

EXTENDING CHARACTERS FROM HALL SUBGROUPS

GUNTER MALLE AND GABRIEL NAVARRO

ABSTRACT. Suppose that G is a finite π -separable group. A classical result asserts that all irreducible characters of a Hall π -subgroup H of G extend to G if and only if H has a normal complement in G . Now, we fix a prime p and analyze when only the p' -degree irreducible characters of H extend to G .

1. INTRODUCTION

We come back to an old result of C. Sah ([18]) that asserts that in a finite π -separable group, all irreducible complex characters of a Hall π -subgroup H of G extend to G if and only if G has a normal π -complement. Now, we fix a prime p and we wish to characterize when only the p' -degree characters of H extend to G .

Theorem A. *Let G be a finite π -separable group. Let H be a Hall π -subgroup of G , let K be a π -complement of G , and let p be a prime. Then every $\alpha \in \text{Irr}(H)$ of p' -degree extends to G if and only if there is $P \in \text{Syl}_p(H)$ such that $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(K)$.*

Of course, Theorem A is far more general than Sah's theorem, although we pay the price of using the Classification of Finite Simple Groups. This is not that surprising, however: in the case where H is normal in G , Theorem A is equivalent to proving a well-known consequence of the (yet unproven) McKay conjecture (see [17, Thm. C]) which therefore now becomes established.

As an easy consequence of our main result we obtain:

Corollary B. *Suppose that A acts coprimely on a finite group G , and let $P \in \text{Syl}_p(G)$ be A -invariant. Then all p' -degree irreducible characters of G are A -invariant if and only if $[\mathbf{N}_G(P), A] = 1$.*

The paper is split into two parts. In Section 2 we prove Theorem A and Corollary B modulo a statement (Theorem 2.2) on finite quasi-simple groups which is then shown in Section 3, using the classification, properties of algebraic groups and Deligne–Lusztig theory.

2. PROOF OF THEOREM A

In our first result we use the Gajendragadkar's π -special characters (whose main properties can be found in [3]).

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Lemma 2.1. *Suppose that G is π -separable and let H be a Hall π -subgroup of G . Let $L \triangleleft G$. Suppose that $\alpha \in \text{Irr}(H)$ extends to G and is such that $H \cap L \subseteq \ker \alpha$. Then there is an extension $\beta \in \text{Irr}(G)$ such that $L \subseteq \ker \beta$. In particular, if all p' -degree irreducible characters of H extend to G , then all p' -degree irreducible characters of HL/L extend to G/L .*

Proof. Let $\tilde{\alpha} \in \text{Irr}(HL)$ be the unique irreducible character of HL that extends α and has L in its kernel. Notice that, by using the definition, $\tilde{\alpha}$ is π -special. By hypothesis, α extends to G . By [6, Thm. F], α has a π -special extension $\hat{\alpha}$ to G . Now by [3, Prop. (6.1)], we have that $\hat{\alpha}_{HL}$ is a π -special extension of α_H to HL . But, again by the uniqueness part in loc.cit., we have that $\hat{\alpha}_{HL} = \tilde{\alpha}$, and we are done. \square

In order to prove Theorem A, we shall need the following non-trivial result whose proof (that uses the Classification of Finite Simple Groups) we defer until Section 3 below.

Theorem 2.2. *Let p be a prime. Suppose that $M \triangleleft G$ has p -power index, and has a normal Hall subgroup S such that M/S is not divisible by p . Assume that S is a non-abelian quasi-simple group of order divisible by p with $\mathbf{C}_G(S) = \mathbf{Z}(S)$ a p' -group. Let $P \in \text{Syl}_p(G)$. Then the following are equivalent:*

- (i) *all p' -degree P -invariant irreducible characters of S are M -invariant, and*
- (ii) *there exists a complement K of S in M normalized by P and with $[\mathbf{N}_S(P), K] = 1$.*

We shall frequently write condition (ii) of the above theorem in the following more convenient form.

Lemma 2.3. *Let $S \triangleleft M$. Suppose that P, K are subgroups of M with $S \cap K = 1$. Then P normalizes K and $[\mathbf{N}_S(P), K] = 1$ if and only if $\mathbf{N}_{SP}(P) \subseteq \mathbf{N}_M(K)$.*

Proof. Since $P \subseteq \mathbf{N}_{SP}(P)$, by Dedekind's lemma we have that $\mathbf{N}_{SP}(P) = \mathbf{N}_S(P)P$. Thus, if P normalizes K and $[\mathbf{N}_S(P), K] = 1$ it is clear that $\mathbf{N}_{SP}(P) \subseteq \mathbf{N}_M(K)$. Conversely, we have that P and $\mathbf{N}_S(P)$ normalize K . Then, using that $S \triangleleft M$, we have that $[\mathbf{N}_S(P), K] \subseteq S \cap K = 1$. \square

Also, the group theoretical conclusion in Theorem A has another convenient form.

Lemma 2.4. *Suppose that G is a π -separable group with a Hall π -subgroup H and a Hall π -complement K . Let $P \in \text{Syl}_p(H)$. If P normalizes K , then $\mathbf{N}_G(P) = \mathbf{N}_H(P)\mathbf{N}_K(P)$. In particular, $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(K)$ if and only if $\mathbf{N}_H(P) \subseteq \mathbf{N}_G(K)$.*

Proof. This is an application of Lemma (2.1) of [10]. \square

Lemma 2.5. *Suppose that P is a p -group acting on a group S that does not have a normal p -complement. Then there exists $1 \neq \chi \in \text{Irr}(S)$ which is P -invariant of p' -degree.*

Proof. The semidirect product SP cannot have a normal p -complement. By Thompson's Theorem (Corollary (12.2) of [5]), there exists a non-linear $\gamma \in \text{Irr}(SP)$ of degree not divisible by p . Now, $\gamma_S = \chi$ is irreducible (by Corollary (11.29) of [5]) and P -invariant. It is clear that χ is not trivial since otherwise γ would be a p' -degree character of a p -group, while γ is not linear. \square

The following is one direction of Theorem A.

Theorem 2.6. *Let G be a finite π -separable group. Let H be a Hall π -subgroup of G , let K be a π -complement of G , and let p be a prime. Suppose that every $\alpha \in \text{Irr}(H)$ of p' -degree extends to G . Then there is $P \in \text{Syl}_p(H)$ such that $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(K)$.*

Proof. If p does not divide $|H|$, then all irreducible characters of H extend to G , and G has a normal π -complement by Sah's Theorem (see [18, Thm. 5]). So we may assume that $p \in \pi$.

We prove that there exists $P \in \text{Syl}_p(H)$ such that $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(K)$ by induction on $|G|$. Since H is a Hall subgroup of G , notice that every Sylow p -subgroup of H is a Sylow p -subgroup of G .

Let N be a fixed but arbitrary minimal normal subgroup of G .

Step 1. We can assume that there is $P \in \text{Syl}_p(H)$ such that $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(K)N$. Also, $N \subseteq H$. In particular, $\mathbf{O}_{\pi'}(G) = 1$.

By Lemma 2.1, we know that all p' -degree irreducible characters of HN/N extend to G/N . Therefore, by induction and using that $\mathbf{N}_G(KN) = \mathbf{N}_G(K)N$, we conclude that there is $P \in \text{Syl}_p(H)$ such that $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(K)N$. Now, since G is π -separable, we have that either N is a π -group or a π' -group. In the second case, $N \subseteq K$, and $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(K)$. So we will assume in the following that N is a π -group. Hence, $N \subseteq H$.

Step 2. It suffices to show that there is $m \in N$ such that $\mathbf{N}_{NP}(P^m)$ normalizes K .

Indeed, suppose that $\mathbf{N}_{NP}(P^m)$ normalizes K for some $m \in N$. Hence P^m , which is contained in NP , normalizes K . Then we have by Step 1 that $\mathbf{N}_G(P^m) \subseteq \mathbf{N}_G(K)N = G_1$. Now, $P^m \in \text{Syl}_p(G_1)$, and we may apply Lemma (2.1) of [10] to conclude that

$$\mathbf{N}_{G_1}(P^m) = \mathbf{N}_N(P^m)\mathbf{N}_{\mathbf{N}_G(K)}(P^m) \subseteq \mathbf{N}_G(K).$$

However $\mathbf{N}_G(P^m) \subseteq G_1$, so $\mathbf{N}_{G_1}(P^m) = \mathbf{N}_G(P^m)$. Now, $P^m \subseteq PN \subseteq H$ is a Sylow p -subgroup of H , and we are done in this case.

