TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

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Algebraic Geometry

Summer Semester 2013 - Problem Set 12

Due July 12, 2013, 1:00 pm

Problem 1. Let k be an algebraically closed field. An n-fold point (over k) is a scheme of the form $X = \operatorname{Spec} R$ such that X has only one point and R is a k-algebra of vector space dimension n over k (i. e. X has length n). Show that every double point is isomorphic to $\operatorname{Spec} k[x]/\langle x^2 \rangle$. On the other hand, find two non-isomorphic triple points over k, and describe them geometrically.

Problem 2. Show that for a scheme X the following are equivalent:

(a) X is reduced, i. e. for every open subset $U \subset X$ the ring $\mathscr{O}_X(U)$ has no nilpotent elements.

(b) For any open subset U_i of an open affine cover $\{U_i\}$ of X, the ring $\mathscr{O}_X(U_i)$ has no nilpotent elements.

(c) For every point $P \in X$ the local ring $\mathscr{O}_{X,P}$ has no nilpotent elements.

Problem 3. Show that $\mathbb{A}^2_{\mathbb{C}} \ncong \mathbb{A}^1_{\mathbb{C}} \times_{\operatorname{Spec} \mathbb{Z}} \mathbb{A}^1_{\mathbb{C}}$.

Problem 4. For a variety X we also call the associated scheme X_{Sch} a variety. Let X be an affine variety, let Y be a closed subscheme of X defined by the ideal $I \subset A(X)$, and let $\widetilde{X}$ be the blow-up of X at I. Show that:

(a) $\widetilde{X} = \operatorname{Proj}(\bigoplus_{d \geq 0} I^d)$, where we set $I^0 := A(X)$.

(b) The projection map $\widetilde{X} \to X$ is the morphism induced by the ring homomorphism $I^0 \to \bigoplus_{d \geq 0} I^d$.

(c) The exceptional divisor of the blow-up, i. e. the fiber $Y \times_X \widetilde{X}$ of the blow-up $\widetilde{X} \to X$ over Y is isomorphic to $\operatorname{Proj}(\bigoplus_{d \geq 0} I^d / I^{d+1})$.