TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

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Algebraic Geometry

Summer Semester 2013 - Problem Set 2

Due May 3, 2013, 1:00 pm

In all exercises, the ground field k is assumed to be algebraically closed.

Problem 1. An algebraic set $X \subset \mathbb{A}^2$ defined by a polynomial of degree 2 is called a conic.

(a) Show that any irreducible conic is isomorphic either to $Z(y - x^2)$ or to $Z(xy - 1)$. (*Hint: Use linear changes of coordinates.*)

(b) Let $X, Y \subset \mathbb{A}^2$ be irreducible conics and assume that $X \neq Y$. Show that X and Y intersect in at most 4 points. For all $n \in \{0, 1, 2, 3, 4\}$, find an example of two conics that intersect in exactly n points.

Problem 2. Which of the following algebraic sets are isomorphic over the complex numbers?

(a) $\mathbb{A}^1$

(b) $Z(x^2 + y^2) \subset \mathbb{A}^2$

(c) $Z(x^2 - y^3) \subset \mathbb{A}^2$

(d) $Z(xy) \subset \mathbb{A}^2$

(e) $Z(y^2 - x^3 - x^2) \subset \mathbb{A}^2$

(f) $Z(y - x^2, z - x^3) \subset \mathbb{A}^3$

Problem 3. Are the following statements true or false: if $f : \mathbb{A}^n \to \mathbb{A}^m$ is a polynomial map (i.e. $f(P) = (f_1(P), \dots, f_m(P))$ with $f_i \in k[x_1, \dots, x_n]$), and ...

(a) $X \subset \mathbb{A}^n$ is an algebraic set, then the image $f(X) \subset \mathbb{A}^m$ is an algebraic set.

(b) $X \subset \mathbb{A}^m$ is an algebraic set, then the inverse image $f^{-1}(X) \subset \mathbb{A}^n$ is an algebraic set.

(c) $X \subset \mathbb{A}^n$ is an algebraic set, then the graph $\Gamma = \{(x, f(x)) \mid x \in X\} \subset \mathbb{A}^{n+m}$ is an algebraic set.

Problem 4. Let $f : X \to Y$ be a morphism between affine varieties, and let $f^* : A(Y) \to A(X)$ be the corresponding map of k-algebras. Which of the following statements are true?

(a) If $P \in X$ and $Q \in Y$, then $f(P) = Q$ if and only if $(f^*)^{-1}(I(P)) = I(Q)$.

(b) f^* is injective if and only f is surjective.

(c) f^* is surjective if and only f is injective.

(d) f^* is an isomorphism if and only f is an isomorphism.

If a statement is false, is there a weaker form of it which is true?