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Algebraic Geometry

Summer Semester 2013 - Problem Set 8

Due June 14, 2013, 1:00 pm

In all exercises, the ground field k is assumed to be algebraically closed.

Problem 1. Let $f : X \to Y$ be a morphism of projective varieties, and let $Z \subset X$ be a closed subset. Assume that $f^{-1}(P) \cap Z$ is irreducible and of the same dimension for all $P \in Y$. Use Theorem 4.2.7$^+$ to prove that then Z is irreducible too. (This is a quite useful criterion to check the irreducibility of closed subsets.)

Show by example that the conclusion is in general false if the $f^{-1}(P) \cap Z$ are irreducible but not all of the same dimension.

Problem 2. Let $X \subset \mathbb{A}^n$ be an affine variety, and let $P \in X$ be a point. Show that the coordinate ring $A(C_{X,P}) := k[x_1, \dots, x_n]/I(X)^{in}$ of the tangent cone to X at P is equal to $\bigoplus_{k \geq 0} I(P)^k / I(P)^{k+1}$, where $I(P)$ is the ideal of P in $A(X)$.

Problem 3. Let $X \subset \mathbb{A}^n$ be an affine variety and $P \in X$ a point (you may assume that $P = (0, \dots, 0) \in \mathbb{A}^n$). Show that the affine cone $C_{X,P}$ is pure-dimensional of dimension $\dim X$. Hint: Use that the projective cone of X at P is a hypersurface in the blow-up $\tilde{X}$ of X in P.

Problem 4. Let $C \subset \mathbb{P}^2$ be a smooth curve, given as the zero locus of a homogeneous polynomial $f \in k[x_0, x_1, x_2]$. Consider the morphism

$$\varphi_C : C \to \mathbb{P}^2, P \mapsto \left(\frac{\partial f}{\partial x_0}(P) : \frac{\partial f}{\partial x_1}(P) : \frac{\partial f}{\partial x_2}(P) \right).$$

The image $\varphi_C(C) \subset \mathbb{P}^2$ is called the *dual curve* to C.

(a) Find a geometric description of φ_C. What does it mean geometrically if $\varphi_C(P) = \varphi_C(Q)$ for two distinct points $P, Q \in C$?

(b) If C is a conic (i.e. f is an irreducible homogeneous polynomial of degree 2), prove that its dual $\varphi_C(C)$ is also a conic.

(c) Show that for any five lines in $\mathbb{P}^2$ in general position there is a unique conic in $\mathbb{P}^2$ that is tangent to these five lines, i.e. show that there is a non-empty open subset $U \subset \times_{i=1}^5 G(1,2)$ such that for any $(l_1, \dots, l_5) \in U$ there exists a unique conic in $\mathbb{P}^2$ tangent to the five lines corresponding to $l_1, \dots, l_5$. (Hint: Use Problem 4 on Problem Set 5.)