Step 3. If $\theta \in \text{Irr}(N)$ of p' -degree extends to PN , then θ is K -invariant.

If $\eta \in \text{Irr}(PN)$ is such an extension, then η^H has p' -degree. Hence there exists $\psi \in \text{Irr}(H)$ over η of p' -degree. By hypothesis, we have that ψ extends to some $\chi \in \text{Irr}(G)$. Let T be the stabilizer of θ in G , so that $PN \subseteq T$. If $\nu \in \text{Irr}(T|\theta)$ is the Clifford correspondent of χ over θ , then $\nu^G = \chi$ has π -degree. It follows that T contains some π -complement of G . Thus K^g is contained in T for some $g \in G$. Now, we know that P normalizes KN by Step 1, and hence $|G : \mathbf{N}_G(K)N|$ is not divisible by p . Thus $|G : \mathbf{N}_G(K^g)N|$ is not divisible by p . By Corollary (1.2) of [19] applied in the group G/N with respect to the subgroup T/N , we have that $|T : \mathbf{N}_T(K^g)N|$ divides $|G : \mathbf{N}_G(K^g)N|$, and therefore $|T : \mathbf{N}_T(K^g)N|$ is not divisible by p either. It follows that there is some $R \in \text{Syl}_p(T)$ normalizing K^gN . Hence $R \in \text{Syl}_p(G)$. Now, R and P^g are Sylow p -subgroups of $\mathbf{N}_G(K^g)N$. Thus $P^{gm n_0} = R$ for some $m \in \mathbf{N}_G(K^g)$ and $n_0 \in N$. Also, $R = P^v$ for some $v \in T$ because R and P are Sylow p -subgroups of T . Now, write $m = x^g$ for some $x \in \mathbf{N}_G(K)$, so that $gm = xg$. We have that $P^{xgn_0} = R = P^v$ and thus $xgn_0v^{-1} \in \mathbf{N}_G(P) \subseteq \mathbf{N}_G(K)N$. Hence, $gn_0v^{-1} \in \mathbf{N}_G(K)N$, and $gn_0v^{-1} = wn$ for some $w \in \mathbf{N}_G(K)$ and $n \in N$. Finally, since $K^g \subseteq T$, we have that $K^{gn_0v^{-1}} \subseteq T$ (because $n_0, v \in T$). Thus $K^{wn} \subseteq T$ and $K = K^w \subseteq T^{n^{-1}} = T$, as claimed.

Step 4. We can assume that $NKP = G$. Thus $H = NP$ and $M = NK \triangleleft G$.

By Step 1, we have that $P \subseteq \mathbf{N}_G(KN) = \mathbf{N}_G(K)N$, so $G_0 = NKP$ is a subgroup of G and $NK \triangleleft G_0$. Write $H_0 = NP$ and $G_0 = NKP$, and notice that H_0 is a Hall π -subgroup of G_0 . Also, K is a π -complement of G_0 . Let $\eta \in \text{Irr}(H_0)$ with p' -degree. Since $|H_0 : N|$ is a power of p , then, by Corollary (11.29) of [5], we have that $\theta = \eta_N \in \text{Irr}(N)$ has p' -degree. Now θ has an extension to H_0 . By Step 3, we conclude that θ is K -invariant. Now, $(|KN : N|, |N|) = 1$ and therefore θ has a canonical extension $\hat{\theta}$ to KN by Corollary (8.16) of [5], which is by uniqueness, therefore P -invariant. Hence, by Corollary (4.2) of [6], it follows that restriction defines a bijection

$$\text{Irr}(G_0|\hat{\theta}) \rightarrow \text{Irr}(H_0|\theta).$$

We conclude that η extends to G_0 . Now suppose that $G_0 < G$. Then, by induction, we conclude that there is $P_0 \in \text{Syl}_p(H_0)$ such that $\mathbf{N}_{G_0}(P_0) \subseteq \mathbf{N}_{G_0}(K)$. Since $P \in \text{Syl}_p(H_0)$, we have that $P_0 = P^n$ for some $n \in N$. Then $\mathbf{N}_{NP}(P^n) \subseteq \mathbf{N}_{G_0}(P_0) \subseteq \mathbf{N}_G(K)$, and we apply Step 2 in this case. Hence, we are reduced to the case where $G_0 = G$, $H = NP$ and $M = NK \triangleleft G$.

Step 5. We can assume that N is a direct product of non-abelian simple groups of order divisible by p which are transitively permuted by G . In particular, $\mathbf{O}_{p'}(G) = \mathbf{O}_p(G) = 1$. Also, we can assume that every p' -degree P -invariant irreducible character of N is M -invariant.

Suppose first that N is a p' -group. Hence, P acts coprimely on NK , and because NK is π -separable, it follows (using, for instance, that the number of Hall ρ -subgroups of KN is not divisible by p , where ρ is the set of prime divisors of $|K|$) that P normalizes some K^n for some $n \in N$. Now every P -invariant character of N extends to PN (by Corollary (8.16) of [5]), and by Step 3 is K -invariant. Therefore, every P -invariant character of N is NK -invariant, and therefore K^n -invariant. Thus every irreducible P -invariant character of N is PK^n -invariant and by Lemma (2.2) of [16], we conclude that $\mathbf{C}_N(PK^n) = \mathbf{C}_N(P)$. Hence, $\mathbf{N}_N(P) = \mathbf{C}_N(P) \subseteq \mathbf{C}_N(K^n) \subseteq \mathbf{N}_G(K^n)$. We had that P^{n-1} normalizes K and now we have that $\mathbf{N}_N(P^{n-1}) \subseteq \mathbf{N}_G(K)$. Thus $\mathbf{N}_{NP}(P^{n-1}) = \mathbf{N}_N(P^{n-1})P^{n-1} \subseteq \mathbf{N}_G(K)$, and this case is complete by Step 2.

Suppose now that N is a p -group. Hence $N \subseteq P$ and $H = P$. In this case, the hypotheses tell us that every linear character of P extends to G . By Tate's Theorem (use, for instance, Theorem (6.31) of [5]), we conclude that G has a normal p -complement. Hence $K \triangleleft G$, and in this case the theorem is proved.

So we may assume that N is a direct product of non-abelian simple groups of order divisible by p , which are transitively permuted by G . In particular $\mathbf{O}^p(N) = N$, and it follows that every P -invariant p' -degree irreducible character of N extends to PN by Corollary (8.16) of [5]. Therefore by Step 3, every P -invariant p' -degree character of N is K -invariant, and hence M -invariant.

Step 6. We can assume that N is a minimal normal subgroup of NP . Hence $N = S^{g_1} \times \cdots \times S^{g_t}$, where $\{S^{g_1}, \dots, S^{g_t}\}$ is the P -orbit of S , a non-abelian simple group of order divisible by p , $g_i \in P$, and $g_1 = 1$. Also, we can assume that $t > 1$.

We can write $N = U \times V$, where $U > 1$ and $V \geq 1$ are P -invariant, and U is the direct product of the P -orbit of a simple group S . That is, $U = S^{g_1} \times \cdots \times S^{g_t}$, where

$\{S^{g_1}, \dots, S^{g_t}\}$ is the P -orbit of S , $g_i \in P$, and $g_1 = 1$. By Lemma 2.5, let $1 \neq \eta \in \text{Irr}(S)$ be $\mathbf{N}_P(S)$ -invariant of p' -degree. Then it is straightforward to show that $\nu = \eta^{g_1} \times \dots \times \eta^{g_t}$ is P -invariant. Now, let $\tau = \nu \times 1_V \in \text{Irr}(N)$, which is P -invariant of p' -degree. Then τ is K -invariant by Step 5. Hence $\ker \tau < N$ is K -invariant, and therefore G -invariant. Since $V \subseteq \ker \tau$, we conclude that $V = 1$, because N is a minimal normal subgroup of G .

Suppose now that N is simple. Since $\mathbf{Z}(N) = 1$, we have that $\mathbf{C}_M(N)$ is a π' -group. Since by Step 1, we know that $\mathbf{O}_{\pi'}(G) = 1$, then we have that $\mathbf{C}_M(N) = 1$. Since G/M is a p -group, then we conclude that $\mathbf{C}_G(N)$ is a p -group. But we know that $\mathbf{O}_p(G) = 1$, and thus we conclude that $\mathbf{C}_G(N) = 1$. Then, and using Step 5, we are in the hypothesis of Theorem 2.2. We conclude by this theorem (and Lemma 2.3) that there is a complement K_1 of N in M such that $\mathbf{N}_{NP}(P) \subseteq \mathbf{N}_G(K_1)$. Now, $K_1 = K^n$ for some $n \in N$, and we have that $\mathbf{N}_{NP}(P^{n^{-1}}) \subseteq \mathbf{N}_G(K)$. Then we are done by Step 2.

