Worst-Case Portfolio Optimization
Optimal Investment in Crash-Threatened Financial Markets

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Seminar on Continuous-Time Portfolio Optimization
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Overview

1. Introduction: Merton’s Optimal Terminal Wealth Problem

2. A Worst-Case Approach to Portfolio Optimization

3. Worst-Case Portfolios under Proportional Transaction Costs
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Market Model: The Black-Scholes Market

Black-Scholes Market

Consider a simple financial market consisting of two assets:

\[
\begin{align*}
 dB(t) &= rB(t)dt \\
 dS(t) &= \alpha S(t)dt + \sigma S(t)dW(t)
\end{align*}
\]

"Bond"  
"Stock"

The strategy of an investor in such a market can be described by the **risky fraction process** \( \pi(t) \):

\[
\pi(t) = \frac{S(t)}{B(t) + S(t)}.
\]

The **total wealth** of an investor with strategy \( \pi(t) \) can then be written as

\[
dX(t) = (1 - \pi(t))X(t)rdt + \pi(t)X(t)(\alpha dt + \sigma dW(t)).
\]
The Optimal Terminal Wealth Problem

For an initial wealth of $x > 0$, our objective is to find an admissible investment strategy $\pi^*(t) \in A(x)$ which yields the highest expected utility of terminal wealth at some terminal time $T$. That is,

$$\sup_{\pi \in A(x)} E_x[U_p(X(T))].$$

We assume that the utility function $U_p$ is of the form

$$U_p(x) = \begin{cases} 
\frac{1}{p} x^p, & \text{if } p < 1, p \neq 0, \\
\log x, & \text{if } p = 0.
\end{cases}$$

The problem was solved by Merton (1969/71) using methods of stochastic control theory.
Solution of the Merton Problem

Define the value function

\[ V(t, x) = \sup_{\pi \in \mathcal{A}(x)} E_{t,x}[U_p(X(T))]. \]

One can show that \( V \) is the unique solution of the HJB equation

\[ 0 = \sup_{\pi} \{ \mathcal{L}V(t, x) \} \]

where the differential operator \( \mathcal{L} \) is given by

\[ \mathcal{L} = \frac{\partial}{\partial t} + (r + \pi(\alpha - r))x \frac{\partial}{\partial x} + \frac{1}{2} \pi^2 \sigma^2 x^2 \frac{\partial^2}{\partial x^2}. \]

The optimal trading strategy is

\[ \pi^*(t) = \frac{1}{1 - p} \frac{\alpha - r}{\sigma^2}. \]

“Merton fraction”
The Optimal Risky Fraction

![Graph showing the optimal risky fraction over time. The line is flat, indicating a constant optimal risky fraction across different time points.]
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In the Merton model, the stock price is modeled as a geometric Brownian motion with **continuous sample paths**.
⇒ The model cannot explain sudden extreme price movements (**crashes**).

**So, how can we model crashes?**
Modeling Market Crashes

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**So, how can we model crashes?**

1. Replace the geometric Brownian motion by a **jump diffusion**.
   ⇒ Crashes are only hedged in the mean, jump intensities may be hard to estimate in practice.
Modeling Market Crashes

In the Merton model, the stock price is modeled as a geometric Brownian motion with \textit{continuous sample paths}.
\implies The model cannot explain sudden extreme price movements (\textit{crashes}).

So, how can we model crashes?

1. Replace the geometric Brownian motion by a \textit{jump diffusion}.
   \implies Crashes are only hedged in the mean, jump intensities may be hard to estimate in practice.

2. Compute a worst-case crash scenario for every admissible trading strategy and optimize over these \textit{worst-case bounds}.
   \implies Indifference about crashes, but leads to very conservative strategies.
The Crash Model

Worst-Case Portfolio Optimization:

Crash-Threatened Financial Market

Assume that the bond and stock dynamics are given by

\[ dB(t) = rB(t)dt \]
\[ dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) \]
The Crash Model

Worst-Case Portfolio Optimization:

Crash-Threatened Financial Market

Assume that the bond and stock dynamics are given by

\[ dB(t) = rB(t)dt \]
\[ dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) - \beta S(t)dN(t) \]

\( N(t) \): Crash process counting the number of crashes.
\( \beta \in (0,1) \): Constant relative crash height.

The key feature of this model is the way in which the crash process \( N(t) \) is modeled.
Properties of the Crash Process

Properties of the crash process:

- \( N(t) \) counts the number of crashes up to time \( t \), i.e.

\[
N(t) = \# \{ u \in [0, T] : S(t-) \neq S(t) \}.
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3. Thus, the problem can be regarded as a **stochastic game** between the investor and the market.
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3. Thus, the problem can be regarded as a **stochastic game** between the investor and the market.

