TECHNISCHE UNIVERSITÄT
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Algebraic Geometry

Summer Semester 2013 - Problem Set 7

Due June 7, 2013, 1:00 pm

Problem 1. A quadric in $\mathbb{P}^n$ is a projective variety in $\mathbb{P}^n$ that can be given as the zero locus of a quadratic polynomial. Show that every quadric in $\mathbb{P}^n$ is birational to $\mathbb{P}^{n-1}$.

Problem 2. Let $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$, $P_3 = (0 : 0 : 1) \in \mathbb{P}^2$, and let $U = \mathbb{P}^2 \backslash \{P_1, P_2, P_3\}$. Consider the morphism

$$f : U \to \mathbb{P}^2, (a_0, a_1, a_2) \mapsto (a_1a_2 : a_0a_2 : a_0a_1)$$

(a) Show that there is no morphism $F : \mathbb{P}^2 \to \mathbb{P}^2$ extending f.

(b) Let $\tilde{\mathbb{P}}^2$ be the blow-up of $\mathbb{P}^2$ in the three points P_1, P_2, P_3. Show that there is an isomorphism $\tilde{f} : \tilde{\mathbb{P}}^2 \to \tilde{\mathbb{P}}^2$ extending f. This is called the *Cremona transformation*.

Problem 3. Let $X \subset \mathbb{A}^n$ be an affine variety. For every $f \in k[x_1, \ldots, x_n]$ denote by f^{in} the initial terms of f, i. e. the terms of f of the lowest occurring degree. Let $I(X)^{in} = \{f^{in} \mid f \in I(X)\}$ be the ideal of the initial terms in $I(X)$. Now let $\pi : \tilde{X} \to X$ be the blow-up of X in the origin $\{0\} = Z(x_1, \ldots, x_n)$. Show that the exceptional hypersurface $\pi^{-1}(0) \subset \mathbb{P}^n$ is precisely the projective zero locus of the homogeneous ideal $I(X)^{in}$.

Problem 4. Let $X \subset \mathbb{A}^n$ be an affine variety, and let $Y_1, Y_2 \subsetneq X$ be irreducible, closed subsets, no-one contained in the other. Let $\tilde{X}$ be the blow-up of X at the (possibly non-radical) ideal $I(Y_1) + I(Y_2)$. Then the strict transforms of Y_1 and Y_2 on $\tilde{X}$ are disjoint.