OKLAHOMA STATE UNIVERSITY
Department of Mathematics

MATH 2144 (Calculus I)
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MIDTERM 1
September 17, 2008

Duration: 50 minutes

No aids allowed.

This examination paper consists of 7 pages and 6 questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth.

Answer 5 of 6 questions.

To obtain credit, you must give arguments to support your answers.

For graders’ use:

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1. [10] True or False? Write a “T” (for true) or an “F” (for false) for each statement.

(a) \( \lim_{x \to 4} \left( \frac{2x}{x-4} - \frac{8}{x-4} \right) = \lim_{x \to 4} \frac{2x}{x-4} - \lim_{x \to 4} \frac{8}{x-4} \)

(b) If \( p \) is a polynomial, then \( \lim_{x \to 1} p(x) = p(1) \).

(c) If \( \lim_{x \to a} [f(x)g(x)] \) exists, then it must be equal to \( f(a)g(a) \).

(d) If \( \lim_{x \to a} f(x) = \infty \) and \( \lim_{x \to a} g(x) = -\infty \), then \( \lim_{x \to a} [f(x) + g(x)] = 0 \).

(e) If \( x = 1 \) is a vertical asymptote of \( y = f(x) \) then \( f \) is not defined at \( 1 \).

(f) If \( f \) is continuous at \( a \), then \( f \) is differentiable at \( a \).

(g) If \( f(x) > 1 \) for all \( x > 0 \) and \( \lim_{x \to 0^+} f(x) \) exists, then \( \lim_{x \to 0^+} f(x) > 1 \).

(h) If \( f'(r) \) exists, then \( \lim_{x \to r} f(x) = f(r) \).

(i) The equation \( x^{10} - 10x^2 + 5 = 0 \) has a root in the interval \((0, 2)\).

(j) A rational function can have two different horizontal asymptotes.

Solution: (a) F (limit law does not apply to infinite limits)

(b) T (polynomials are continuous)

(c) F (example: \( f(x) = x, g(x) = x^{-1}, a = 0 \))

(d) F (example: \( f(x) = x^{-2}, g(x) = -x^{-4}, a = 0 \))

(e) F (limits in the definition of the vertical asymptote “ignore” \( f(1) \))

(f) F (example: \( f(x) = |x| \))

(g) F (example: \( f(x) = x + 1 \))

(h) T (differentiable implies continous)

(i) T (apply Intermediate Value Theorem to \([0, 1]\))
2. [10] Give a simple example or write “N/A” if there is no such example.

(a) A polynomial that is a power function.
(b) A polynomial that is not a rational function.
(c) A rational function that is not a polynomial.
(d) An inverse trigonometric function that is not an algebraic function.
(e) A continuous function without horizontal and vertical asymptotes.
(f) A function with 3 vertical asymptotes.
(g) A root function whose domain does not include $-1$.
(h) An algebraic function with domain $(-1, 1)$.
(i) A function with infinitely many discontinuities.
(j) A continuous but not differentiable function.

Solution:

(a) $f(x) = 1 \ (= x^0)$
(b) N/A (by definition any polynomial is a rational function)
(c) $f(x) = \frac{1}{x}$
(d) $\sin^{-1}$ (essentially any inverse trigonometric function works)
(e) $f(x) = x$
(f) $f(x) = \frac{1}{x^3-x} \ (= \frac{1}{(x-(-1))(x-0)(x-1)})$
(g) $f(x) = \sqrt{x}$
(h) $f(x) = 1/\sqrt{1-x^2}$
(i) $f(x) = \lfloor x \rfloor$ (discontinuous at all integers)
(j) $f(x) = |x|$
3. [10]

(a) For \( f(x) = \frac{2x^2 - 18}{x^2 + 2x - 3} \), find all asymptotes and the limits that describe the asymptotic behavior of the function.

(b) Find the horizontal asymptotes of the function \( f(x) = \frac{\sqrt{x^6 - 1}}{x^3 + 7x^2 + 4x - 8} \).

**Solution:** (a) Dropping the terms with not highest exponents in the numerator and denominator of \( f \) (as explained in the lecture) yields \( y = \frac{2x^2}{2x} = 2 \) as horizontal asymptote for both \( x \to \infty \) and \( x \to -\infty \). So we have \( \lim_{x \to \pm \infty} f(x) = 2 \).

To find the vertical asymptotes, we factorize and cancel factors if possible:

\[
\frac{2x^2 - 18}{x^2 + 2x - 3} = \frac{2(x + 3)(x - 3)}{(x + 3)(x - 1)} = \frac{2x - 3}{x - 1}
\]

So, \( x = 1 \) is the only vertical asymptote. As \( x - 3 < 0 \) for \( x \) close to 1, we have \( \lim_{x \to -1^-} = \infty \) and \( \lim_{x \to -1^+} = -\infty \).

(b) Dropping the terms with not highest exponents under the root and in the denominator of \( f \) (as above) yields \( \sqrt{\frac{x^6}{x^3}} = |x|^3 = |x|/x \) which has the same asymptotic behavior as \( f(x) \) for large \( |x| \). So \( \lim_{x \to \infty} f(x) = \lim_{x \to -\infty} |x|/x = 1 \) and similarly \( \lim_{x \to -\infty} f(x) = -1 \). In other words, \( y = 1 \) and \( y = -1 \) are two (different) horizontal asymptotes.
4. [10]

(a) Find all values for $a$ and $b$ such that the function

$$f(x) = \begin{cases} \frac{x^2 - 9}{x - 3} & \text{if } x < 1 \\ (x - a)^2 & \text{if } 1 \leq x < 2 \\ 2ax - b & \text{if } 2 \leq x \end{cases}$$

becomes continuous.

