. A]. When $p \neq 2$, the restriction to $G_2(q)$ of the natural 8-dimensional representation of ${}^3D_4(q)$ over $\mathbb{F}_{q^3}$ contains a trivial constituent and the 7-dimensional representation of $G_2(q)$. The character table of $G_2(2)$ shows that its Brauer character vanishes on the involutions of this group. For $p = 2$ the 26-dimensional Brauer character of ${}^3D_4(2)$ vanishes on all elements of order 13, by [9].

The heart of the natural permutation representation embeds the alternating group $\mathfrak{A}_{10}$ into a 9-dimensional orthogonal group. This lifts to an embedding of the 2-fold cover $2.\mathfrak{A}_{10}$ into the simply-connected group of type B_4. The latter is a subsystem subgroup of F_4, so the 2-fold cover of $\mathfrak{A}_{10}$ embeds into a group of type F_4. Since the representation of $2.\mathfrak{A}_{10}$ is defined over the integers, reduction modulo p gives a homomorphism $2.\mathfrak{A}_{10} \to \mathrm{Spin}_9(p) \to F_4(p)$, with kernel of order $\gcd(2,p)$. We now claim that the restriction to Spin_9 of the irreducible character of the adjoint representation of F_4 is the sum of the characters of the spin representation and the adjoint representation of Spin_9. Clearly, the restriction contains the adjoint representation of Spin_9. Since Spin_9 is of maximal rank in F_4, its centralizer on the adjoint module for F_4 is trivial and so the restriction does not contain trivial constituents. The table of dimensions in [12] now shows that the only possibility for the remaining 16-dimensional subquotient is the spin representation of Spin_9.

So the Brauer character of $2.\mathfrak{A}_{10}$ on the adjoint module for $F_4(p)$ is the spin character plus the exterior square of the natural 9-dimensional character. From the character table of $2.\mathfrak{A}_{10}$ one sees that the latter vanishes on some elements of order 4, as well as on some elements of order 21. This yields a zero of the Brauer character of the adjoint representation of $F_4(q)$.

The groups $E_6(q)$ and ${}^2E_6(q)$ both contain a subsystem subgroup $F_4(q)$, hence by the above the 2-fold cover $2.\mathfrak{A}_{10}$ of the alternating group $\mathfrak{A}_{10}$. Arguing as above for F_4 we see that the restriction of the irreducible adjoint representation of E_6 to F_4 contains the adjoint representation and a 26-dimensional representation. We claim that the latter when restricted to B_4 contains the natural 9-dimensional representation, the spin representation and a trivial constituent. Indeed, B_4 is contained in the subsystem subgroup of type D_5, which centralizes a one-dimensional torus, so the adjoint representation for E_6 will contain at least one trivial constituent. By our result for F_4, this is not contained in the adjoint representation for F_4. On the other hand, the centralizer of B_4 is only 1-dimensional, so there are no further trivial composition factors. By dimension reasons (see [12]) the remaining 25-dimensional module must have composition factors as claimed.

Thus, the adjoint representation of E_6 restricts to $2.\mathfrak{A}_{10}$ as the p-modular reduction of the heart of the permutation module, the exterior square of this, two copies of the spin module, and one trivial constituent. The character table of $2.\mathfrak{A}_{10}$ shows that this vanishes on certain elements of order 8. If $p = 2$, then the restriction to ${}^3D_4(2) < F_4(2)$ of the 27-dimensional irreducible module of ${}^{(2)}E_6(2^f)$ only splits off one trivial composition factor. The Brauer character table of ${}^3D_4(2)$ shows that this restriction vanishes on all elements of order 3.

According to the observation of Serre in [17, p. 417], (the p-modular reduction of) any of its 133-dimensional rational representations embeds the simple group $\mathrm{U}_3(8)$ into the adjoint representation of $E_7(p)$, for any prime p. This is irreducible for $p > 2$, and the 133-dimensional character of $\mathrm{U}_3(8)$ vanishes on elements of orders 7 and 19. In characteristic 2, the adjoint module splits off a 1-dimensional trivial constituent. The Brauer character of the remaining irreducible part vanishes on elements of order 9 of $\mathrm{U}_3(8)$.

