Elastic Properties of Laguerre Approximations of Random Foams

A. Liebscher*a, C. Redenbach*a

Department of Mathematics, University of Kaiserslautern, Box 3049, 67653 Kaiserslautern, Germany

Abstract

Stochastic models are valuable tools to study the mechanical behaviour of foams whose characteristic features include random cellular morphology and cells of different size (polydispersity). Random Laguerre tessellations, additively weighted generalisations of Voronoi tessellations, possess these features which makes them natural models for foams. The Young’s modulus of low-density solid foams with open cells is calculated for two different classes of random, polydisperse foams based on Laguerre tessellations and soap froth simulated with the Surface Evolver. For the special case of monodisperse foams, we compare Laguerre foams with the well known Kelvin and Weaire–Phelan foams.

Keywords: Open-cell foams, Mechanical properties, Stochastic modelling, Random tessellation, Voronoi tessellation, Laguerre tessellation

*Corresponding author

Email address: liebscher@mathematik.uni-kl.de (A. Liebscher)
1. Introduction

Solid foams are two-phase systems with a random cellular morphology. This special microstructure induces some remarkable functional properties like a high mechanical stiffness and strength (Ashby et al., 2000). Random tessellations are valuable tools to study the functional behaviour of foams. An overview can be found in Liebscher (2014).

Solid foams are closely related to soap froths and thus both share many geometric properties. In fact, we can consider a solid foam as a liquid foam that was frozen in a specific state. Mathematically, a low-density foam with a relative density \( \phi \to 0 \), can be interpreted as a subdivision of space into random (slightly curved) polyhedral cells whose edges form an interconnected network.

The local structure then obeys Plateau’s laws (Plateau, 1873; Taylor, 1976), that is, exactly three faces may intersect in an edge at a dihedral angle of exactly 120°. Consequently, four edges meet at a perfectly symmetric tetrahedral vertex. If we, for a moment, neglect the curvature of the cells, we can perceive a foam as a subdivision of space into convex polyhedra whose local topology is induced by Plateau’s laws. In stochastic geometry such a subdivision of space is known as a normal tessellation. It can be shown that any normal tessellation of the three-dimensional Euclidean space \( \mathbb{R}^3 \) is a Laguerre tessellation (Aurenhammer, 1987; Lautensack and Zuyev, 2008).

A Laguerre tessellation is an additively weighted generalisation of the well known Voronoi tessellation. The latter is generated by a locally finite set of
points $\psi \subset \mathbb{R}^3$ by assigning to each point $x \in \psi$ the polyhedron $C(x, \psi)$ composed of those points in space that have $x$ as nearest neighbour in $\psi$. Locally finite means that any bounded subset of $\mathbb{R}^3$ may only contain a finite number of elements of $\psi$.

To generalise the concept of Voronoi tessellations, we attach positive weights to the points of $\psi$. Each pair $[x, r] \in \psi$ can then be interpreted as a sphere with centre $x$ and radius $r$, whose corresponding Laguerre polyhedron is defined by

$$C([x, r], \psi) = \{ y \in \mathbb{R}^3 : ||y - x||^2 - r^2 \leq ||y - x'||^2 - r'^2 \text{ for all } [x', r'] \in \psi \}.$$  

If all radii are equal, we obtain the Voronoi polyhedron as special case.

Random Laguerre tessellations appear to be natural models for low-density foams as both share the same topology and a similar random cellular morphology. Consequently, Laguerre tessellations are common models for foams (Kanaun and Tkachenko, 2006, 2007; Lautensack, 2008; Jaeger et al., 2011; Beckmann and Hohe, 2012; Liebscher, 2014; Chen et al., 2015; Liebscher et al., 2015a; Randrianalisoa et al., 2015; Marvi-Mashhadi et al., 2018) and other cellular materials (Telley et al., 1992; Xue et al., 1997; Fan et al., 2004; Kühn and Steinhauser, 2008; Liebscher et al., 2015b; Spettl et al., 2016; Falco et al., 2017). However, the cell curvature of foams only allows for an approximate representation by Laguerre tessellations. So recently foam models based on the simulation of the physical formation process of soap froth became popular (Jang et al., 2010; Gaitanaros et al., 2012, 2018; Gaitanaros
and Kyriakides, 2014; Köll and Hallström, 2016; Vecchio et al., 2016; Settgast et al., 2019). We call such models simulated foams.

