Technische Universität Kaiserslautern

Prof. Dr. Mathias Schulze
Dipl.-Math. Cornelia Rottner
Fachbereich Mathematik

Algebraic Geometry

Summer Semester 2013 - Problem Set 3

Due May 10, 2013, 1:00 pm

In all exercises, the ground field k is assumed to be algebraically closed.

Problem 1. Let X be a prevariety. Show that:

(a) X is a Noetherian topological space,

(b) any open subset of X is a prevariety.

Problem 2.

(a) Show that every isomorphism $f : \mathbb{A}^1 \to \mathbb{A}^1$ is of the form $f(x) = ax + b$ for some $a, b \in k$, where x is the coordinate on $\mathbb{A}^1$.

(b) Show that every isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ is of the form $f(x) = \frac{ax+b}{cx+d}$ for some $a, b, c, d \in k$, where x is the affine coordinate on $\mathbb{A}^1 \subset \mathbb{P}^1$.

(c) Given three distinct points $P_1, P_2, P_3 \in \mathbb{P}^1$ and three distinct points $Q_1, Q_2, Q_3 \in \mathbb{P}^1$, show that there is a unique isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $f(P_i) = Q_i$ for $i = 1, 2, 3$.

Problem 3. Let X and Y be prevarieties with affine open covers $\{U_i\}$ and $\{V_j\}$, respectively. Construct the product prevariety $X \times Y$ by glueing the affine varieties $U_i \times V_j$ together. Moreover, show that there are projection morphisms $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ satisfying the usual universal property for morphisms: given any two morphisms $f : Z \to X$ and $g : Z \to Y$ of prevarieties from any prevariety Z, there is a unique morphism $h : Z \to X \times Y$ such that $f = \pi_X \circ h$ and $g = \pi_Y \circ h$.

Problem 4. Let $(X, \mathscr{O}_X)$ be a prevariety, and $Y \subset X$ be an irreducible closed subset. For every open subset $U \subset Y$ define $\mathscr{O}_Y(U)$ to be the ring of k-valued functions f on U with the following property: for every point $P \in U$ there is a neighborhood V of P in X and a section $F \in \mathscr{O}_X(V)$ such that f coincides with F on $U \cap V$.

(a) Show that the rings $\mathscr{O}_Y(U)$ together with the obvious restriction maps define a sheaf $\mathscr{O}_Y$ on Y.

(b) Show that $(Y, \mathscr{O}_Y)$ is a prevariety.