4. The maximal number of crashes in $[0, T]$ is assumed to be **bounded**.

5. **Multiple crashes** at the same time instant are not allowed.

6. The market **does not** have to exercise any crash option if it is not optimal.
The optimization problem in this model is

$$\sup_{\pi \in \mathcal{A}} \inf_{N \in \mathcal{B}} E_x \left[ U_p(X(T)) \right],$$

with corresponding value function

$$V^n(t, x) = \sup_{\pi \in \mathcal{A}} \inf_{N \in \mathcal{B}} E_{t,x,n} \left[ U_p(X(T)) \right].$$

Also, recall the differential operator $\mathcal{L}$:

$$\mathcal{L} = \frac{\partial}{\partial t} + (r + \pi(\alpha - r)) x \frac{\partial}{\partial x} + \frac{1}{2} \pi^2 \sigma^2 x^2 \frac{\partial^2}{\partial x^2}.$$
The Bellman System

Verification Theorem

Define for $v^n \in C^{1,2}$

$$\mathcal{A}'_{n}(t, x) = \{\pi \in \mathcal{A} : 0 \leq \mathcal{L}v^n(t, x)\},$$

$$\mathcal{A}''_{n}(t, x) = \{\pi \in \mathcal{A} : v^n(t, x) \leq v^{n-1}(t, (1 - \beta \pi)x)\}.$$  

If $v^n$ is polynomially bounded, $v^n(T, x) = U_p(x)$, and

$$0 \leq \sup_{\pi \in \mathcal{A}''_{n}(t, x)} \mathcal{L}v^n(t, x),$$

$$0 \leq \sup_{\pi \in \mathcal{A}'_{n}(t, x)} v^{n-1}(t, (1 - \beta \pi)x) - v^n(t, x),$$

$$0 = \sup_{\pi \in \mathcal{A}''_{n}(t, x)} \mathcal{L}v^n(t, x) \sup_{\pi \in \mathcal{A}'_{n}(t, x)} v^{n-1}(t, (1 - \beta \pi)x) - v^n(t, x),$$

then $v^n$ must be the value function $V^n$. 
The Optimal Trading and Crash Strategies

The **optimal trading strategy** is then

\[ \pi_n^*(t) = \arg \sup_{\pi \in A''(t,x)} \mathcal{L}V^n(t,x). \]

The **optimal crash time** is given by the stopping time

\[ \tau_n^* := \inf\{u > t : V^n(t,x) \geq V^{n-1}(t,(1 - \beta \pi_n^*)x)\}. \]

As it turns out, the crash constraint holds with equality for all \((t,x)\). This leads to an ordinary differential equation for \(\pi_n^*\):

\[
\frac{\partial}{\partial t} \pi_n^* = \frac{1}{\beta}(1 - \beta \pi_n^*) \left( (\alpha - r)(\pi_n^* - \pi_{n-1}^*) \right.
\]

\[
- \frac{1}{2}(1 - p)\sigma^2 \left( (\pi_n^*)^2 - (\pi_{n-1}^*)^2 \right)
\]

\[ \pi_n^*(T) = 0. \]
The Optimal Risky Fraction in the Worst-Case Model

The graph shows the optimal risky fraction over time for different values of n. The red line represents n = 0, the blue line represents n = 1, and the black line represents n = 2. The x-axis represents time, ranging from 0 to 1, and the y-axis represents the risky fraction, ranging from 0 to 1.
Conclusions

We draw the following conclusions:

1. The solution of the worst-case problem is obtained by balancing reasonable performance with protection against the impact of crashes.

2. The worst-case approach leads to optimal strategies dependent on the time to maturity.

3. In the presence of crashes it is still optimal to keep a fraction of the wealth invested in the stock.

4. The optimal trading strategy makes the investor indifferent about the occurrence of the worst-case crash scenario at any time instant.
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Crash Market with Proportional Transaction Costs

Proportional Transaction Costs:

Crash-Threatened Financial Market with Transaction Costs
Assume that the amounts of money invested in bond and stock follow

\[ dB(t) = rB(t)dt \]
\[ dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) - \beta S(t)dN(t) \]
Crash Market with Proportional Transaction Costs

Proportional Transaction Costs: 

Crash-Threatened Financial Market with Transaction Costs
Assume that the amounts of money invested in bond and stock follow

\[ dB(t) = rB(t)dt - (1 + \lambda)dL(t) \]
\[ dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) - \beta S(t)dN(t) + dL(t) \]

\( L(t) \): Increasing, càdlàg process recording the cumulative amount of money used for buying stock.