Step 7. Conclusion.

As before, write $N = S_1 \times \dots \times S_t$, where $S_i = S^{g_i}$, $g_i \in P$ and $S = S_1$. Also recall that $t > 1$ and therefore $\mathbf{N}_G(S) < G$.

By using Steps 5 and 6, we have that $G = P\mathbf{N}_G(S)$. Thus $|G : \mathbf{N}_G(S)|$ is a power of p . Now, since G/N has a normal p -complement M/N , and $N \subseteq \mathbf{N}_G(S)$, we conclude that $M \subseteq \mathbf{N}_G(S)$. In particular, $M \subseteq \mathbf{N}_G(S^{g_i})$ for all i .

Now, using that G/M is a p -group, let $G_2 \triangleleft G$ be containing $\mathbf{N}_G(S)$ with $|G_2| < |G|$. Notice that $\mathbf{N}_P(S)$ is a Sylow p -subgroup of $\mathbf{N}_G(S)$. Also $\mathbf{N}_P(S) \subseteq P_2 = P \cap G_2 \in \text{Syl}_p(G_2)$. Now, let $\eta \in \text{Irr}(NP_2)$ of p' -degree. We have that $\eta_N \in \text{Irr}(N)$ is P_2 -invariant of p' -degree. Write $\eta_N = \theta_1 \times \dots \times \theta_t$, where $\theta_i \in \text{Irr}(S_i)$. Then θ_1 is $\mathbf{N}_P(S)$ -invariant, because we can write $N = S \times S'$, where S and S' are $\mathbf{N}_P(S)$ -invariant. Now, let

$$\theta = \theta_1 \times (\theta_1)^{g_2} \times \dots \times (\theta_1)^{g_t},$$

which is P -invariant. By Step 5, the character θ is M -invariant. In particular θ_1 is M -invariant. Since $\mathbf{N}_P(S^{g_i}) \subseteq P_2 \triangleleft P$, we can repeat the same argument with every S^{g_i} and every θ_i to conclude that η_N is M -invariant. By induction applied in the group G_2 with respect to the Hall π -subgroup NP_2 and Hall π -complement K , we conclude that there exists $P_3 \in \text{Syl}_p(P_2N)$ such that $\mathbf{N}_{G_2}(P_3) \subseteq \mathbf{N}_G(K)$. Hence P_3 normalizes K and is such that $[\mathbf{N}_N(P_3), K] = 1$ by Lemma 2.3. Now, $P_3 \cap N \in \text{Syl}_p(N)$, and also $[P_3 \cap N, K] = 1$. In particular, $P_3 \cap N \subseteq \mathbf{N}_N(K)$ and by the Frattini argument, we see that $|G : \mathbf{N}_G(K)|$ is not divisible by p .

Now, $P_3 \subseteq P_1 \in \text{Syl}_p(\mathbf{N}_G(K))$, and we have that $P_1 \in \text{Syl}_p(G)$. Recall that $G = MP = (KN)P$. Since $P \in \text{Syl}_p(G)$, we may write $(P_1)^{kn^{-1}x} = P$ for some $k \in K$, $x \in P$ and $n \in N$. Hence $(P_1)^k = P^n$. Since P_1 normalizes K , we have that $(P_1)^k$ normalizes K and thus P^n normalizes K . Also, since $[\mathbf{N}_N(P_3), K] = 1$, we have that $[\mathbf{N}_N((P_3)^k), K] = 1$, where $(P_3)^k \subseteq P^n$.

Finally, let $Q = (P_3)^k \cap N = (P_3 \cap N)^k \in \text{Syl}_p(N)$. Now, notice that, by elementary group theory, if R is any p -subgroup of G such that $R \cap N = Q$, then $\mathbf{N}_G(R) \subseteq \mathbf{N}_G(Q)$ and $\mathbf{N}_N(R)/Q = \mathbf{C}_{\mathbf{N}_N(Q)/Q}(R)$.

Since $(P_3)^k \cap N = Q \in \text{Syl}_p(N)$, and $(P_3)^k \subseteq (P_1)^k$, we also have that $(P_1)^k \cap N = Q$. In particular,

$$\mathbf{N}_N((P_1)^k)/Q = \mathbf{C}_{\mathbf{N}_N(Q)/Q}((P_1)^k) \subseteq \mathbf{C}_{\mathbf{N}_N(Q)/Q}((P_3)^k) = \mathbf{N}_N((P_3)^k)/Q$$

and we conclude that $\mathbf{N}_N((P_1)^k) \subseteq \mathbf{N}_N((P_3)^k) \subseteq \mathbf{C}_G(K)$. Hence, we have found $(P_1)^k \in \text{Syl}_p(G)$ such that $(P_1)^k$ normalizes K and $[\mathbf{N}_N((P_1)^k), K] = 1$. Hence

$$\mathbf{N}_{N(P_1)^k}((P_1)^k) \subseteq \mathbf{N}_G(K)$$

using Lemma 2.3. Since $(P_1)^k = P^n \in \text{Syl}_p(H)$, we use Step 2 to finish the proof of the theorem. \square

In order to prove the remaining direction of Theorem A, we need one more lemma.

Lemma 2.7. *Suppose that A acts as automorphisms on a finite group G , where G is a direct product of a set of subgroups \mathcal{X} , which are permuted by A . Let $B \subseteq A$, and suppose that B acts transitively on \mathcal{X} . Let $S \in \mathcal{X}$. Then $\mathbf{C}_G(A) = \mathbf{C}_G(B)$ if and only if $\mathbf{C}_S(\mathbf{N}_A(S)) = \mathbf{C}_S(\mathbf{N}_B(S))$.*

Proof. By [9, Lemma 2.2], we have that $\mathbf{C}_G(A) \cong \mathbf{C}_S(\mathbf{N}_A(S))$ and $\mathbf{C}_G(B) \cong \mathbf{C}_S(\mathbf{N}_B(S))$. Since $\mathbf{C}_G(A) \subseteq \mathbf{C}_G(B)$ and $\mathbf{C}_S(\mathbf{N}_A(S)) \subseteq \mathbf{C}_S(\mathbf{N}_B(S))$, the proof easily follows. \square

The following completes the proof of Theorem A.

Theorem 2.8. *Let G be a finite π -separable group. Let H be a Hall π -subgroup of G , let K be a π -complement of G , and let p be a prime. Suppose that there is $P \in \text{Syl}_p(H)$ such that $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(K)$. Then every $\alpha \in \text{Irr}(H)$ of p' -degree extends to G .*

Proof. Again, we can assume that $p \in \pi$. We argue by double induction, first on $|G : \mathbf{O}_\pi(G)|$, and second on $|G|$. If $U \leq G$ and $N \triangleleft G$, then notice that $|U : \mathbf{O}_\pi(U)| \leq |G : \mathbf{O}_\pi(G)|$, and that $|G/N : \mathbf{O}_\pi(G/N)| \leq |G : \mathbf{O}_\pi(G)|$.

By hypothesis, we have that $K \triangleleft KP \leq G$. Also, if $KP \leq U < G$, by induction, we have that the theorem is valid for U with respect to any Hall subgroup of U containing P . Let $\alpha \in \text{Irr}(H)$ be of p' -degree. We want to show that α extends to G . If $N \triangleleft G$, then we have that $\mathbf{N}_{G/N}(PN/N) \subseteq \mathbf{N}_{G/N}(KN/N)$. Therefore, if $1 < N$ is a π' -group, we easily see that $|G/N : \mathbf{O}_\pi(G/N)| < |G : \mathbf{O}_\pi(G)|$, and we deduce that $\hat{\alpha} \in \text{Irr}(HN/N)$ (the unique extension of α to HN having N in its kernel) extends to G/N , by induction. Hence, we may assume that $\mathbf{O}_{\pi'}(G) = 1$.

