Existence and Bounds for the Half Moment Entropy Approximation to Radiative Transfer*

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Abstract

We establish existence, uniqueness and a priori bounds for the half moment entropy approximation to radiative heat transfer. The bounds are physically reasonable and underline the approximation properties of the system. We show numerical results to illustrate the bounds.

Keywords. Radiative transfer equations; moment methods; half fluxes; a priori estimates

1 Introduction and Main Result

Radiative transfer plays a prominent role in many applications in physics and engineering. Especially when high temperatures are involved the effect of radiation becomes very important. Applications include astrophysics [16] (stellar atmospheres), reentry of space vehicles [19], combustion in gas turbine combustion chambers [7], industrial glass cooling [13], and external photon beam radiotherapy [9], among others.

Because of the nature of photon transport, the radiative heat transfer equations are still computationally very expensive. Thus several approximate models have been introduced, cf. [5, 17] for a review. Recently, the partial moment minimum entropy approach has been developed in a series of papers [4, 3, 20, 6]. It is our purpose in this paper to prove existence, uniqueness and physically reasonable bounds for the first member of this model hierarchy. Analytical results concerning the existence and uniqueness of solutions to the transfer equation itself and to the radiative heat transfer equations, where also energy conservation and additionally heat conduction are considered, have been obtained by many authors. A rather recent review on methods for transport equations can be found in [2], cf. also [1]. The transfer equation together with energy conservation is considered in [8, 15]. The issue of heat conduction is addressed in [11, 12, 10]. Convection, conduction and radiation is treated in [18, 14].

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The half moment entropy approximation to radiative transfer on the interval $\Omega = (0,1)$ reads [3]

$$\partial_x F_+ = \kappa \left(T^4 - E_+\right), \qquad \partial_x(\chi_+ E_+) = \kappa \left(\frac{1}{2}T^4 - F_+\right), \tag{1a}$$

$$\partial_x F_- = \kappa \left(T^4 - E_-\right), \qquad \partial_x(\chi_- E_-) = \kappa \left(-\frac{1}{2}T^4 - F_-\right). \tag{1b}$$

To get a well posed problem we supplement this system with boundary data

$$F_+(0) = \frac{1}{2}T_l^4, \qquad F_-(1) = -\frac{1}{2}T_r^4, \tag{2a}$$

$$(\chi_+ E_+)(0) = \frac{1}{3}T_l^4, \qquad (\chi_- E_-)(1) = \frac{1}{3}T_r^4. \tag{2b}$$

The unknowns are macroscopic moments of the photon distribution I,

$$E_+ = \int_0^1 I(\mu)d\mu, \quad E_- = \int_{-1}^0 I(\mu)d\mu, \quad F_+ = \int_0^1 \mu I(\mu)d\mu, \quad F_- = \int_{-1}^0 \mu I(\mu)d\mu$$

and μ is the direction cosine. The Eddington factors are defined by

$$\chi_\pm(f_\pm) = \frac{8f_\pm^2}{1 \pm 6f_\pm + \sqrt{1 \pm 12f_\pm - 12f_\pm^2}}$$

for the relative fluxes $f_\pm = F_\pm/E_\pm$. The absorption coefficient is denoted by $\kappa > 0$. For simplicity, we have set Stefan-Boltzmann's constant equal to one.

We denote the minimum and maximum temperature by

$$\underline{T} = \min\left(T_l, T_r, \min_{x\in\Omega} T(x)\right) \qquad \text{and} \qquad \overline{T} = \max\left(T_l, T_r, \max_{x\in\Omega} T(x)\right).$$

The main result of this paper is

Theorem 1. *Let $T \in C([0,1])$ be given. There exists a unique solution $(E_+, F_+, E_-, F_-) \in [C^1([0,1])]^4$ to the half moment minimum entropy system (1-2). Furthermore, the solution satisfies the bounds*

$$\begin{aligned} \delta\underline{T}^4 \le E_+ \le \frac{1}{\delta}\overline{T}^4, &\qquad \delta\underline{T}^4 \le E_- \le \frac{1}{\delta}\overline{T}^4, \\ \delta\underline{T}^4 \le F_+ \le \overline{T}^4, &\qquad -\overline{T}^4 \le F_- \le \delta\underline{T}^4, \\ \delta \le f_+ \le 1, &\qquad 1 \le f_- \le -\delta. \end{aligned}$$

Here, δ is the unique solution to $\chi_+(\delta) = \frac{\delta}{2}$ in $]0, \frac{1}{2}[$.

Especially the bounds on the relative fluxes $f_\pm$ are responsible for the success of the half moment minimum entropy model. These bounds ensure that the model has the correct free-streaming limit. Many prominent models, like P_N and diffusion type models do not have this feature.

In the next section, we will rewrite the model equations amd state some properties. After that, we will establish the a priori bounds, which will also ensure existence and uniqueness by a continuation argument.