A simulated foam can be considered as a Laguerre tessellation which is ‘relaxed’ such that it fulfils Plateau’s laws (Almgren, 1991; Kusner and Sullivan, 1996). Physically this means to reduce free energy by minimising the surface area as illustrated in Fig. 1. A software that is dedicated to the solution of this and other problems involving energy minimisation is Brakke’s Surface Evolver (Brakke, 1992). Unfortunately, generating simulated foams is computationally expensive as the whole foam evolution process has to be modelled in software—including a large number of topological transitions that result in cell neighbour switching (Kraynik et al., 2003, 2004, 2005).

Considering the mentioned approaches to foam modelling, one question remains: How good can an approximation of a foam with a Laguerre tessellation be? To answer that question we compute a set of spheres whose Laguerre tessellation best fits the cell system (and thus the edge system) of a foam by minimising the discrepancy between the individual cells of the foam and the Laguerre tessellation (Liebscher, 2015). We call such a tessellation Laguerre approximation. Concerning cellular morphology the results presented in Liebscher (2015) are promising. In this work we analyse the error in predicting the elasticity of foams by Laguerre approximations. Additionally, we compare the prediction capabilities of Laguerre approximations for monodisperse foams to some well known deterministic foam models (i.e. ordered structures like the Kelvin cell (Thomson, 1887) and the Weaire–Phelan
Figure 1: Effect of the relaxation process transforming a Laguerre polyhedron (a) into a foam cell (b). The Laguerre approximation of (b) is shown in (c). Note the absence of the small white faces of the Laguerre polyhedron in the foam cell and its approximation.

2. Computing the Laguerre approximation

Following Liebscher (2015), we formulate the approximation of a simulated foam in a bounded subset $P \subset \mathbb{R}^3$ by a Laguerre tessellation as a minimisation problem with the objective to minimise the discrepancy between the individual cells of the foam and the Laguerre tessellation. Denote by $\mathcal{P} = \{P_1, \ldots, P_m\}$ a set of foam cells such that $P = \bigcup_{i=1}^{m} P_i$. Our objective is now to approximate $\mathcal{P}$ by the cells $C_i$ of a Laguerre tessellation $\mathcal{L} = \mathcal{L}|_P = \{C_i \cap P\}_{i=1}^{m}$ in $P$.

The misfit of $\mathcal{L}$ for cell $P_j \in \mathcal{P}$ is expressed by the accumulated overlap between $P_j$ and the corresponding adjacent cells in $\mathcal{L}$ (see Fig. 2 for an illustration). Let us denote by $y_i = [x_i, r_i] \in \mathbb{R}^3 \times \mathbb{R}_+$ the sphere that generates cell $C_i$, where $\mathbb{R}_+$ stands for the positive real numbers. Then the overall discrepancy between a foam and its Laguerre approximation is given
Figure 2: Illustration of the error (shaded grey) between a cell of a Laguerre approximation (- - -) and a foam cell (—–) as defined in Eq. (2).

by

\[ f(y_1, \ldots, y_m) = \sum_{j=1}^{m} \sum_{i \neq j} \nu(P_j \cap C_i), \]

(2)

where \( \nu(\cdot) \) denotes the volume. Hence, the Laguerre approximation of \( \mathcal{P} \) is the minimiser of (2) that can be computed numerically by gradient decent methods.