Now, let $N = \mathbf{O}_\pi(G) \subseteq H$. Suppose that $Z \subseteq N$ is normal in G . Since α has p' -degree, there exists $\hat{\eta} \in \text{Irr}(ZP)$ of p' -degree under α . Hence $\eta = \hat{\eta}_Z \in \text{Irr}(Z)$, because ZP/Z is a p -group. Now, if $ZPK < G$, by induction, the group ZPK with Hall subgroup ZP and complement K satisfies the hypothesis of the theorem. We conclude that $\hat{\eta}$ extends to ZPK , and therefore that η is K -invariant. Hence $PK \subseteq T = I_G(\eta)$, the stabilizer of η in G . We have that $HT = G$ because $K \subseteq T$. Let $\nu \in \text{Irr}(T \cap H)$ be the Clifford correspondent of α over η . If $T < G$, by induction, we conclude that ν has an extension $\hat{\nu} \in \text{Irr}(T)$. Then $(\hat{\nu}^G)_H = \nu^H = \alpha$ (using Mackey), and we are done in this case. Hence, we conclude that whenever $Z \subseteq N$ is normal in G then either $ZKP = G$ or η is G -invariant.

In the latter case, where η is G -invariant, we use the theory of character triple isomorphisms, as developed in [7]. Since η extends to ZP , then by using Theorem 5.2 of [7] and its proof, we can find a character triple (G^*, Z^*, η^*) isomorphic to (G, Z, η) , where Z^* is a p' -group, and also a π -group. Now, $(PZ)^* = P^* \times Z^*$ for a unique Sylow p -subgroup P^* of G^* . Also, H^* is a Hall π -subgroup of G^* and, using the Schur–Zassenhaus theorem,

we have that $(KZ)^* = K^* \times Z^*$ for a unique subgroup K^* of G^* , which turns out to be a Hall π -complement of G^* . Also, using that $\mathbf{N}_G(PZ)^* = \mathbf{N}_{G^*}(P^* \times Z^*) = \mathbf{N}_{G^*}(P^*)$ and that $\mathbf{N}_{G^*}(K^* \times Z^*) = \mathbf{N}_{G^*}(K^*)$, we see that the hypotheses of the theorem are satisfied in G^* . Furthermore, $N^* = \mathbf{O}_\pi(G^*)$, and therefore $|G : \mathbf{O}_\pi(G)| = |G^* : \mathbf{O}_\pi(G^*)|$.

Now let $\theta \in \text{Irr}(N)$ be an irreducible constituent of α_N . Suppose that θ is G -invariant. Then, using the notation of the previous paragraph with $Z = N$, we have that $N^* = \mathbf{O}_\pi(G^*)$ is central in G^* . By the Hall–Higman’s 1.2.3 Lemma, it follows that $V = \mathbf{O}_{\pi'}(G^*) > 1$. Then $|G^*/V : \mathbf{O}_\pi(G^*/V)| < |G^* : N^*| = |G : \mathbf{O}_\pi(G)|$, and by induction, and arguing as in the first paragraph of the proof, we conclude that α^* extends to G^* , and therefore that α extends to G . Hence, by the previous paragraph, we may assume that $NPK = G$. Thus $H = NP$.

Suppose now that θ is K -invariant. In this case, θ has a canonical extension ρ to $M = NK \triangleleft G$, using that $(|M : N|, |N|) = 1$. Also ρ is P -invariant by uniqueness. Also, in this case, we know by Corollary (4.2) of [6] that restriction defines a bijection $\text{Irr}(G|\rho) \rightarrow \text{Irr}(NP|\theta)$, and we conclude that α extends to G . Hence, it is enough to show that θ is K -invariant.

Now, let N/Z be a chief factor of G . Since $ZKP < G$, we conclude by the second paragraph of the proof that α_Z has a G -invariant irreducible constituent. Hence, by using again character triple isomorphisms, it is no loss to assume that Z is a central p' -subgroup of G .

In our present situation, and using Lemma 2.3, notice that our hypotheses now are that P normalizes K (that is, $M \triangleleft G$) and that $[\mathbf{N}_N(P), K] = 1$. If N/Z is a p' -group, then $\mathbf{N}_N(P) = \mathbf{C}_N(P)$, and thus $\mathbf{C}_N(KP) = \mathbf{C}_N(P)$. In this case, and using that θ is P -invariant, θ is K -invariant by Lemma (2.2) of [16]. If N/Z is a p -group, since P normalizes K , we have that $[\mathbf{O}_p(G), K] = 1$, and in this case we have that θ is K -invariant too (since $N \subseteq \mathbf{O}_p(G) \times Z$).

Hence we may assume that N/Z is the direct product $S_1/Z \times \dots \times S_t/Z$ of non-abelian simple groups of order divisible by p which are transitively permuted by G . In particular $\mathbf{O}_p(G) = 1$. Since $\mathbf{O}_{\pi'}(G) = 1$, we easily deduce that $\mathbf{C}_M(N) = \mathbf{Z}(N) = Z$. Now, $\mathbf{C}_H(N)$ is a Hall π -subgroup of $\mathbf{C}_G(N)$, and $\mathbf{C}_K(N)$ is a Hall π -complement of $\mathbf{C}_G(N)$. Hence $\mathbf{C}_G(N) = \mathbf{C}_H(N)\mathbf{C}_K(N)$. Since $\mathbf{C}_K(N) = Z$, we see that $\mathbf{C}_G(N) \subseteq H = NP$. In particular, $\mathbf{C}_G(N)/Z$ is a p -group. Since $\mathbf{O}_p(G) = 1$, we see that $\mathbf{C}_G(N) = Z$. Now, since N/Z is a direct product of non-abelian simple groups, we have that $N'Z = N$. Since Z is a p' -group, then N/N' is a p' -group, and $N \cap P = N' \cap P$. Suppose that N is not perfect. If $N'P = NP$, then $N = N \cap N'P = N'(N \cap P) \subseteq N'$. Thus $N'P < P$. By Corollary (4.2) of [6], restriction then defines a bijection $\text{Irr}(NP|\lambda) \rightarrow \text{Irr}(N'P|\lambda_{Z \cap N'})$, and also $\text{Irr}(N|\lambda) \rightarrow \text{Irr}(N'|\lambda_{Z \cap N'})$. By the inductive hypothesis applied in $N'PK$, we conclude that $\alpha_{N'P}$ extends to $N'PK$. Thus $\alpha_{N'}$ is K -invariant. By uniqueness in the restriction map, we deduce that $\alpha_N = \theta$ is also K -invariant, and we are done in this case too. Thus we may assume that N is perfect.

If $t = 1$, that is, if N/Z is simple, then we may apply Theorem 2.2 to conclude that θ is K -invariant. So we may assume that $t > 1$. Hence $\mathbf{N}_G(S_1) < G$.

Let $Q = P \cap N \in \text{Syl}_p(N)$, and let $Q_i = Q \cap S_i = P \cap S_i \in \text{Syl}_p(S_i)$. Since $[\mathbf{N}_N(P), K] = 1$, we have that $[Q, K] = 1$. Now, let $1 \neq x \in Q_i$, and let $k \in K$. Then $x = x^k \in S_i \cap (S_i)^k$. If $(S_i)^k \neq S_i$, then $(S_i)^k \cap S_i = Z$, a p' -group, and this is not possible. We conclude that

$K \subseteq \mathbf{N}_G(S_i)$ for all i . Since $S_i \triangleleft N$, then $M \subseteq \mathbf{N}_G(S_i)$ and we conclude that all the S_i 's are P -conjugate, and $P\mathbf{N}_G(S_i) = G$.

Now, using that Z is a p' -group, we have that $Q = Q_1 \times \cdots \times Q_t$. Write $S_i = S^{g_i}$ for some $g_i \in P$, where $S = S_1$. Let $P_0 = \mathbf{N}_P(S) \in \text{Syl}_p(\mathbf{N}_G(S))$. Since $N \subseteq \mathbf{N}_G(S)$, we have that $P_0 \cap N \in \text{Syl}_p(N)$ and $P_0 \cap S \in \text{Syl}_p(S)$. Necessarily, $P_0 \cap N = Q$ and $P_0 \cap S = Q_1$. Also, since P normalizes K , then P_0 normalizes KS . Thus $KS \triangleleft KSP_0 = G_0 < G$.

We wish to apply the inductive hypothesis in G_0 , where here $H_0 = SP_0$ is a Hall π -subgroup of G_0 , and K is a π -complement of G_0 . Notice that $P_0 \in \text{Syl}_p(SP_0)$, since $P_0 \cap S \in \text{Syl}_p(S)$ and $|SP_0 : P_0| = |S : S \cap P_0|$.