2 Preliminaries

In the following discussion, we will restict ourselves to the "+" equations. All the results for the "-" equations can be proven in analogy.

We define the functions $h_+(f_+) \stackrel{\text{def}}{=} \partial_f \chi_+(f_+)$ and $g_+(f_+) \stackrel{\text{def}}{=} \chi_+ - f_+ h_+(f_+)$. A direct calculation shows

Lemma 2. *For $0 \leq f_+ \leq 1$ the estimates*

$$\begin{aligned} f_+^2 \leq \chi_+(f_+) \leq 1 \\ -1 \leq g_+(f_+) \leq 0 \\ 0 \leq h_+(f_+) \leq 2 \end{aligned}$$

hold.

Using h_+ and g_+, the system can be written as

$$\begin{aligned} \partial_x F_+ &= \kappa \left(T^4 - E_+\right) \\ (g_+ \partial_x E_+ + h_+ \partial_x F_+) &= \kappa \left(\frac{1}{2} T^4 - F_+\right). \end{aligned}$$

We introduce for $\delta \in (0,1)$ the cut–off functions

$$h_+^\delta(f_+) = \begin{cases} \partial_f \chi_+(\delta), & f_+ \leq \delta, \\ \partial_f \chi_+(f_+), & \delta \leq f_+ \leq 1, \\ 2, & f_+ \geq 1, \end{cases}$$

$$g_+^\delta(f_+) = \begin{cases} \chi_+(\delta) - f_+ h_+^\delta(\delta), & f_+ \leq \delta, \\ g_+(f_+), & \delta \leq f_+ \leq 1, \\ -1, & f_+ \geq 1, \end{cases}$$

and consider the auxiliary system

$$\partial_x F_+ = \kappa \left(T^4 - E_+\right) \tag{3a}$$

$$\partial_x E_+ = \kappa \left(-\frac{h_+^\delta}{g_+^\delta} \left(T^4 - E_+\right) + \frac{1}{g_+^\delta} \left(\frac{1}{2} T^4 - F_+\right) \right). \tag{3b}$$

3 A priori bounds

Lemma 3. *Let $T \in C([0,1])$ be given. A solution $(F_+, E_+) \in [C^1([0,1])]^2$ of system (3) fulfills*

$$F_+ \leq E_+.$$

Proof. Assume that $F_+ \geq E_+$ at one point $x_0 \in (0,1)$. Then there is a point $x_1 \in (0,1)$ in which

$$(F_+ - E_+) = 0 \quad \text{and} \quad \partial_x (F_+ - E_+) > 0.$$

But at x_1 we have $f_+ = 1$, thus

$$\begin{aligned}\partial_x(F_+ - E_+) &= \kappa\left(\left(1+\frac{h_+^\delta}{g_+^\delta}\right)\left(T^4 - E_+\right) - \frac{1}{g_+^\delta}\left(\frac{1}{2}T^4 - F_+\right)\right)\\ &= \kappa\left(-\left(T^4 - E_+\right) + \left(\frac{1}{2}T^4 - F_+\right)\right) = -\frac{1}{2}T^4 \le 0,\end{aligned}$$

which contradicts the assumption. □

Lemma 4. *Let $T \in C([0,1])$ be given. A solution $(F_+, E_+) \in [C^1([0,1])]^2$ of system (3) satisfies*

$$F_+ \ge \delta E_+,$$

where $\delta \in]0, \frac{1}{2}[$ is uniquely defined by $\chi_+(\delta) = \frac{\delta}{2}$.

Proof. Let us assume that there is a point in which

$$(F_+ - \delta E_+) = 0 \quad \text{and} \quad \partial_x(F_+ - \delta E_+) < 0.$$

But at this point

$$\begin{aligned}\partial_x(F_+ - \delta E_+) =& \kappa\underbrace{\left(1+\frac{\delta}{g_+^\delta}\left(h_+^\delta - \frac{1}{2}\right)\right)}_{=0 \text{ for } F_+=\delta E_+} T^4 + \kappa\left(\frac{\delta}{g_+^\delta}F_+ - \left(1+\frac{\delta h_+^\delta}{g_+^\delta}\right)E_+\right)\\ &= \kappa\frac{\delta F_+ - g_+^\delta E_+ - h_+^\delta \delta E_+}{g_+^\delta} = \kappa\frac{\delta F_+ - \chi_+(\delta)E_+}{g_+^\delta}\\ &\ge \kappa\frac{\delta^2 - \chi_+(\delta)}{g_+^\delta}E_+ \ge 0.\end{aligned}$$

□

Lemma 5. *Let $T \in C([0,1])$ be given. A solution $(F_+, E_+) \in [C^1([0,1])]^2$ of system (3) fulfills*

(a) $\delta \underline{T}^4 \le F_+ \le \overline{T}^4$,

(b) $\delta \underline{T}^4 \le E_+ \le \frac{1}{\delta}\overline{T}^4$.