The presented approach provides an exact reconstruction of a tessellation if it is known to be a Laguerre tessellation. However, as foam cells have curved faces, an exact representation by Laguerre polyhedra is impossible. Nevertheless, the results in Liebscher (2015) show that the geometric cell properties (volume, surface area, mean breadth, and edge length distribution) as well as the local topology are reproduced quite well by Laguerre approximations.

We quantify the topological approximation error by comparing the individual foam cells \( P_i \) with their approximations \( C_i \) and count the absolute number of extra or missing cell neighbours. As during the generation of simulated foams all tiny edges and faces have been eliminated, no cell neigh-
bour switching is possible. Hence, we may define the fraction of cells with
incorrectly assigned neighbours for a foam with $m$ cells as

$$\mathcal{F}_{\text{diff}} = \frac{1}{m} \sum_{i=1}^{m} 1_{[1,\infty)}(\mathcal{F}(P_i) - \mathcal{F}(C_i)), \quad (3)$$

where $\mathcal{F}(\cdot)$ denotes the number of faces per cell and $1_{[1,\infty)}(x)$ equals one if $x \geq 1$ and zero, otherwise.

3. Computing the Elastic Properties

3.1. Simulated Random Foams

To compare the linear elastic behaviour of the simulated foams with their
Laguerre approximations, we consider open-cell foams at the low-density
limit with a relative density $\phi \to 0$. In this setting, foams are usually re-
presented by simplified beam models, that is, a network where straight edges
with uniform cross section connect the cell vertices. Owing to the low density,
we may neglect the correction for material at the vertices. The mechanical
stiffness is then explained by a deformable edge model with appropriate me-
chanical properties.

We discretise the edges with finite elements in ABAQUS using 2-node
cubic Bernoulli beam elements with uniform cross section. The cross sections
are modelled based on ideal Plateau borders. Their typical deltoidal shape is
conceived as the complement of three circles with the same radius $r$ that are
placed at different corners of an equilateral triangle as illustrated in Fig. 3
(Weaire and Hutzler, 1999). The area $A$, moments of inertia $I_y, I_z$ (Gong
et al., 2005b), and torsion constant $J$ (Warren and Kraynik, 1997) of the resulting geometry are

$$A = \left( \sqrt{3} - \frac{\pi}{2} \right) r^2, \quad I_y = I_z = \frac{1}{24} \left( 20\sqrt{3} - 11\pi \right) r^4, \quad J = 0.0021 r^2. \quad (4)$$

The mechanical properties of the beams are defined by the quantities in Eq. (4) along with the beam general section feature of ABAQUS. As we consider a foam in the low-density limit, we chose $r$ such that the resulting relative density of the foam model is $\phi = 1/1000$. The elastic moduli are then computed in a cubical domain with periodic boundaries for incompressible struts with a Poisson’s ratio $\nu = 0.49$ (see Gong et al., 2005a).

By definition the simulated foams are geometrically isotropic for large systems. Consequently, for each foam the effective elastic moduli computed in x-, y-, and z-direction ($E_{100}$, $E_{010}$, and $E_{001}$) should nearly coincide. To avoid errors by choosing one direction (e.g. due to minor structure imperfections), we define the isotropic effective elastic modulus $\bar{E}$ for one structure as the average of $E_{100}$, $E_{010}$, and $E_{001}$, i.e. $\bar{E} = (E_{100} + E_{010} + E_{001})/3$.

Recall, that the effective elastic modulus of a low-density open-cell foam with struts of uniform cross section scales with the squared relative density $\phi$ following

$$\bar{E} = c E \phi^2, \quad (5)$$

where $c$ is a constant depending on strut geometry and $E$ is Young’s modulus of the strut material (Warren and Kraynik, 1997; Gibson and Ashby, 1999). For fixed $E$ and $\phi$, $c = \bar{E}/E \phi^2$ is proportional to $\bar{E}$. Hence we may consider
c as a scale free measure of the effective elastic modulus. To quantify the ability of Laguerre tessellations to reproduce the elastic properties of a foam, we study the deviation (in percent) between the resulting constant c of a simulated foam and its Laguerre approximation. We denote this quantity $E_{\text{diff}}$.

