TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

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Algebraic Geometry

Summer Semester 2013 - Problem Set 5

Due May 24, 2013, 1:00 pm

In all exercises, the ground field k is assumed to be algebraically closed.

Problem 1. Let $X \subset \mathbb{P}^n$ be a non-empty projective algebraic set. Show that $I(X) \subset k[x_0, \dots, x_n]$ is prime if and only if X is irreducible.

Problem 2. Let $C \subset \mathbb{P}^3$ be the "twisted cubic curve" given by the parametrization

$$\mathbb{P}^1 \to \mathbb{P}^3, (s,t) \mapsto (x : y : z : w) = (s^3 : s^2t : st^2 : t^3).$$

Let $P = (0 : 0 : 1 : 0) \in \mathbb{P}^3$, and let H be the hyperplane defined by $z = 0$. Let φ be the projection from P to H, i.e. the map associating to a point Q of C the intersection point of the unique line through P and Q with H.

(a) Show that φ is a morphism.

(b) Determine the equation of the curve $\varphi(C)$ in $H \cong \mathbb{P}^2$.

(c) Is $\varphi : C \to \varphi(C)$ an isomorphism onto its image?

Problem 3. In this exercise we will make the space of all lines in $\mathbb{P}^n$ into a projective variety. Fix $n \geq 1$. We define a set-theoretic map

$$\varphi : \{\text{lines in } \mathbb{P}^n\} \to \mathbb{P}^N$$

with $N = \binom{n+1}{2} - 1$ as follows. For every line $L \subset \mathbb{P}^n$ choose two distinct points $P = (a_0 : \dots : a_n)$ and $Q = (b_0 : \dots : b_n)$ on L and define $\varphi(L)$ to be the point in $\mathbb{P}^N$ whose homogeneous coordinates are the $\binom{n+1}{2}$ maximal minors of the matrix

$$\begin{pmatrix} a_0 & a_1 & \dots & a_n \\ b_0 & a_1 & \dots & b_n \end{pmatrix}$$

in any fixed order. Show that:

(a) The map φ is well-defined and injective.

(b) The image of φ is a projective variety that has a finite open cover by affine spaces $\mathbb{A}^{2(n-1)}$. It is called the *Grassmannian* $G(1,n)$. Hint: recall that by the Gaussian algorithm most matrices (what does this mean?) are equivalent to one of the form

$$\begin{pmatrix} 1 & 0 & a'_2 & \dots & a'_n \\ 0 & 1 & b'_2 & \dots & b'_n \end{pmatrix}$$

for some a'_i, b'_i.

(c) $G(1,1)$ is a point, $G(1,2) \cong \mathbb{P}^2$, and $G(1,3)$ is the zero locus of a quadratic equation in $\mathbb{P}^5$.

Problem 4. Let V be the vector space over k of homogeneous degree-2 polynomials in three variables x_0, x_1, x_2, and let $\mathbb{P}(V) \cong \mathbb{P}^5$ be its projectivization.

(a) Show that the space of conics in $\mathbb{P}^2$ can be identified with an open subset U of $\mathbb{P}^5$. (One says that U is a "moduli space" for conics in $\mathbb{P}^2$ and that $\mathbb{P}^5$ is a "compactified moduli space".) What geometric objects can be associated to the points in $\mathbb{P}^5 \backslash U$?

(b) Show that it is a linear condition in $\mathbb{P}^5$ for the conics to pass through a given point in $\mathbb{P}^2$. More precisely, if $P \in \mathbb{P}^2$ is a point, show that there is a linear subspace $L \subset \mathbb{P}^5$ such that the conics passing through P are exactly those in $U \cap L$. What happens in $\mathbb{P}^5 \backslash U$, i. e. what do the points in $(\mathbb{P}^5 \backslash U) \cap L$ correspond to?

(c) Prove that there is a unique conic through any five given points in $\mathbb{P}^2$, as long as no three of them lie on a line. What happens if three of them do lie on a line?