(b) Is $f$ differentiable for some choice of $a$ and $b$?

Solution:

(a) First, note that $f(x) = x + 3$ for $x < 1$. Continuity is clear at $x \neq 1, 2$. The following two conditions are equivalent to continuity at 1 and 2 respectively:

$$4 = \lim_{x \to 1^-} f(x) = f(1) = (1 - a)^2,$$

$$(2 - a)^2 = \lim_{x \to 2^-} f(x) = f(2) = 4a - b.$$  

The first equality gives $a = 1 \mp 2$, so $a = -1$ or $a = 3$. Then the second equality reads $5 \pm 4 = (1 \pm 2)^2 = 4 \mp 8 - b$ which gives $b = -1 \mp 12$. So either $a = -1$ and $b = -13$ or $a = 3$ and $b = 11$.

(b) Note that if $f$ is continuous, then $f(1) = 4$ by the first part. For $f'(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h}$ to exist, the corresponding left- and right-sided limits

$$\lim_{h \to 0^-} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^-} \frac{1 + h + 3 - 4}{h} = 1,$$

$$\lim_{h \to 0^+} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^+} \frac{(1 + h - a)^2 - 4}{h}$$

must be equal. But, for $a = 1 \mp 2$, the right-sided limit equals

$$\lim_{h \to 0^+} \frac{(h \pm 2)^2 - 4}{h} = \lim_{h \to 0^+} \frac{h^2 \pm 4h}{h} = \pm 4$$

which is not equal to the left-sided limit. So the answer is “no”.

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5. [10]

(a) Compute the derivative of \( f(x) = \frac{1-x}{1+x} \) (using the limit definition).

(b) Find the domains of \( f(x) \) and \( f'(x) \).

**Solution:**

(a) 

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
= \lim_{h \to 0} \frac{(1-x-h)(1+x) - (1-x)(1+x+h)}{h(1+x+h)(1+x)} \\
= \lim_{h \to 0} \frac{1-x-h+x-x^2-hx-1-x-h+x+x^2+hx}{h(1+x+h)(1+x)} \\
= \lim_{h \to 0} \frac{-2h}{h(1+x+h)(1+x)} = -\frac{2}{(1+x)^2}
\]

(b) Both domains are obviously \( \mathbb{R} \setminus \{-1\} \).

(a) \( \lim_{x \to -\infty} \frac{\sqrt{4x^6} - x}{x^3 + 9} \)

(b) \( \lim_{t \to \infty} \frac{t^2 - t}{2t^2 + t + 7} \)

(c) \( \lim_{x \to 0} \left( x^4 \cos \frac{x}{2} \right) \) (Hint: use the Squeeze Theorem)

(d) \( \lim_{x \to \frac{\pi}{8}} \arctan \left( \frac{64x^2 - \pi^2}{64x - 8\pi} \right) \)

(e) \( \lim_{x \to \pi} \sin(x + \sin(x + \sin(x + \sin(x + \sin x)))) \)

Solution:

(a) Note that for \( x < 0 \), we have \( x^{-3} = -\sqrt{x^{-6}} \). Using this, we compute

\[
\lim_{x \to -\infty} \frac{\sqrt{4x^6} - x}{x^3 + 9} = \lim_{x \to -\infty} -\frac{\sqrt{4x^6} - x\sqrt{x^{-6}}}{(x^3 + 9)x^{-3}} = \lim_{x \to -\infty} -\frac{\sqrt{\frac{1}{x^{-6}}}}{1 + \frac{9}{x^3}} = -2.
\]

(b) \( \lim_{t \to \infty} \frac{t^2 - t}{2t^2 + t + 7} = \lim_{t \to \infty} \frac{t^2}{2t^2} = \frac{1}{2} \)

(c) We have \(-x^4 \leq x^4 \cos \frac{x}{2} \leq x^4 \) and \( \lim_{x \to 0} x^4 = 0 \). So, by the Squeeze Theorem, it follows that also \( \lim_{x \to 0} \left( x^4 \cos \frac{x}{2} \right) = 0 \).

(d) By Theorem 8 in Section 2.5, \( \lim_{x \to \frac{\pi}{8}} \arctan \left( \frac{64x^2 - \pi^2}{64x - 8\pi} \right) = \arctan \left( \lim_{x \to \frac{\pi}{8}} \frac{64x^2 - \pi^2}{64x - 8\pi} \right) \).

But \( \lim_{x \to \frac{\pi}{8}} \frac{64x^2 - \pi^2}{64x - 8\pi} = \lim_{x \to \frac{\pi}{8}} \left( x + \frac{\pi}{8} \right) = \frac{\pi}{4} \) and hence the result is \( \arctan \frac{\pi}{4} = 1 \).

(e) By continuity, \( \lim_{x \to \pi} \sin(x + \sin(x + \sin(x + \sin(x + \sin x)))) = \sin(\pi + \sin(\pi + \sin(\pi + \sin(\pi + \sin \pi))) = 0. \)

End of examination

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