### 3.2. Deterministic Foam Models

The elastic properties of deterministic models are usually anisotropic, i.e. they are orientation dependent. In our study, we only consider models with cubic symmetry. This means the orientation dependence can be characterised by probing two principal directions (Nye, 1972, Chap. 8), namely $E_{100}$ and $E_{111}$. An effective elastic modulus is then calculated by orientation averaging. Following Warren and Kraynik (1997), we use the Hashin–Shtrikman average. It is the mean of the upper $E_{\text{HS}}^u$ and lower $E_{\text{HS}}^l$ Hashin–Shtrikman bound (Hashin and Shtrikman, 1962, 1963). Those bounds characterise the effective elastic properties of a polycrystal based on individual grain properties. Formally the upper and lower bound for materials with cubic symmetry are given by

$$E_{\text{HS}}^u = \mu_1 + 3 \left( \frac{5}{\mu_2 - \mu_1} - 6h_1 \right)^{-1} \quad \text{and} \quad E_{\text{HS}}^l = \mu_2 + 2 \left( \frac{5}{\mu_1 - \mu_2} - 6h_2 \right)^{-1}$$

(6)

where $\mu_1$ corresponds to $E_{100}$, $\mu_2$ corresponds to $E_{111}$,

$$h_i = \frac{3(\kappa + 2\mu_i)}{3\mu_i(3\kappa + 4\mu_i)},$$

(7)
and $\kappa$ denotes the bulk modulus. That means that for deterministic foam models with cubic symmetry we have $(E_{L}^{\text{HS}} + E_{U}^{\text{HS}})/2 = cE\phi^2$, and thus

$$c = \frac{E_{L}^{\text{HS}} + E_{U}^{\text{HS}}}{2E\phi^2}. \quad (8)$$

4. Results

In the following two subsections we study the properties of Laguerre approximated random foams. The foams were simulated with the Surface Evolver based on Laguerre tessellations generated from dense polydisperse sphere packings with lognormally distributed volumes. Choosing a dense packing allows, to a certain degree, to control the polydispersity of the resulting foam via the volume distribution of the spheres (Gervois et al., 2002; Fan et al., 2004; Redenbach, 2009). We define the polydispersity $p$ of the foam as the coefficient of variation of the equivalent sphere radius $r$ of the foam cells. That is, for each cell of the foam, $r$ is determined from the volume $V$ by setting $V = \frac{4}{3}\pi r^3$. Then, $p = \sigma_r/\bar{r}$, where $\bar{r}$ denotes the mean and $\sigma_r$
Figure 4: Median of error percentage in local topology $F_{\text{diff}}$ (i.e. fraction of cells with extra or missing cell neighbours) between a simulated foam, its Laguerre approximation (LA), and original Laguerre tessellation (LT) with increasing polydispersity $p$. The error bars indicate the standard deviation.

denotes the standard deviation of the equivalent radius distribution. Note that $p$ equals zero only for monodisperse foams.

In our study, foams with increasing polydispersity were generated by increasing the variance of the sphere volume distribution. The empirical median polydispersities of the foams are $p \in \{0, 0.07, 0.13, 0.18, 0.25, 0.33, 0.44, 0.59\}$. For each polydispersity, we generated six foams with 2197 cells. As we expected an increased variance in the mechanical response for high polydispersities, we added two additional foams for polydispersity $p > 0.3$. For $p = 0.33$ and $p = 0.44$ these foams stemmed from a previous study and contain 1728 cells.
4.1. Topological Approximation Error of Tessellation Models

To compare the topological differences between foam and tessellation model, we first study the topological approximation error $\mathcal{F}_{\text{diff}}$ as introduced in Eq. (3). Fig. 4 shows the median of $\mathcal{F}_{\text{diff}}$ when comparing simulated foams, their Laguerre approximation and the original Laguerre tessellation with respect to polydispersity $p$. Note that $\mathcal{F}_{\text{diff}}$ is almost identical when comparing foams or Laguerre approximations with Laguerre tessellations. This implies that the main topological differences arise from the relaxation step. For polydisperse structures with $0 < p < 0.6$ the median error percentage is around 60% $\pm$ 2% and increases to about 65% for $p \rightarrow 0.6$. For the monodisperse case (i.e. $p \approx 0$), the median topological error percentage becomes 73%.

