Worst-Case Portfolio Optimization
Optimal Investment in Crash-Threatened Financial Markets

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Introduction. The Merton model has certain drawbacks, e.g.

- The stock price is modeled as a continuous geometric Brownian motion. Hence it cannot explain extreme sudden price fluctuations (crashes).
- The optimal strategy forces the investor to adjust the stock position at every time instant. In the presence of transaction costs this leads to immediate bankruptcy.

Worst-Case Portfolio Optimization. Consider the financial market

\[
\begin{align*}
    dB(t) &= rB(t)dt, \\
    dS(t) &= \alpha S(t)dt + \sigma S(t)dW(t) - \beta S(t)dN(t), \\
    X(t) &= B(t) + S(t),
\end{align*}
\]

where \( \beta \in (0, 1) \) is the crash height and \( N(t) \) is a counting process modeling the occurrence of crashes. It is assumed to have the following properties:

- No distributional assumptions, instead it is considered as a control variable held by the market used to minimize expected utility.
- The maximal number of crashes in \([0, T]\) is assumed to be bounded.
- Multiple crashes at the same time instant are not allowed.
- The market does not have to exercise any crash option if it is not optimal.

The objective in this model is to find for initial wealth \( x > 0 \) a trading strategy \( \pi^* \) in the set of admissible strategies \( \mathcal{A} \) with the best worst-case performance if there are at most \( n \) crashes left:

\[
V^n(t, x) = \sup_{\pi \in \mathcal{A}} \inf_N E_{t,x,n}[U_p(X(T))],
\]

where \( U_p(x) = \frac{1}{p} x^p, \ p < 1 \). Let \( v^n \in C^{1,2} \) and set

\[
\begin{align*}
    \mathcal{L}v^n &= v^n_t + (r + \pi(\alpha - r))xv^n_x + \frac{1}{2}\pi^2\sigma^2x^2v^n_{xx}, \\
    \mathcal{A}_n(t, x) &= \{ \pi \in \mathcal{A} : 0 \leq \mathcal{L}v^n(t, x) \}, \\
    \mathcal{A}_n^*(t, x) &= \{ \pi \in \mathcal{A} : v^n(t, x) \leq v^{n-1}(t, (1 - \beta)x) \}.
\end{align*}
\]
One can show that if $v^n$ is polynomially bounded, $v^n(T, x) = U_p(x)$, and

$$0 \leq \sup_{\pi \in A_n'(t, x)} \mathcal{L}v^n(t, x),$$

$$0 \leq \sup_{\pi \in A_n'(t, x)} v^{n-1}(t, (1 - \beta \pi)x) - v^n(t, x),$$

$$0 = \sup_{\pi \in A_n'(t, x)} \mathcal{L}v^n(t, x) \sup_{\pi \in A_n'(t, x)} v^{n-1}(t, (1 - \beta \pi)x) - v^n(t, x),$$

then $v^n$ must be the value function $V^n$. This so-called Bellman system can be used to deduce an ordinary differential equation for the optimal trading strategy:

$$\frac{\partial}{\partial t} \pi^*_n = \frac{1}{\beta} \left(1 - \beta \pi^*_n\right) \left((\alpha - r)(\pi^*_n - \pi^*_{n-1}) - \frac{1}{2}(1 - p)\sigma^2((\pi^*_n)^2 - (\pi^*_{n-1})^2)\right)$$

$$\pi^*_n(T) = 0.$$

The optimal crash time is

$$\tau^*_n := \inf\{u > t : v^n(t, x) \geq v^{n-1}(t, (1 - \beta \pi^*_n)x)\}.$$

**Worst-Case Portfolios with Transaction Costs.** The previous financial market can be extended to include proportional transaction costs:

$$dB(t) = rB(t)dt - (1 + \lambda)dL(t) + (1 - \mu)dM(t),$$

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) - \beta S(t)dN(t) + dL(t) - dM(t),$$

$$X(t) = B(t) + (1 - \mu)S(t)1_{\{S(t) > 0\}} + (1 + \lambda)S(t)1_{\{S(t) \leq 0\}},$$

where $L(t)$ and $M(t)$ are increasing, right-continuous processes recording the cumulative amount of money used for buying and selling stock, respectively, and where $\lambda \in (0, \infty)$ and $\mu \in (0, 1)$ are the proportional transaction costs. The pair $(L(t), M(t))$ is admissible if $X(t) \geq 0$ a.s. for every $t$. The optimization problem is given by

$$V^n(t, b, s) = \sup_{(L, M)^N} \inf_{E_t,b,s,n} [U_p(X(T))].$$
The value function $V^n$ is a viscosity solution of the HJB equation

$$0 = \min \left\{ -V^n_t - \alpha_s V^n_s - rbV^n_b - \frac{1}{2} \sigma^2 s^2 V^n_{ss}, \right.$$ 

$$\left. - (1 - \mu)V^n_b + V^n_s, (1 + \lambda)V^n_b - V^n_s \right\}$$

under the constraint

$$V^n(t, b, s) \leq V^{n-1}(t, b, (1 - \beta)s), \quad n > 0.$$ 

The optimal crash time is given by

$$\tau^*_n = \inf \{ u > t : V^n(u, b, s) = V^{n-1}(u, b, (1 - \beta)s) \}.$$ 

The operators in the HJB equation determine the optimal action of the investor. Define

$$\mathcal{R}^{nt} = \{(t, b, s) : 0 = -V^n_t - \alpha_s V^n_s - rbV^n_b - \frac{1}{2} \sigma^2 s^2 V^n_{ss}\},$$

$$\mathcal{R}^{sell} = \{(t, b, s) : 0 = -(1 - \mu)V^n_b + V^n_s\},$$

$$\mathcal{R}^{buy} = \{(t, b, s) : 0 = (1 + \lambda)V^n_b - V^n_s\}.$$ 

In $\mathcal{R}^{nt}$ it is optimal not to trade at all. In $\mathcal{R}^{buy}$ the investor should buy stock until he is back inside $\mathcal{R}^{nt}$. Similarly, in $\mathcal{R}^{sell}$ the optimal action is to sell stock until the bond/stock position is back inside $\mathcal{R}^{nt}$.