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Algebraic Geometry

Summer Semester 2013 - Problem Set 1

Due April 26, 2013, 1:00 pm

In all exercises, the ground field k is assumed to be algebraically closed.

Problem 1. Let $X_1, X_2 \subset \mathbb{A}^n$ be algebraic sets and $X \subset \mathbb{A}^n$ any subset. Show that

(a) $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$,

(b) $Z(I(X)) = \overline{X}$,

(c) $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$. Show by example that the radical cannot be omitted, i.e. find algebraic sets X_1, X_2 such that $I(X_1 \cap X_2) \neq I(X_1) + I(X_2)$. Find a geometric reason for this inequality.

Problem 2.

(a) Let $X \subset \mathbb{A}^3$ be the union of the three coordinate axes. Determine generators for the ideal $I(X)$. Show that $I(X)$ cannot be generated by fewer than 3 elements and that X has dimension 1 in $\mathbb{A}^3$.

(b) Let $X = \{(t, t^3, t^5) \mid t \in k\} \subset \mathbb{A}^3$. Show that X is an affine variety of dimension 1 and compute $I(X)$.

Problem 3.

(a) Let Y be a subspace of a topological space X. Show that Y is irreducible if and only if the closure of Y in X is irreducible.

(b) If we identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way, show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1$.

Problem 4. Let $X \subset \mathbb{A}^2$ be an irreducible algebraic set. Show that either

- $X = Z(0)$, i.e. X is the whole space $\mathbb{A}^2$, or
- $X = Z(f)$ for some irreducible polynomial f in $k[x, y]$, or
- $X = Z(x - a, y - b)$ for some $a, b \in k$, i.e. X is a single point.

Deduce that $\dim(\mathbb{A}^2) = 2$. (Hint: Show that the common zero locus of two polynomials $f, g \in k[x, y]$ without common factor is finite using e.g. the Gauss Lemma or resultants.)