For $p \neq 2$ the Brauer character of the adjoint representation of $E_8(q)$ vanishes on certain elements of order 4, see Cohen–Griess [3, Table 4]. By Parker–Saxl [16] the group $E_8(2)$ contains a subgroup $\mathrm{L}_3(5)$, to which the adjoint representation restricts as the sum of two 124-dimensional irreducible 2-Brauer characters. These vanish on elements of order 31.

Now for all exceptional groups $G(q)$ we produced an irreducible Brauer character φ vanishing on some p-regular element. Let μ be the image of φ under some automorphism of $G(q)$. Both μ and φ restrict irreducibly to the group $G(p)$ over the prime field. It can be checked that $G(p)$ has at most two representations of the given degree $\varphi(1) = \mu(1)$, and that both vanish on the given p-regular class. Hence μ has a zero at the same class as φ, so this is an invariant zero. □

The desired result for linear and unitary groups will essentially follow from a property of symmetric groups:

Lemma 3.6. *Let $n \geq 4$, χ the character of the heart of the natural permutation module of $\mathbb{Q}\mathfrak{S}_{n+1}$. Then*

(a) *the exterior powers $\chi_k := \Lambda^k(\chi)$ are irreducible for $1 \leq k \leq n$,*
(b) *$\chi_k(g) = 0$ for g an $n-1$-cycle, for $2 \leq k \leq n-1$,*
(c) *if n is odd, $\chi_k(g) = 0$ for g of cycle shape $((n-1)/2)^2$, for $1 \leq k \leq (n-1)/2$.*

Proof. The first statement holds more generally for the reflection character of any irreducible reflection group, see for example [5, Thm. 5.1.4]. For the second and third statements note that χ_k is labeled by the hook partition $(n+1-k)(1)^k$ of $n+1$. The assertion now follows by an easy application of the Murnaghan–Nakayama rule, see e.g. [7, 2.4.7]. □

The following is well-known (see [12] for example):

Lemma 3.7. *Let V denote the natural n-dimensional module of $G = \mathrm{SL}_n(q)$ over $\mathbb{F}_q$, resp. of $G = \mathrm{SU}_n(q)$ over $\mathbb{F}_{q^2}$. Then we have:*

(a) *$\Lambda^k(V)$ is irreducible for G, for $1 \leq k \leq n$. Moreover, the action on $\Lambda^k(V)$ has $Z(G)$ in its kernel whenever k is a multiple of $\gcd(n, q-1)$, resp. $\gcd(n, q+1)$.*
(b) *If $p \nmid n$, the adjoint module $A(V)$, that is, $V \otimes V^*$ modulo the trivial submodule, is irreducible for G, with trivial action of $Z(G)$.*

Proposition 3.8. *Theorem 1.2 holds for $\mathrm{L}_n(q)$, $q = p^f$, and the prime p.*

Proof. For $G = \mathrm{L}_2(q)$ with q odd consider the 3-dimensional adjoint representation identifying G with a 3-dimensional orthogonal group. Its Brauer character vanishes on elements of order 3, a set which is stable under automorphisms, so we are done for $p \neq 3$. If $p = 3$ then $q \geq 9$. The tensor product of any two different Frobenius twists of the natural representation vanishes on all involutions of G (since elements of order 4 have trace 0 in the natural representation).

In the case that $p = 2$ by Steinberg's tensor product theorem, all irreducible 2-modular representations are tensor products of Frobenius twists of the natural 2-dimensional representation, hence of 2-power degree. This gives exception (a) in Theorem 1.2.

So now assume that $n \geq 3$. Let V denote the natural module for $\mathrm{GL}_n(q)$. The symmetric group $\mathfrak{S}_{n+1}$ embeds into $\mathrm{GL}(V)$ via its natural permutation representation modulo the trivial submodule. Since this representation is the p-modular reduction of a rational representation, its image is already contained in the general linear group over the prime field, hence invariant under field automorphisms of $\mathbb{F}_q$. By the same reason, it is invariant under the transpose inverse automorphism of $\mathrm{GL}_n(V)$. This representation restricts to a representation of $\mathfrak{A}_{n+1}$ into $\mathrm{SL}(V)$. The Brauer character of $g \in \mathfrak{A}_{n+1}$ on V is just the number of fixed points of g minus 1. We'll produce elements for which the restriction of the Brauer character on V, $\Lambda^2(V)$ or $A(V)$ to $\mathfrak{A}_{n+1}$ takes value 0.