\( \lambda \in (0, \infty) \) : proportional transaction costs for buying
Crash Market with Proportional Transaction Costs

Proportional Transaction Costs:

Crash-Threatened Financial Market with Transaction Costs

Assume that the amounts of money invested in bond and stock follow

\begin{align*}
    dB(t) &= rB(t)dt - (1 + \lambda)dL(t) + (1 - \mu)dM(t) \\
    dS(t) &= \alpha S(t)dt + \sigma S(t)dW(t) - \beta S(t)dN(t) + dL(t) - dM(t)
\end{align*}

\( L(t) \): Increasing, càdlàg process recording the cumulative amount of money used for buying stock.

\( M(t) \): Increasing, càdlàg process recording the cumulative amount of money obtained from selling stock.

\( \lambda \in (0, \infty), \mu \in (0, 1) \): proportional transaction costs for buying and selling, respectively.
Admissible Trading Strategies

We are interested in the **total net wealth after liquidation** of the stock position.

\[ X(t) = \begin{cases} 
B(t) + (1 - \mu)S(t), & \text{if } S(t) > 0, \\
B(t) + (1 + \lambda)S(t), & \text{if } S(t) \leq 0.
\end{cases} \]

This inspires the following definitions of a **solvency region**:

\[ S^0 = \{(b, s) \in \mathbb{R}^2 : b + (1 + \lambda)s > 0, b + (1 - \mu)s > 0\}, \]
\[ S^n = \{(b, s) \in \mathbb{R}^2 : b + (1 + \lambda)s > 0, b + (1 - \beta)(1 - \mu)s > 0\}. \]

The set of **admissible strategies** is given by

\[ \mathcal{A}(t, b, s) = \{(L, M) : X^{L, M}(u) \in \overline{S^n} \text{ for all } u \in [t, T]\}. \]
Problem Formulation

For an initial position \((b, s) \in \mathbb{S}^n\), the worst-case terminal wealth problem is given by

\[
\sup_{(L,M) \in A(0,b,s)} \inf_{N \in \mathcal{B}} E_{b,s} \left[ U_p(X(T)) \right],
\]

with value function

\[
V^n(t, b, s) = \sup_{(L,M) \in A(t,b,s)} \inf_{N \in \mathcal{B}} E_{t,b,s,n} \left[ U_p(X(T)) \right].
\]

The value function in this model can be characterized in a similar way as the in the case without transaction costs:

1. In between crash times, the classical HJB equation should hold.
2. The value function for \(n\) crashes should be bounded by the value function with \(n - 1\) crashes:

\[
V^n(t, b, s) \leq V^{n-1}(t, b, (1 - \beta)s).
\]
The HJB Equation \((n = 0)\)

The value function \(V^0\) satisfies the HJB equation

\[
0 = \min \left\{ -V_t^0 - \alpha s V_s^0 - rb V_b^0 - \frac{1}{2} \sigma^2 s^2 V_{ss}^0, \right. \\
\left. - (1 - \mu) V_b^0 + V_s^0, (1 + \lambda) V_b^0 - V_s^0 \right\}. 
\]

in the sense of \textit{viscosity solutions} with terminal condition

\[
V^0(T, b, s) = \begin{cases} 
  b + (1 - \mu)s, & \text{if } s > 0, \\
  b + (1 + \lambda)s, & \text{if } s \leq 0.
\end{cases}
\]

Depending on which operator in the HJB equation vanishes determines the optimal action of the investor.
Trading Regions under Transaction Costs ($n = 0$)
The HJB Equation \((n > 0)\)

**Characterization of the Value Function**

The value function \(V^n\) is a viscosity solution of the HJB equation

\[
0 = \min \left\{ -V_t^n - \alpha s V_s^n - rb V_b^n - \frac{1}{2} \sigma^2 s^2 V_{ss}^n, \right. \\
\left. - (1 - \mu) V_b^n + V_s^n, (1 + \lambda) V_b^n - V_s^n \right\}
\]

under the constraint

\[
V^n(t, b, s) \leq V^{n-1}(t, b, (1 - \beta)s), \quad n > 0.
\]

Depending on which operator in the HJB equation vanishes determines the optimal action of the investor. The optimal crash time is given by

\[
\tau^*_n = \inf \{ u > t : V^n(u, b, s) = V^{n-1}(u, b, (1 - \beta)s) \}.
\]
Trading Regions under Transaction Costs \((n = 1)\)
Trading Regions under Transaction Costs ($n = 2$)
We draw the following conclusions:

1. Again, we obtain the solution by balancing performance and crash-protection.

2. For short investment periods it may not be optimal to invest any money in the stock!

3. It is not (yet) clear under which conditions the optimal strategies are admissible.
Any Questions??
Thank you for your attention!!!