By Lemma 2.4, we need to check that $\mathbf{N}_{SP_0}(P_0)$ normalizes K . By Lemma 2.3, we need to check that P_0 normalizes K and $[\mathbf{N}_S(P_0), K] = 1$. Since P normalizes K , we only need to check that $[\mathbf{N}_S(P_0), K] = 1$. By hypothesis, we know that $[\mathbf{N}_N(P), K] = 1$.

Now, $P_0 \cap \mathbf{N}_S(Q_1) = Q_1$ and therefore $\mathbf{N}_{G_0}(P_0) \subseteq \mathbf{N}_{G_0}(Q_1)$. We easily conclude that

$$\mathbf{N}_S(P_0)/Q_1 = \mathbf{C}_{\mathbf{N}_S(Q_1)/Q_1}(P_0).$$

By the same argument,

$$\mathbf{N}_N(P)/Q = \mathbf{C}_{\mathbf{N}_N(Q)/Q}(P).$$

Since $[\mathbf{N}_N(P), K] = 1$, then $\mathbf{C}_{\mathbf{N}_N(Q)/Q}(P) = \mathbf{C}_{\mathbf{N}_N(Q)/Q}(PK)$.

Now, we have that $\mathbf{N}_N(Q)/Q$ is KP -isomorphic to the direct product of $\mathbf{N}_{S_i}(Q_i)/Q_i$ and that these factors are transitively permuted by P . By Lemma 2.7, we know that

$$\mathbf{C}_{\mathbf{N}_N(Q)/Q}(KP) = \mathbf{C}_{\mathbf{N}_N(Q)/Q}(P)$$

if and only if

$$\mathbf{C}_{\mathbf{N}_S(Q_1)/Q_1}(KP_0) = \mathbf{C}_{\mathbf{N}_S(Q_1)/Q_1}(P_0) = \mathbf{N}_S(P_0)/Q_1.$$

We conclude that K acts trivially on $\mathbf{N}_S(P_0)/Q_1$. Since $[Q_1, K] = 1$, by coprime action we have that $[K, \mathbf{N}_S(P_0)] = 1$, as desired.

Hence, we can apply the inductive hypothesis in G_0 . Now, by using the notation of central products used in Section 5.1 of [8], we can write $\theta = \theta_1 \cdots \theta_t$, where $\theta_i \in \text{Irr}(S_i)$. Since θ is P -invariant, then we conclude that θ_1 is P_0 -invariant. Since the determinantal order and the degree of θ_1 are coprime to p , we see that θ_1 extends to a p' -degree character of SP_0 . By induction, this character extends to G_0 , and we conclude that θ_1 is K -invariant. The same argument applies to every θ_i , and we conclude that θ is K -invariant. This finishes the proof of the theorem. \square

The proof of Corollary B now is immediate:

Proof of Corollary B. Let $\Gamma = GA$ be the semidirect product. Then Γ has a Hall π -subgroup G and a π -complement A . By coprime action, G has an A -invariant Sylow p -subgroup P , and all of them are $\mathbf{C}_G(A)$ -conjugate. Also $\mathbf{N}_\Gamma(P) = \mathbf{N}_G(P)A$ and $\mathbf{N}_\Gamma(A) = \mathbf{C}_G(A) \times A$. Suppose that $[\mathbf{N}_G(P), A] = 1$. Then $\mathbf{N}_\Gamma(P) \subseteq \mathbf{N}_\Gamma(A)$ and by Theorem A, we have that all irreducible p' -degree characters of G extend to Γ and are, therefore, A -invariant.

Conversely, if all p' -degree irreducible characters of G are A -invariant, then all of them extend to Γ since $(|\Gamma : G|, |G|) = 1$. Hence, by Theorem A, there is a Sylow p -subgroup P_1 of G such that $\mathbf{N}_\Gamma(P_1) \subseteq \mathbf{N}_\Gamma(A)$. Hence $[\mathbf{N}_G(P_1), A] = 1$. In particular, P_1 is A -invariant,

and we conclude that $(P_1)^c = P$ for some $c \in \mathbf{C}_G(A)$. Then $[\mathbf{N}_G(P), A] = 1$, and the proof is complete. \square

3. PROOF OF THEOREM 2.2

The aim of this section is the proof of Theorem 2.2 which we restate as follows:

Theorem 3.1. *Let S be quasi-simple, normal in a group X with $S/Z(S) \leq X/Z(S) \leq \text{Aut}(S/Z(S))$, and let p be a prime dividing $|S|$, with $|Z(S)|$ prime to p . Let M/S be normal in X/S of order prime to $|S|$, such that X/M is a p -group. Let $P \in \text{Syl}_p(X)$. Then the following are equivalent:*

- (i) *all P -invariant characters in $\text{Irr}_{p'}(S)$ are M -invariant;*
- (ii) *there exists a complement K of S in M normalized by P with $[N_S(P), K] = 1$.*

The statement is trivially true when $M = S$, so we may assume that the quasi-simple group S has outer automorphisms of order prime to $|S|$, which by the classification forces S to be of Lie type and M/S to consist of field automorphisms:

Proposition 3.2. *Let S be finite quasi-simple and $\sigma \in \text{Aut}(S)$ with $\gcd(o(\sigma), |S|) = 1$. Then S is of Lie type and σ is a field automorphism. In particular, $\text{Out}(S)$ has a cyclic central π' -Hall subgroup, where $\pi = \pi(|S|)$.*

Proof. According to [4, Cor. 5.1.4], any automorphism of a finite quasi-simple group S is induced by an automorphism of the simple quotient $S/Z(S)$, so it suffices to deal with the case that S is simple. Since the outer automorphism group of finite simple groups not of Lie type has order a power of 2, clearly S must be of Lie type. In this case, $\text{Out}(S)$ is described in [4, §2.5]: it is an extension of the normal subgroup D of diagonal automorphisms (which is cyclic or a Klein four group) by the commuting product of the cyclic group of field automorphisms with the group of graph automorphisms (the latter being a subgroup of \mathfrak{S}_3). Now the order of D is only divisible by prime divisors of the order of the Weyl group of S . But by Remark 3.3 below, since $o(\sigma)$ is prime to $|S|$, it is only divisible by primes larger than those occurring in $|D|$, so σ can only act trivially on D . This shows the claim. \square

Remark 3.3. In the preceding proof, we used the following observation: if W is a Weyl group, and p a prime divisor of its order, then all primes dividing $p - 1$ also divide $|W|$. This can easily be checked by inspection. It also follows from the rationality of Weyl groups.

In fact, even more is true: all primes smaller than p divide $|W|$; no a priori proof of this is known to the authors.

3.1. The general setup. We fix the following notation throughout this section. Let \mathbf{G} be a simple linear algebraic group of simply connected type over the algebraic closure of a finite field of characteristic r , and $\mathbf{T} \leq \mathbf{G}$ a maximal torus. Then for any graph automorphism of the Dynkin diagram of \mathbf{G} and any integral power $q = r^a$ there exists a Steinberg endomorphism $F_a : \mathbf{G} \rightarrow \mathbf{G}$ such that F_a acts as $q\phi$ on the character group $X(\mathbf{T})$ of \mathbf{T} , with ϕ of finite order inducing the given graph automorphism on the Weyl group. Similarly, if \mathbf{G} is of type B_2 , G_2 or F_4 , and $r = 2, 3$, respectively 2, then for any odd power $q = \sqrt{r}^a$ of \sqrt{r} there exists a Steinberg endomorphism $F_a : \mathbf{G} \rightarrow \mathbf{G}$ such that

F_a acts as $q\phi$ on $X(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ and ϕ induces the non-trivial graph automorphism of the Coxeter diagram of \mathbf{G} (see [15, Thm. 22.5], for example).

Now for a Steinberg endomorphism $F = F_a$ as above, let $G = \mathbf{G}^F$ be the corresponding finite group of fixed points. Then, with a finite number of exceptions, G is a quasi-simple group of Lie type, and all finite simple groups of Lie type arise by this setup as $G/Z(G)$, except for ${}^2F_4(2)'$. Since the latter group has no coprime automorphisms, this exception is of no importance for our question.

So we may and will now assume that $G = \mathbf{G}^F$ is perfect. Then, except for a finite number of cases, G is the universal Schur cover of the simple group $G/Z(G)$. Since none of the simple groups of Lie type with exceptional Schur multiplier have coprime automorphisms (see [4, §6.1]), we may assume that $G/Z(G)$ is none of these. Thus, the quasi-simple group S from the statement of Theorem 3.1 can be realized as a central quotient $S = G/Z$ of a group G as above, for some $Z \leq Z(G)$, and in particular all irreducible characters of S occur among the characters of G .

