Technische Universität Kaiserslautern

Prof. Dr. Mathias Schulze
Dipl.-Math. Cornelia Rottner
Fachbereich Mathematik

Algebraic Geometry

Summer Semester 2013 - Problem Set 6

Due May 31, 2013, 1:00 pm

Problem 1. Let $G(1,n)$ be the Grassmannian of lines in $\mathbb{P}^n$ as in Problem 3 on Problem Set 5. Show that:

(a) The set $\{(L,P); P \in L\} \subset G(1,n) \times \mathbb{P}^n$ is closed.

(b) If $Z \subset G(1,n)$ is any closed subset then the union of all lines $L \subset \mathbb{P}^n$ such that $L \in Z$ is closed in $\mathbb{P}^n$.

(c) Let $X, Y \subset \mathbb{P}^n$ be disjoint projective varieties. Then the union of all lines in $\mathbb{P}^n$ intersecting X and Y is a closed subset of $\mathbb{P}^n$. It is called the *join* $J(X,Y)$ of X and Y.

Problem 2. Let $X, Y \subset \mathbb{P}^n$ be projective varieties. Show that $X \cap Y$ is not empty if $\dim X + \dim Y \geq n$.

On the other hand, give an example of a projective variety Z and closed subsets $X, Y \subset Z$ with $\dim X + \dim Y \geq \dim Z$ and $X \cap Y = \emptyset$.

(Hint: Let H_1, H_2 be two disjoint linear subspaces of dimension n in $\mathbb{P}^{2n+1}$, and consider $X \subset \mathbb{P}^n \cong H_1 \subset \mathbb{P}^{2n+1}$ and $Y \subset \mathbb{P}^n \cong H_2 \subset \mathbb{P}^{2n+1}$ as subvarieties of $\mathbb{P}^{2n+1}$. Show that the join $J(X,Y) \subset \mathbb{P}^{2n+1}$ has dimension $\dim X + \dim Y + 1$. Then construct $X \cap Y$ as a suitable intersection of $J(X,Y)$ with $n+1$ hyperplanes.)

Problem 3. Let $C_1, C_2 \subset \mathbb{P}^4$ be two conic curves given by

$$C_1 : x_0 = x_1 = x_2^2 - x_3x_4 = 0 \text{ and } C_2 : x_3 = x_4 = x_2^2 - x_0x_1 = 0.$$

Show that $J(C_1, C_2)$ is a quartic hypersurface.