Proof. We have

$$\partial_x F_+ = \kappa\left(T^4 - E_+\right) \ge \kappa\left(\underline{\mathrm{T}}^4 - \frac{1}{\delta}F_+\right) = \frac{\kappa}{\delta}\left(\delta\underline{\mathrm{T}}^4 - F_+\right).$$

Thus $\delta\underline{\mathrm{T}}^4$ is a subsolution and we conclude $F_+ \ge \delta\underline{\mathrm{T}}^4$. The second inequality in assertion (a) follows from the estimate

$$\partial_x F_+ = \kappa\left(T^4 - E_+\right) \le \kappa\left(\overline{T}^4 - F_+\right).$$

Assertion (b) is a direct consequence of assertion (a) and Lemmas 3-4. □

Proof of Theorem 1. From Lemmas 3-5 we conclude that the systems (1) and (3) are indeed equivalent. Furthermore, any solution $(F_+, E_+) \in [C^1([0,1])]^2$ of system (1) is uniformly bounded. Thus, the unique local solution to (1) can be continued to a solution on the whole interval.

4 Numerical Results

In this section, we present numerical results which illustrate the upper and lower bounds on the relative flux.

For the upper bound on the relative half flux, $f_+ \leq 1$, we consider a highly absorbing medium with absorption coefficient $\kappa = 20$. The temperature $T = 1$ of the medium can be neglected compared to the temperature $T_l = 1000$ of the left wall. A heat wave propagates into the medium. While that happens, the photon distribution is weakened and becomes forward-peaked, thus leading to a relative half flux close to 1.

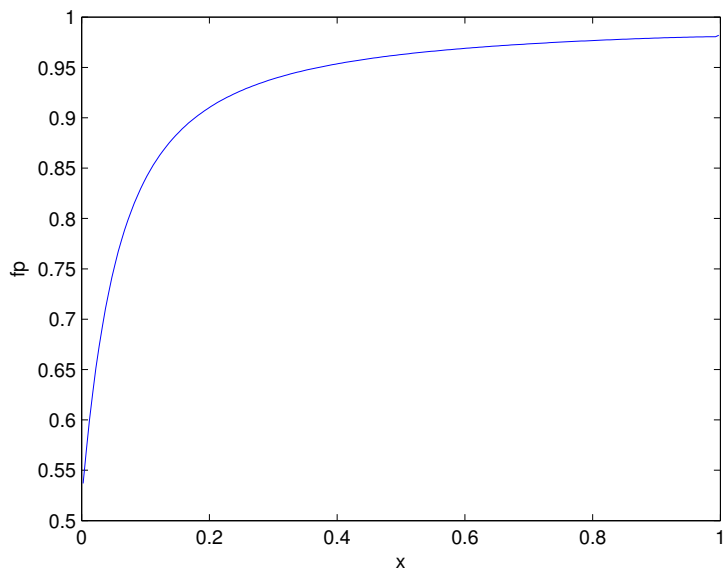

Figure 1: Upper bound $f_+ \leq 1$.

The lower bound is reached in an opposite physical setting. The temperature $T_l = 1$ of the left wall is small and there is a steep temperature gradient inside the medium, $T(x) = 1 + 9999x^4$, and little absorption, $\kappa = 0.2$. This means that there is basically no heat transfer from left to right. Thus f_+ approaches its minimum value $\delta \approx 0.2857$.

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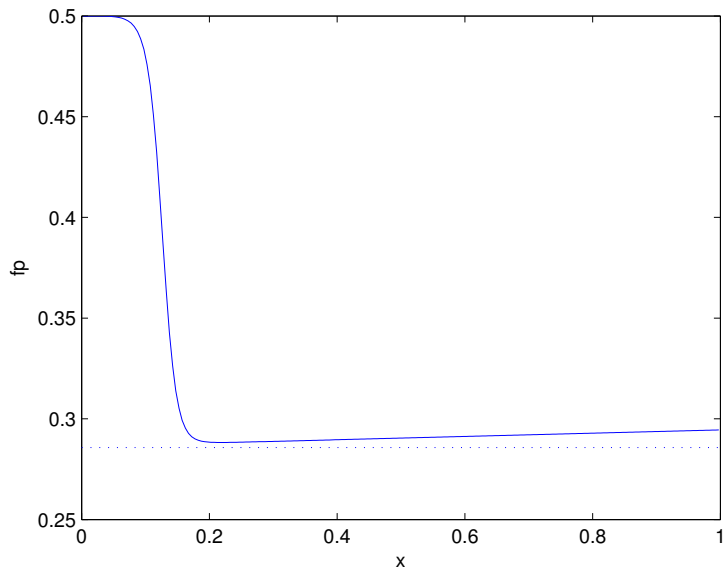

Figure 2: Lower bound $f_+ \geq \delta \approx 0.2857$.

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