Let first $n = 3$. If $q \equiv 2 \pmod 3$ then $\mathrm{L}_3(q) = \mathrm{SL}_3(q)$, and the Brauer character on V vanishes on 3-cycles in $\mathfrak{A}_4$. If $p = 3$ then it suffices to locate a zero in the 3-modular Brauer table of $\mathrm{L}_3(3)$, since all irreducible representations extend to the algebraic closure of the base field. One finds in [9] that the 6-dimensional representations vanish on elements of order 4. If $q \equiv 1 \pmod 3$, then the Brauer character of V has value -1 on double transpositions, so the Brauer character on $A(V)$, which is irreducible by Lemma 3.7(b), vanishes on these elements. It hence remains to treat the case $p = 2$, $q = 2^{2m}$. Here, 3-cycles have Brauer character value 0 on the natural module, so the tensor product of V with the mth Frobenius twist, which is irreducible and has $Z(\mathrm{SL}_3(q))$ in its kernel, also takes value 0 on 3-cycles.

So now let $n \geq 4$ and first assume that n is even. If $p \nmid n$ then in particular $p \neq 2$, so that both $n-1$-cycles and $n+1$-cycles lie in $\mathfrak{A}_{n+1}$ and at least one of them is a p-regular class. The Brauer character has value ± 1 on V, so it vanishes on $A(V)$ by Lemma 3.7(b). If $p|n$, then $d := \gcd(n, q-1) \leq n/p \leq n-1$. Then, with $k := \max\{2, d\}$, $\Lambda^k(V)$ is an irreducible representation of $\mathrm{L}_n(q)$ having Brauer character value 0 on $n-1$-cycles by Lemma 3.6(b).

If n is odd, first assume that $p \nmid n$. Then $(n-1)/2$, $(n+1)/2$ are coprime, and at least one of them is prime to p. The Brauer character of $A(V)$ vanishes on elements of cycle shape $((n-1)/2)^2$ or $((n+1)/2)^2$. If $p|n$, so p is odd, we have

$$d := \gcd(n, q-1) \leq n/p \leq n/3 < (n-1)/2.$$

By Lemma 3.6(c) the exterior power $\Lambda^k(V)$, $k := \max\{2, d\}$, has Brauer character value 0 on elements of cycle shape $((n-1)/2)^2$. □

Remark 3.9. It can be checked that for example in $\mathrm{L}_2(27)$ there does not exist a 3-modular Brauer character taking value zero and invariant under all automorphisms. The same holds for $\mathrm{L}_2(27)$ in characteristic 2.

Proposition 3.10. *Theorem 1.2 holds for* $\mathrm{U}_n(q)$, $q = p^f$, *and the prime* p.

Proof. We again use the embedding $\mathfrak{A}_{n+1} \hookrightarrow \mathrm{SU}_n(q)$ and the fact that $\Lambda^k(V)$ and $A(V)$ are irreducible for $\mathrm{SU}(V)$ under the conditions formulated in Lemma 3.7.

Let first $n = 3$. If $q \equiv 1 \pmod 3$ then $\mathrm{U}_3(q) = \mathrm{SU}_3(q)$, and the Brauer character on V vanishes on 3-cycles in $\mathfrak{A}_4$. If $p = 3$ then it suffices to locate a zero in the 3-modular Brauer table of $\mathrm{U}_3(3)$. Indeed, the 7-dimensional representation vanishes on elements of order 7. If $q \equiv 2 \pmod 3$, then the Brauer character of V has value -1 on double transpositions, so the Brauer character on $A(V)$ vanishes on these elements. Finally, for $p = 2$ and $q = 2^{2m+1} > 2$, 3-cycles have Brauer character value 0 on the natural module, so the tensor product of V with some Frobenius twist also takes value 0 on 3-cycles.