According to Proposition 3.2 we will have to consider field automorphisms of S . Now any coprime field automorphism γ of G , and hence of S , is induced by a Steinberg endomorphism of \mathbf{G} as follows. For fixed r and \mathbf{G} , we may assume the F_a to be chosen compatibly such that $F_{an} = F_a^n$ for any n coprime to $o(\phi)$. In this setting, if γ is a coprime field automorphism of G of order f , then in particular, f is prime to the order of ϕ , and thus there exists a Steinberg endomorphism $F_c : \mathbf{G} \rightarrow \mathbf{G}$ such that $F = F_{cf} = F_c^f$ and $\langle \gamma \rangle = \langle F_c|_G \rangle$ in $\text{Out}(G)$. The centralizer of γ in G is then just the fixed point group \mathbf{G}^{F_c} under F_c , a finite group of Lie type of the same type as G .

3.2. Action on $\text{Irr}(G)$. In order to determine the action of coprime automorphisms on $\text{Irr}(G)$, we first recall Lusztig's parametrization. For this, let \mathbf{G}^* be a group in duality with \mathbf{G} , with compatible Steinberg endomorphisms $F_a^* : \mathbf{G}^* \rightarrow \mathbf{G}^*$, $a \geq 1$ (respectively a odd, when G is a Suzuki or Ree group). We'll write $G^* := \mathbf{G}^{*F^*}$ for the group of fixed points of $F^* = F_a^*$. Then by the results of Lusztig, there is a partition

$$\text{Irr}(G) = \bigsqcup_{s \in G^*/\sim} \mathcal{E}(G, s)$$

into Lusztig series $\mathcal{E}(G, s)$, indexed by semisimple elements s of the dual group G^* modulo conjugation. Now recall that $S = G/Z$ for some subgroup $Z \leq Z(G)$. Let $T \leq G$ be a maximal torus, with dual torus $T^* \leq G^*$. So any $s \in T^*$ is a linear character of T , and $\chi \in \mathcal{E}(G, s)$ is trivial on $Z \leq T$ if and only if $Z \leq \ker(s)$ (see [13, Lemma 2.2]). This defines a subgroup of T^* of index $|Z|$; we denote by S^* the normal subgroup of G^* generated by all these subgroups, so $G^*/S^* \cong Z$ (and hence $[G^*, G^*] \leq S^*$). Thus, for $s \in G^*$ semisimple, the characters in $\mathcal{E}(G, s)$ have Z in their center, so descend to $S = G/Z$, if and only if $s \in S^*$.

$$\begin{array}{ccc} G_{\text{ss}}^* & \xrightarrow{\text{param. irred. of}} & G \\ \uparrow & & \downarrow \\ S_{\text{ss}}^* & \xrightarrow{\text{param. irred. of}} & S \end{array}$$

Now any field automorphism of G permutes the various Lusztig series $\mathcal{E}(G, s)$. More precisely, if γ is induced by F_c as described above, then by Lusztig $\mathcal{E}(G, s)$ is γ -stable if and only if the G^* -class of s is F_c^* -stable. We need the following:

Lemma 3.4. *In the above situation, assume that $\mathcal{E}(G, s)$ is γ -stable. Then γ fixes each orbit in $\mathcal{E}(G, s)$ under diagonal automorphisms.*

Proof. Let $\mathbf{H} = C_{\mathbf{G}^*}(s)$. Lusztig [12, Prop. 5.1] proves the existence of a surjection

$$\psi : \mathcal{E}(G, s) \rightarrow \mathcal{E}(\mathbf{H}^{\circ F}, 1) \pmod{\mathbf{H}^F/\mathbf{H}^{\circ F}}$$

with fibres the orbits under the action of diagonal automorphisms, such that multiplicities of $\rho \in \mathcal{E}(G, s)$ in Deligne–Lusztig characters are determined by those of elements of $\psi(\rho)$. We claim that ψ can be chosen γ -equivariant. The result then follows since field automorphisms act trivially on $\mathcal{E}(\mathbf{H}^{\circ F}, 1)$ (see e.g. [2, Prop. 6.6]).

Now unipotent characters are uniquely determined by their multiplicities in Deligne–Lusztig characters, and thus the claim follows, unless H has a component of exceptional type (see [2, Prop. 6.3]). But in the latter case, characters with same multiplicities in all Deligne–Lusztig characters have distinct eigenvalues of Frobenius attached, and since γ commutes with F , these are respected by γ . \square

In our situation, as remarked in the proof of Proposition 3.2, any prime divisor of $o(\gamma)$ is larger than the order of the group of diagonal automorphisms. Thus γ must in fact fix all elements in $\mathcal{E}(G, s)$. We hence get the following characterization of γ -stable characters of S (recall that all irreducible characters of S occur among those of G):

Proposition 3.5. *Let γ be a coprime automorphism of S , where $S = G/Z$ with $G = \mathbf{G}^F$, $Z \leq Z(G)$ as above. Then $\chi \in \text{Irr}(S)$ is γ -stable if and only if $\chi \in \mathcal{E}(G, s)$ for some F_c^* -stable semisimple element $s \in S^*$, where γ is induced by F_c .*

Proof. This follows from the above, using the fact that any F_c^* -stable conjugacy class of the connected group \mathbf{G} contains an F_c^* -stable (and hence $F^* = F_c^{*f}$ -stable) element (see e.g., [15, Thm. 21.11]). \square

3.3. Defining characteristic. We now first dispose of an easy case; observe that in defining characteristic, all semisimple characters lie in $\text{Irr}_{p'}(G)$, by Lusztig’s Jordan decomposition of characters (but note that when r is bad $\text{Irr}_{p'}(G)$ may contain non-semisimple characters).

Proposition 3.6. *Let S be quasi-simple of Lie type, γ a coprime automorphism of S . Let $p = r$, the defining characteristic of S . Let A be a p -group of automorphisms of S . Then γ does not centralize a Sylow p -subgroup of S , and moreover not all A -invariant elements of $\text{Irr}_{p'}(G)$ are γ -invariant. In particular, Theorem 3.1 holds in this situation.*

Proof. Let f denote the order of γ . As above, we may choose $F, F_c : \mathbf{G} \rightarrow \mathbf{G}$ such that S is a central quotient of $G = \mathbf{G}^F$, $F = F_c^f$ and γ is induced by $F_c|_G$. Then $G_0 := \mathbf{G}^{F_c}$ is a group of the same type as G , but over a subfield \mathbb{F}_{q_0} of \mathbb{F}_q , of index $[\mathbb{F}_q : \mathbb{F}_{q_0}] = f$. The order formula for groups of Lie type (see e.g., [15, Table 24.1]) then shows that the p -parts of G and G_0 differ. Since $Z(G)$ has order prime to p , the same holds for the p -parts of S and $C_S(\gamma)$. In particular, γ does not centralize a Sylow p -subgroup of S , whence (ii) of Theorem 3.1 does not hold.

Since diagonal automorphisms have order prime to the defining characteristic, A consists of graph and field automorphisms only. Thus, A induces a group of automorphisms A^* on the dual G^* of G , and we write $H := C_{G^*}(A^*)$ for its fixed point group. Now let $s \in H \cap [G^*, G^*]$ be semisimple, not centralized by γ^* (which exists by Zsigmondy's theorem). Then by Proposition 3.5 the Lusztig series $\mathcal{E}(G, s)$ consists of characters with $Z(G)$ in their kernel, invariant under A , but not invariant under γ . Moreover, the semisimple character in $\mathcal{E}(G, s)$ is of p' -degree. This shows that (i) of Theorem 3.1 is not satisfied either, thus completing the proof. \square

3.4. Sylow subgroups. We now turn to the non-defining primes. There we require the following crucial result from [13, Prop. 7.3]:

Proposition 3.7. *Let G be as above, $p \neq r$ and $\chi \in \text{Irr}_{p'}(G)$. Then there is some semisimple $s \in G^*$ centralizing a Sylow p -subgroup of G^* with $\chi \in \mathcal{E}(G, s)$. Conversely, if $s \in G^*$ centralizes a Sylow p -subgroup, then there is some $\chi \in \mathcal{E}(G, s)$ of p -height 0.*

We next consider the following generic case, which occurs for all large enough primes p :

Proposition 3.8. *Let $S = G/Z$ be quasi-simple of Lie type, with $G = \mathbf{G}^F$, $Z \leq Z(G)$ as above. Let $p \neq r$ and P a Sylow p -subgroup of G , P^* a Sylow p -subgroup of G^* .*

- (a) *Then P is contained in a proper F -stable Levi subgroup of \mathbf{G} if and only if P^* is contained in a proper F^* -stable Levi subgroup of \mathbf{G}^* .*
- (b) *Assume that the condition in (a) is satisfied, and that moreover S has no diagonal or graph automorphism of order p . Then neither (i) nor (ii) in Theorem 3.1 holds.*

Proof. Let \mathbf{L} be a proper F -stable Levi subgroup of \mathbf{G} containing P . If \mathbf{L}^* denotes an F^* -stable Levi subgroup of \mathbf{G}^* dual to \mathbf{L} , then $|G^*| = |G|$ and $|\mathbf{L}^{*F^*}| = |\mathbf{L}^F|$, so \mathbf{L}^* contains a Sylow p -subgroup P^* of G^* . Since we may exchange the roles of \mathbf{G} , \mathbf{G}^* in this argument, we obtain (a).