Laguerre approximations with a polydispersities between 0 and 0.45 topologically differ from simulated foams by at most 5%. The minimal error of about 1% is obtained around $p \approx 0.2$. For polydispersities above 0.45 the error increases to about 14% at $p \approx 0.6$.

To explain the drastic increase in the approximation error for polydisperse structures, recall that we approximate curved foam cells with flat polyhedra by minimising the excessive volume. The curvature of foam cells is related to the volume differences between neighbouring cells (owing to the law of Laplace–Young (see e.g. Weaire and Hutzler, 1999)) and, thus, increases with increasing polydispersity. Consequently, it becomes harder to correctly approximate foam cells by flat polyhedra. This difficulty is reflected by the increase in the approximation error $\mathcal{F}_{\text{diff}}$. 

12
Figure 5: Comparison of the effective Young’s moduli of simulated foams, their corresponding Laguerre approximations and the original Laguerre tessellations with increasing polydispersity $p$. The error bars indicate the standard deviation.

4.2. Elastic Properties of Tessellation Models

Next we study how the topological approximation error influences the elastic properties of the resulting geometry model. Fig. 5 compares the effective Young’s modulus of Laguerre tessellations to the ones of the simulated foams and their Laguerre approximations. The simulated foams were always softer than the Laguerre tessellations and the Laguerre approximations, i.e. the median constant $c$ of the effective elastic moduli of a foam is always smaller than the one of its corresponding tessellations. This is again caused by the modelling of curved foam cells by flat Laguerre polyhedra. Consequently, the vertices of the faces of a Laguerre cell must lie in a common plane. This is not the case for foam cells where curvature allows for more freedom in vertex placement. Also note that the mechanical behaviour of the original Laguerre tessellations is additionally affected by very short edges (see Fig. 1) which leads to stiffening in general.
LA vs. LT: $h(p) = -195.16p^3 + 201.01p^2 - 50.67p + 14.91$

Foam vs. LT: $g(p) = 106.07p^2 - 34.36p + 21.53$

Foam vs. LA: $f(p) = 60.65p^3 + 4.26$

Fig. 6 shows the influence of the cellular morphology on elasticity by illustrating the difference (in percent) $E_{\text{diff}}$ in Young’s modulus for each foam and its corresponding Laguerre tessellation and Laguerre approximation, respectively. In both cases the foams serve as reference for computing $E_{\text{diff}}$. For Laguerre tessellations with polydispersities $p$ up to 0.4 the effective Young’s moduli are 15% to 25% higher than for the corresponding foam with the minimum around $p \approx 0.15$. Increasing $p$ above 0.4 further increases the difference in Young’s modulus up to 35%. This relation follows a polynomial in $p$, i.e.

$$g(p) = 106.07p^2 - 34.36p + 21.53,$$

where the coefficients were obtained by regression analysis with a resulting
adjusted $R^2$ value of 0.94.

For Laguerre approximations with a polydispersity of up to about 0.25, the error induced in the effective Young's moduli by the Laguerre approximations is insensitive with respect to polydispersity. Within this range it is almost constant and smaller than 5%. The error then rapidly increases to about 17% with increasing polydispersity. This relation is well described by a cubic function of the polydispersity $p$, more precisely,

$$f(p) = 60.65p^3 + 4.26.$$  \hfill (10)

The coefficients of Eq. (10) were determined by regression analysis using weighted least squares with a resulting adjusted $R^2$ value of 0.89.