So now let $n \geq 4$. In the proof of Proposition 3.8 we have produced elements $g \in \mathfrak{A}_{n+1}$ on which the Brauer character of $\Lambda^k(V)$ or $A(V)$ vanishes. By our embedding of $\mathfrak{A}_{n+1}$ into $\mathrm{SU}_n(q)$ this provides the required zeros for $\mathrm{SU}_n(q)$ as well. □

Proposition 3.11. *Theorem 1.2 holds for* $\mathrm{O}_{2n+1}(q)$, $n \geq 2$, $q = p^f$ *odd, and the prime* p.

Proof. The alternating group $\mathfrak{A}_{2n+2}$ embeds into $\mathrm{O}_{2n+1}(q)$ via the natural permutation representation modulo the trivial submodule. If $p \nmid (2n+1)$ then $2n+1$-cycles have Brauer character value 0 on this module, while for $p|(2n+1)$, elements of cycle shape $(2n-3)(2)^2$ have Brauer character value 0, unless $n = 2$. In the latter case $p = 5$, and then the 13-dimensional representations vanish on elements of order 13, as can be seen on the 5-Brauer character table of $\mathrm{S}_4(5)$ [9]. □

Proposition 3.12. *Theorem 1.2 holds for* $\mathrm{S}_{2n}(q)$, $n \geq 2$, $q = p^f$, *and the prime* p.

Proof. The groups $\mathrm{S}_4(q) \cong \mathrm{O}_5(q)$ for q odd have been treated in Proposition 3.11. For q even, all basic representations of $\mathrm{S}_4(q)$ have degree 4, and all other 2-modular irreducibles are obtained as tensor products, so all Brauer character degrees are powers of 2, leading to the exception (d) in Theorem 1.2.

For $n \geq 3$ we use the embeddings $\mathfrak{S}_n \hookrightarrow \mathrm{GL}_n(q) \hookrightarrow \mathrm{Sp}_{2n}(q)$ via the stabilizer of a maximal isotropic subspace. This identifies the natural module V of $\mathrm{Sp}_{2n}(q)$ with two copies of the permutation module of $\mathfrak{S}_n$. For $p \neq 2$ the symmetric square of V is irreducible for $\mathrm{S}_{2n}(q)$ (see [12, Table 2]). Thus its Brauer character vanishes on n-cycles or on elements of cycle shape $(n-2)(2)$, at least one of which is p-regular, unless $n = p = 3$. In the latter case, the 13-dimensional Brauer character of $\mathrm{S}_6(3)$ (and hence of $\mathrm{S}_6(3^f)$) vanishes on elements of order 13. If $p = 2$ then $\mathfrak{A}_{2n} \hookrightarrow \mathrm{O}_{2n+1}(q) \cong \mathrm{Sp}_{2n}(q)$, so the Brauer character of the natural representation of $\mathrm{Sp}_{2n}(q)$ becomes the permutation character of $\mathfrak{A}_{2n}$, which vanishes for example on elements of cycle shape $(2n-3)(3)$. □

Proposition 3.13. *Theorem 1.2 holds for* $\mathrm{O}_{2n}^{\pm}(q)$, $n \geq 4$, $q = p^f$, *and the prime* p.

Proof. By [11, p.186–187] the alternating group $\mathfrak{A}_{2n}$ embeds into $\mathrm{O}_{2n-1}(q)$, hence into both $\mathrm{SO}_{2n}^{\pm}(q)$ (for $p = 2$ use that $\mathrm{S}_{2n-2}(q) \cong \mathrm{O}_{2n-1}(q)$). If $p \neq 2$ the exterior square of the natural module V of $\mathrm{SO}_{2n}^{\pm}(q)$ is irreducible for the simple group $\mathrm{O}_{2n}^{\pm}(q)$, by [12, Thm. 5.1], and its Brauer character vanishes on $2n-1$-cycles or on elements of cycle shape $(2n-5)(2)^2$, at least one of which is p-regular. If $p = 2$, the Brauer character of the natural module vanishes on elements of cycle shape $(2n-3)(3)$. □