Now assume that P is contained in the F -stable proper Levi subgroup \mathbf{L} . Then $\mathbf{T} := Z(\mathbf{L})$ is an F -stable torus of dimension at least 1 (see [15, Prop. 12.6]). Thus, for a coprime automorphism γ of S , induced by F_c , $\mathbf{T}^F \setminus \mathbf{T}^{F_c}$ is non-empty. Let $\kappa : G \rightarrow G/Z = S$ denote the canonical epimorphism. Since $\kappa(\mathbf{T}^F) \leq C_S(\kappa(P)) \leq N_S(\kappa(P))$, this shows that $N_S(\kappa(P))$ is not centralized by γ .

On the other hand, the non-trivial torus $\mathbf{T}^* := Z(\mathbf{L}^*)$ lies in $C_{\mathbf{G}^*}(P^*)$. It follows that there exists a semisimple $s \in \mathbf{T}^{*F^*} \setminus \mathbf{T}^{*F_c^*} \subseteq C_{\mathbf{G}^*}(P^*)$ which hence is not F_c^* -invariant. Then, by Proposition 3.5, the elements of $\mathcal{E}(G, s)$ are not γ -invariant. But $\mathcal{E}(G, s)$ contains characters of height 0 by Proposition 3.7 above.

Furthermore, by assumption the Sylow p -subgroup B of $\text{Out}(S)$ consists of field automorphisms, hence is cyclic. Thus there exists $F_b : \mathbf{G} \rightarrow \mathbf{G}$ such that $\beta := F_b|_G$ generates B modulo inner automorphisms. Then we may argue as before with semisimple elements in $\mathbf{T}^{F_b} \setminus \mathbf{T}^{\langle F_b, F_c \rangle}$, respectively $s \in \mathbf{T}^{*F_b^*} \setminus \mathbf{T}^{*\langle F_b^*, F_c^* \rangle}$ to see that neither (i) nor (ii) of Theorem 3.1 are satisfied. \square

3.5. Bad primes and torsion primes. It remains to deal with the primes p not satisfying the condition in Proposition 3.8(a). As announced before, these are small:

Proposition 3.9. *Let \mathbf{H} be a simple algebraic group in characteristic r with Steinberg endomorphism $F : \mathbf{H} \rightarrow \mathbf{H}$. Let $p \neq r$ be a prime and P a Sylow p -subgroup of $H := \mathbf{H}^F$.*

If P is not contained in any proper F -stable Levi subgroup of \mathbf{H} , then every semisimple element centralizing P is quasi-isolated, and in particular, p is a bad prime or torsion prime for \mathbf{H} .

Proof. Clearly we have that $P \neq 1$. Let $g \in C_H(P)$ be semisimple, so $P \leq C_{\mathbf{H}}(g)$. Now the connected group $C_{\mathbf{H}}^{\circ}(g)$ acts by conjugation on the set Ω of proper Levi subgroups of \mathbf{H} containing $C_{\mathbf{H}}(g)$, and any orbit therein is F -stable; hence by [15, Thm. 21.11], if Ω is non-empty then it contains an F -stable element \mathbf{L} , and then $P \leq C_{\mathbf{H}}(g) \leq \mathbf{L}$, which is not the case. So $C_{\mathbf{H}}(g)$ is not contained in any proper Levi subgroup of \mathbf{H} , whence, by definition, g is quasi-isolated in \mathbf{H} (see [15, Exmp. 14.4(2)]).

Apply this to $1 \neq g \in Z(P)$. If p is not a torsion prime for \mathbf{H} , the index $|C_{\mathbf{H}}(g) : C_{\mathbf{H}}^{\circ}(g)|$ is prime to p (see [15, Prop. 14.20]), so $P \leq C_{\mathbf{H}}^{\circ}(g)$. By assumption, this does not lie in any proper Levi subgroup of \mathbf{H} , so by definition g is isolated. Now by a result of Deriziotis (see [15, Rem. 14.5]) and the algorithm of Borel and de Siebenthal (see [15, Thm. 13.12]), this implies that all prime divisors of $o(g)$ are bad for \mathbf{H} . \square

We need the following two elementary results, where we write Φ_m for the m th cyclotomic polynomial:

Lemma 3.10. *Let $q \geq 1$, p a prime not dividing q , and e the multiplicative order of q modulo p . Then $p|\Phi_d(q)$ if and only if $d = ep^i$ for some $i \geq 0$.*

See e.g. [13, Lemma 5.2] for a proof.

Lemma 3.11. *Let $m, f \geq 1$. Then*

$$\Phi_m(X^f) = \prod_d \Phi_d(X)$$

where the product runs over all divisors d of mf which do not divide $m'f$ for any $m' < m$. In particular, when $\gcd(m, f) = 1$ then $\Phi_m(X^f) = \prod_{d|f} \Phi_{md}(X)$.

Proof. The roots of $\Phi_m(X)$ are the primitive m th roots of unity. Thus the roots of $\Phi_m(X^f)$ are mf th roots of unity whose f th power is a primitive m th root of unity, hence whose order is not a divisor of $m'f$, for any $m' < m$. \square

Proposition 3.12. *Let \mathbf{G} be connected reductive with Steinberg endomorphisms $F_c, F = F_c^f$ such that $\gcd(f, |\mathbf{G}^F|) = 1$, $\mathbf{H} \leq \mathbf{G}$ a connected reductive F_c -stable subgroup, p a prime dividing the order of the Weyl group of \mathbf{G} , $p \neq r$. Then \mathbf{H}^{F_c} contains a Sylow p -subgroup of \mathbf{H}^F .*

Proof. The group $H_0 = \mathbf{H}^{F_c}$ is a group of the same type as $H = \mathbf{H}^F$. Thus there are non-negative integers $a(d)$ such that $|H_0|_{r'} = \prod_d \Phi_d(q_0)^{a(d)}$ and $|H|_{r'} = \prod_d \Phi_d(q)^{a(d)}$, with $q_0^f = q$, and d divides the order of the Weyl group W of \mathbf{G} whenever $a(d) > 0$ (see [15, Cor. 24.6 and 24.7]). Since f is coprime to $|H|$, it's coprime to $p - 1$ by Remark 3.3, so q, q_0 have the same order e modulo p . Now, if p divides $\Phi_d(q) = \Phi_d(q_0^f)$, then $d = ep^i$ for some $i \geq 0$ by Lemma 3.10, and hence $\gcd(f, d) = 1$. Then the only factor of $\Phi_d(q_0^f)$ (in the factorization given by Lemma 3.11), which is divisible by p , is $\Phi_d(q_0)$ (again using Lemma 3.10). Thus the p -parts of $\Phi_d(q)$ and $\Phi_d(q_0)$ coincide, and hence $|H_0|_p = |H|_p$. \square

Corollary 3.13. *Let S be quasi-simple of Lie type as above, γ a coprime automorphism and p a prime dividing the order of the Weyl group of S . Then the fixed group $C_S(\gamma)$ contains a Sylow p -subgroup of S .*

Proof. Write $S = G/Z$ with $G = \mathbf{G}^F$, $Z \leq Z(G)$. Now γ is induced by some Steinberg endomorphism F_c with $F_c^f = F$, where $f = o(\gamma)$ is coprime to $|S|$. Then $|G_0|_p = |G|_p$ by the preceding result, with $G_0 = \mathbf{G}^{F_c}$. Since the subgroup $Z \leq Z(G)$ has order only divisible by prime divisors of the order of the Weyl group, it is fixed by F_c (again by Remark 3.3) and hence $G_0/(Z \cap G_0) = G_0/Z \leq C_S(\gamma)$, which shows that $|S|_p = |C_S(\gamma)|_p$. \square

After these preparations we can return to the proof of Theorem 3.1.