In summary, a Laguerre approximation model of a foam decreases the prediction error of the effective elastic modulus on average by 16% when compared to a Laguerre tessellation model.

We observe a similar decrease in effective Young's modulus when comparing Laguerre approximations and Laguerre tessellations, where the Laguerre approximations serve as reference for computing $E_{\text{diff}}$. Independently of polydispersity $p$, $E_{\text{diff}}$ stays in between 10% and 15% following a polynomial

$$h(p) = -195.16p^3 + 201.01p^2 - 50.67p + 14.91,$$  \hfill (11)

where the coefficients were obtained by regression analysis using weighted least squares with a resulting adjusted $R^2$ value of 0.75. Fig. 6 also indicates that Laguerre approximations are always softer than Laguerre tessellations due to the absence of very short edges (see Fig. 1) which increase stiffness and
may lead to numerical instabilities (Kanaun and Tkachenko, 2007; van der Burg et al., 1997).

4.3. Elastic Properties of Deterministic Models

For monodisperse foams, Laguerre approximations overpredict Young’s modulus by about 4% (see Figs. 6 and 7). However, in this special case, it might be viable to consider simpler deterministic models (i.e. models with ordered cell structure). Deterministic foam models are closely related to Kelvin’s problem (Thomson, 1887). It is simply stated: Partition the space into equal-volume polyhedral cells with minimal interface area.

For brevity, we restrict our study to structures that appeared in the context of solving Kelvin’s problem. The considered structures are illustrated in Fig. 7 along with their resulting constants for the effective elastic moduli as defined in Eq. (8). All models are studied in their relaxed form, i.e. the Surface Evolver was used to minimise interface area before the effective elastic moduli were computed. The values for the simulated foam and its Laguerre approximation are the median from the study presented in Section 4.1.

Kelvin conjectured that the orthic tetradecahedron is the most efficient space filling polyhedron. Its relaxed form, known as Kelvin cell, consists of eight slightly curved hexagonal and six flat quadrilateral faces. Warren and Kraynik (1997) already found $c = 0.979$ as the constant for the effective Young’s modulus of a Kelvin foam at the low-density limit. In the same year, Zhu et al. (1997) independently report a value of $c \approx 1$. Both results support the Gibson–Ashby scaling model for open-cell foams, where $c = 1$ was
Figure 7: Error when predicting Young’s modulus of a random monodisperse foam by deterministic models, i.e. the Kelvin (K) and Weaire–Phelan (A15) foam. For comparison we included the median result for monodisperse Laguerre approximations (LA) from Section 4.1.

estimated from empirical data (Gibson and Ashby, 1999). These results are in good agreement with our finding of $c = 0.978$. However, this underpredicts the effective Young’s modulus of a monodisperse random foam by about 14%.

The best known solution to the Kelvin problem was discovered by Weaire and Phelan (1994). They found that the cubic tetrahedrally closed-packed structure called A15 (in its relaxed form) has 0.34% less interface area than the Kelvin cell. A15, henceforth known as Weaire–Phelan foam, consists of two pyritohedra and six irregular tetradecahedra with two hexagonal and twelve pentagonal faces.

Fig. 7 shows the elastic properties of the the Weaire–Phelan foam in its relaxed form. It is easy to see that it is highly anisotropic with respect to elasticity. However, when considering its Hashin–Shtrikman average of
the orientation dependent moduli, the Weaire–Phelan foam overpredicts the effective Young’s modulus of a monodisperse foam only by about 2%. That means, despite of its deterministic cell morphology and anisotropic elastic features, it predicts the effective elastic properties of a random monodisperse foam quite well.

