Technische Universität Kaiserslautern

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Algebraic Geometry

Summer Semester 2013 - Problem Set 4

Due May 17, 2013, 1:00 pm

In all exercises, the ground field k is assumed to be algebraically closed.

Problem 1. Let $(X, \mathscr{O}_X)$ be a prevariety. Consider pairs (U, f) where U is an open subset of X and $f \in \mathscr{O}_X(U)$ a regular function on U. We call two such pairs (U, f) and (U', f') equivalent if there is an open subset V in X with $V \subset U \cap U'$ such that $f|_V = f'|_V$.

(a) Show that this defines an equivalence relation.

(b) Show that the set of all such pairs modulo this equivalence relation is a field. It is called the *field of rational functions* on X and denoted $K(X)$.

(c) If $X \subset \mathbb{A}^m$ is an affine variety, show that $K(X)$ is just the field of rational functions as defined in definition 2.1.3.

(d) If $U \subset X$ is any non-empty open subset, show that $K(U) = K(X)$.

Problem 2. Show that the prevariety $\mathbb{P}^1$ is a variety.

Problem 3. Let L_1 and L_2 be two disjoint lines in $\mathbb{P}^3$, and let $P \in \mathbb{P}^3 \backslash (L_1 \cup L_2)$ be a point. Show that there is a unique line $L \subset \mathbb{P}^3$ meeting L_1, L_2, and P (i. e. such that $P \in L$ and $L \cap L_i \neq \emptyset$ for $i = 1, 2$).

Problem 4. Let $I \subset k[x_1, \ldots, x_n]$ be an ideal. Define I^h to be the ideal generated by $\{f^h \mid f \in I\} \subset k[x_0, \ldots, x_n]$, where

$$f^h(x_0, \ldots, x_n) := x_0^{\deg(f)} \cdot f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$$

denotes the homogenization of f with respect to x_0. Show that:

(a) I^h is a homogeneous ideal that can be generated by finitely many homogeneous polynomials.

(b) $Z(I^h) \subset \mathbb{P}^n$ is the closure of $Z(I) \subset \mathbb{A}^n$ in $\mathbb{P}^n$. We call $Z(I^h)$ the *projective closure* of $Z(I)$.

(c) Let $I = (f_1, \ldots, f_k)$. Show by an example that $I^h \neq (f_1^h, \ldots, f_k^h)$ in general.