Proposition 3.14. *Let $S = G/Z$ be quasi-simple of Lie type, with $G = \mathbf{G}^F$ and $Z \leq Z(G)$ as above, $p \neq r$ a prime such that no proper F -stable Levi subgroup of \mathbf{G} contains a Sylow p -subgroup of G . Then (i) in Theorem 3.1 holds.*

Proof. Let $\chi \in \text{Irr}_p(S)$, so $\chi \in \mathcal{E}(G, s)$ for some semisimple element $s \in G^*$ centralizing a Sylow p -subgroup P^* of S^* , by Proposition 3.7. Now by Proposition 3.8, P^* is not contained in a proper F^* -stable Levi subgroup of G^* , so by Proposition 3.9, applied to $\mathbf{H} = \mathbf{G}^*$, we conclude that all semisimple elements in $C_{G^*}(P^*)$ are quasi-isolated, their order is only divisible by torsion primes and bad primes, so in particular only by divisors of the order of the Weyl group of \mathbf{G}^* . Let \mathbf{T}^* be an F^* -stable torus of \mathbf{G}^* containing s . Then Proposition 3.12 applied with $\mathbf{H} = \mathbf{T}^*$ shows that $s \in \mathbf{T}^{*F_c}$. But then χ is F_c -stable by Proposition 3.5. \square

For p odd let $e_p(q)$ denote the order of q modulo p , respectively the order of q modulo 4 when $p = 2$.

Proposition 3.15. *Let $S = G/Z$ be quasi-simple of Lie type, with $G = \mathbf{G}^F$ and $Z \leq Z(G)$ as above, $p \neq r$ a prime such that no proper F -stable Levi subgroup of \mathbf{G} contains a Sylow p -subgroup of G . Then (ii) in Theorem 3.1 holds.*

Proof. By Proposition 3.9 our assumptions imply that p is a torsion or bad prime for S . By [13, Thms. 5.14, 5.19, 8.4] the normalizer of a Sylow p -subgroup S_p of S is contained in the normalizer of a Φ_e -torus \mathbf{T}_e , where $e = e_p(q)$, unless $S = \text{SL}_3(q), \text{SU}_3(q), G_2(q), {}^2F_4(q^2)$ with $p = 3$ or $S = \text{Sp}_{2n}(q), {}^2G_2(q^2)$ with $p = 2$.

Now first assume that S is of exceptional type. Since a Sylow 2-subgroup of ${}^{(2)}E_6(q)$ (with q odd) is contained in a Levi subgroup of type ${}^{(2)}D_5(q)$, and a Sylow 3-subgroup of $E_7(q)$ (with $3 \nmid q$) is contained in a Levi subgroup of type $E_6(q)$ or ${}^2E_6(q)$, these situations do not arise here. In the remaining cases, which are collected in Table 1, the Sylow normalizers may easily be worked out explicitly inside $N_S(\mathbf{T}_e)$, respectively they are already given in [14, Sect. 3 and 4] (see also [11, Sect. 4] for the case $p = 2$). In particular they depend only on the integer a , which is the same for S and for the centralizer of any coprime automorphism by Corollary 3.13.

Now consider the case where S is of classical type. If S is not of type A , then $p = 2$ is the only torsion prime, and from the above mentioned result on $N_S(S_p)$ (or from [1, Thm. 4]) it follows that a Sylow 2-subgroup S_2 of S is self-normalizing, or an extension

TABLE 1. Normalizers for Sylow subgroups of exceptional groups not contained in proper Levi subgroups

| G | p | $N(S_p)$ |
|---------------------|-----|--|
| ${}^2G_2(3^{2f+1})$ | 2 | $2^3.7.3$ |
| $G_2(q)$ | 2 | $(2^a)^2.2^2$ |
| | 3 | $\begin{cases} (3^a)^2.W(G_2) & q \equiv 1, 8 \pmod{9} \\ 3^{1+2}.Q_8.2 & q \equiv 2, 4, 5, 7 \pmod{9} \end{cases}$ |
| ${}^3D_4(q)$ | 2 | $(2^a)^2.2^2$ |
| | 3 | $(3^a \times 3^{a+1}).W(G_2)$ |
| ${}^2F_4(2^{2f+1})$ | 3 | $\begin{cases} (3^a)^2.W(G_2) & 2^{2f+1} \equiv 8 \pmod{9} \\ 3^{1+2}.Q_8.2 & 2^{2f+1} \equiv 2, 5 \pmod{9} \end{cases}$ |
| | 2 | $(2^a)^4.S_2(W(F_4))$ |
| $F_4(q)$ | 2 | $(2^a)^4.S_2(W(F_4))$ |
| | 3 | $(3^a)^4.3^2.2^3$ |
| $E_6(q)$ | 3 | $\begin{cases} (3^a)^6.S_3(W(E_6)).2^2 & q \equiv 1 \pmod{3} \\ (3^a)^4.3^2.2^3 & q \equiv 2 \pmod{3} \end{cases}$ |
| | 3 | $\begin{cases} (3^a)^6.S_3(W(E_6)).2^2 & q \equiv 2 \pmod{3} \\ (3^a)^4.3^2.2^3 & q \equiv 1 \pmod{3} \end{cases}$ |
| $E_7(q)$ | 2 | $(2^a)^7.S_2(W(E_7))$ |
| $E_8(q)$ | 2 | $(2^a)^8.S_2(W(E_8))$ |
| | 3 | $(3^a)^8.S_3(W(E_8)).2^4$ |
| | 5 | $\begin{cases} (5^a)^8.S_5(W(E_8)).[2^6] & q \equiv \pm 1 \pmod{5} \\ (5^a)^4.5.4^2 & q \equiv \pm 2 \pmod{5} \end{cases}$ |

Here, $p^a = |\Phi_e(q)|_p$ with $e = e_p(q)$, respectively $3^a = |\Phi_4(2^{2f+1})|_3$ for the Ree groups, and $S_p(H)$ denotes a Sylow p -subgroup of H . Moreover, p^a stands for a cyclic group of that order, $[p^k]$ for an unspecified group of order p^k , H^k for a direct product of k copies of H .

of S_2 of degree 3^t where t is the number of summands in the 2-adic expansion of n for $S = \text{Sp}_{2n}(q)$ with $q \equiv \pm 3 \pmod{8}$. In particular, it's the same in S and in $C_S(\gamma)$.

Finally, when S is linear the possible torsion primes are divisors of $\gcd(n, q-1)$. For $\text{SL}_3(q)$ with $p=3$ and $q \equiv 4, 7 \pmod{9}$ the Sylow p -normalizers is isomorphic to $3^{1+2}.Q_8$, independently of q (see [14, Sect. 3.1]), whence the claim holds. Exclude this case. Let $n = \sum_i p^{k_i}$ be the p -adic expansion of n (thus, any p^{k_i} occurs at most $p-1$ times). If this has at least two summands, then a proper Levi subgroup $\text{GL}_{n_1}(q) \circ \text{GL}_{n_2}(q)$, with $n_1 = p^{k_1}$, $n_2 = n - n_1$, contains a Sylow p -subgroup of S . Thus we may assume that $n = p^k$ is a p -power. For $p=2$ the Sylow 2-subgroups are self-normalizing by [1, Thm. 4]. For odd p an easy matrix calculation shows that the normalizer of S_p (modulo the center) is an extension of a homocyclic group $(p^a)^{n-1}$ by the normalizer in \mathfrak{S}_n of one of its Sylow p -subgroups, where $p^a = |q-1|_p$. So again, it is the same in S as in the centralizer of any coprime automorphism. The case of unitary groups is entirely similar. \square

The proof of Theorem 3.1 is now complete by Propositions 3.6, 3.8, 3.14 and 3.15. In the course of the proof we have established the following: if S is of Lie type and p not the defining characteristic, then (i) and (ii) in Theorem 3.1 are equivalent to

(iii) a Sylow p -subgroup of S is not contained in any proper Levi subgroup of S .

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FB MATHEMATIK, TU KAISERSLAUTERN, POSTFACH 3049, 67653 KAISERSLAUTERN, GERMANY.
E-mail address: malle@mathematik.uni-kl.de

DEPARTAMENT D’ÀLGEBRA, UNIVERSITAT DE VALÈNCIA, DR. MOLINER 50, 46100 BURJASSOT, SPAIN.

E-mail address: gabriel.navarro@uv.es