3.4. Sporadic groups.

Proposition 3.14. *The assertion of Theorem 1.2 holds for sporadic simple groups.*

Proof. By Proposition 2.3 and Lemma 3.2 we only need to consider

$$G \in \{M_{11}, M_{12}, M_{22}, J_2, HS, M_{24}, He, Ru, Suz, Co_3, Co_1, B\}.$$

Browsing the ordinary character tables of sporadic simple groups [4] it can be checked that the criterion in Lemma 3.1 is satisfied except for

$$M_{11}, M_{12}, M_{22}, J_2, HS, M_{24}, Ru, Suz, Co_3, Co_1, B$$

with $p = 2$ and for Suz, Co_3 with $p = 3$. The Brauer character tables for all these cases except for those of Co_1 and B modulo 2 are contained in [9], and the assertion can be checked directly.

Now the 24-dimensional representation of the 2-fold cover $2.Co_1$ on the Leech lattice restricts to an irreducible 2-modular Brauer character of Co_1, which vanishes for example on class 3D, according to [4]. Finally, by Jansen [8, 4.3.29] the 2-modular reduction of the smallest faithful character of degree 4371 of B contains a trivial constituent and an irreducible 2-modular Brauer character of degree 4370 which vanishes on elements of order 19. □

This completes the proof of Theorem 1.2.

References

[1] R. Burkhardt, Die Zerlegungsmatrizen der Gruppen $\mathrm{PSL}(2, p^f)$. J. Algebra **40** (1976), 75–96.

[2] R. W. Carter, Finite groups of Lie type. Conjugacy classes and complex characters. Wiley Classics Library. John Wiley & Sons, Chichester, 1993.

[3] A. Cohen, R. Griess, On finite simple subgroups of the complex Lie group of type E_8. Pp. 367–405 in: *The Arcata Conference on Representations of Finite Groups*, Amer. Math. Soc., Providence, RI, 1987.

[4] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, *Atlas of Finite Groups.* Clarendon Press, Oxford, 1985.

[5] M. Geck, G. Pfeiffer, *Characters of finite Coxeter groups and Iwahori-Hecke algebras.* Clarendon Press, New York, 2000.

[6] A. Granville, K. Ono, Defect zero p-blocks for finite simple groups. Trans. Amer. Math. Soc. **348** (1996), 331–347.

[7] G. James, A. Kerber, *The representation theory of the symmetric group.* Addison-Wesley, Reading, 1981.

[8] C. Jansen, The minimal degrees of faithful representations of the sporadic simple groups and their covering groups. LMS J. Comput. Math. **8** (2005), 122–144.

[9] C. Jansen, K. Lux, R. Parker, R. Wilson, *An Atlas of Brauer Characters.* Clarendon Press, Oxford, 1995.

[10] P. Kleidman, The maximal subgroups of the Chevalley groups $G_2(q)$ with q odd, the Ree groups ${}^2G_2(q)$, and their automorphism groups. J. Algebra **117** (1988), 30–71.

[11] P. Kleidman, M. Liebeck, *The subgroup structure of the finite classical groups.* Cambridge University Press, Cambridge, 1990.

[12] F. Lübeck, Small degree representations of finite Chevalley groups in defining characteristic. LMS J. Comput. Math. **4** (2001), 135–169.

[13] G. Lusztig, On the representations of reductive groups with disconnected centre. Astérisque **168** (1988), 157–166.

[14] G. Malle, Almost irreducible tensor squares. Comm. Algebra **27** (1999), 1033–1051.

[15] O. Manz, R. Staszewski, W. Willems, On the number of components of a graph related to character degrees. Proc. Amer. Math. Soc. **103** (1988), 31–37.

[16] C. Parker, J. Saxl, Two maximal subgroups of $E_8(2)$. Israel J. Math. **153** (2006), 307–318.

[17] A. Ryba, Short proofs of embeddings into exceptional groups of Lie type. J. Algebra **249** (2002), 402–418.

FB Mathematik, TU Kaiserslautern, Postfach 3049, 67653 Kaiserslautern, Germany.
E-mail address: malle@mathematik.uni-kl.de