However, this result only holds for the relaxed Weaire–Phelan foam where cell curvature leads to out-of-plane vertex placement and, thus, nonconvex cells. When we consider the unrelaxed form of the monodisperse Weaire–Phelan foam with convex cells, it overpredicts the effective Young’s modulus by approximately 8%. It is easy to show that the unrelaxed monodisperse Weaire–Phelan foam is a normal tessellation of space, i.e. a subdivision of space into convex polyhedra that obeys the topology induced by Plateau’s laws (see Kusner and Sullivan, 1996). Consequently, it can be represented as a Laguerre tessellation (see Lautensack and Sych (2008) on how to define the generating spheres). As the prediction error of a Laguerre approximation of a random monodisperse foam (about 4%) lies in between the results of a relaxed and an unrelaxed Weaire–Phelan foam, we may conclude that the Laguerre approximation may be preferable when convex cells are desirable, the random cell morphology cannot be neglected or isotropic elastic properties are strictly necessary.
5. Summary and Conclusions

It was shown in Liebscher (2015) that the cellular morphology of foams is reproduced quite well by Laguerre approximations. For polydispersities below 0.45, Fig. 4 illustrates that even the local topology of the approximations is in astonishing agreement with the simulated foams. Within this range we observed at most 5% cells with incorrectly assigned neighbours in the complete structure. Hence, we may conclude that from a geometric point of view Laguerre approximations provide good models for mono- and low- to mid-polydisperse (low-density) foams. In the presented study we additionally considered the mechanical response of open-cell foams.

For monodisperse foams, we found the elastic behaviour of the Laguerre approximations in good agreement with the one of the simulated foams. The approximations were about 4% stiffer than the simulated foams. Hence, the percentage error is reduced by a factor of 2 compared to the 8% of an unrelaxed Weaire–Phelan foam, which itself can be represented as a Laguerre tessellation. However, the absolute deviation of the Laguerre approximations is about twice as high as the one of the relaxed Weaire–Phelan (A15) foam which yields an error of 2%.

Considering this result, A15 may seem as the ideal deterministic model for monodisperse foams. On the other hand, A15 is mechanically much more anisotropic than the Kelvin foam while also being geometrically more complex. Also both lack the morphological cell variations of a random monodisperse foam. So the optimal model choice depends much on the desired ap-
Moreover, deterministic models lack the ability to reproduce the cellular morphology of polydisperse foams which makes more complex models necessary. For low-polydisperse foams with $0 \leq p < 0.25$, we found the elastic behaviour of the Laguerre approximations in very good agreement with the one of the simulated foams. The approximations were at most 5% stiffer than the simulated foams. For mid-polydisperse foams with $0.25 \leq p < 0.5$, the error in Young’s modulus increases to 12% following a cubic function, despite the good geometric approximation.

This result indicates that with increasing polydispersity the influence of vertex displacement due to cell curvature (see Section 4.1) affects the elastic behaviour. In practice, however, foams barely exceed a polydispersity of 0.3. Laguerre approximations then provide realistic models for low-density (open-cell) foams. Consequently, our study shows (together with Liebscher, 2015) that it is possible to realistically model (low-density) foams by Laguerre tessellations.

Additionally the presented method provides a nonparametric approach to model foams where only the model class (Laguerre tessellation) needs to be specified but the parameters are adapted automatically. On average this approach leads to a 16% better prediction of Young’s moduls compared to the classical parametric model fitting approach that uses Laguerre tessellations parametrised by sphere packings (Lautensack, 2008), but comes at the price of a higher computational cost. The parametric models contain tiny
edges and faces as illustrated in Fig. 1(a) that may lead to numerical instabilities and blow-ups (see e.g. Kanaun and Tkachenko, 2007; van der Burg et al., 1997). These features can be removed by relaxing the structure as proposed by Vecchio et al. (2016). Studying the generating sets of Laguerre approximations of simulated foams may allow to suggest parametric models which yield more realistic cell structures.

Acknowledgements

We would like to thank A. M. Kraynik for valuable comments and proof reading.

References


Hashin, Z., Shtrikman, S., 1962. A variational approach to the theory of


Thomson, W., 1887. On the division of space with minimum partitional area. Philosophical Magazine 24, 503–514.


