Second order directional analysis of point processes revisited

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Abstract

Various methods for directional analysis of spatial point patterns have been introduced in the literature. In this paper, we formulate a unifying framework for methods based on integral transforms of the second order product density. Examples include directional versions of Ripley’s K-function, wavelet transforms, and spectral analysis. Furthermore, we propose an additional method based on the projection of the Fry points of the point pattern on the unit sphere. This method solves some of the practical problems appearing in the application of the integral transform. In particular, it can be easily applied in 3D and provides a definition of an isotropy test without the need of replicated data. The method is evaluated in a simulation study.

Keywords: Anisotropy test, Fry plot, Geometric anisotropy, Projection, Second-order characteristics, Spatial summary statistics, Spectral analysis, Wavelets

1. Introduction

Rajala et al. (2018) give an overview of non-parametric methods for a directional analysis of 2D and 3D stationary point processes based on nearest neighbour and second order summary statistics, spectral theory and wavelets. The authors illustrate the performance of the methods in detecting isotropy and finding preferred directions both in clustered and regular point patterns and review the existing isotropy tests found in the literature. Two types of anisotropy, anisotropy caused by a linear transformation of a stationary and isotropic process (geometric anisotropy), and anisotropy caused by increased intensity of points along directed lines, are considered.

This paper concerns directional analysis of geometric anisotropic stationary point processes by using second order methods. We introduce a unifying framework for methods based on integral transforms of the second-order product density, including methods based on second order summary statistics, spectral theory and wavelets. There are some difficulties when using these methods, namely how to choose the tuning parameters and grids of directions, how to orthogonalize the estimates of the transformation axes (compression and dilation directions) in 3D, and how to find the distribution of the test statistics under the general isotropy hypothesis which is needed to define an isotropy test.

To overcome some of the difficulties mentioned above, we introduce a new second order method which is based on projecting the Fry points of the point pattern, i.e. the pairwise difference vectors between all point pairs, on the unit sphere. A similar idea in 2D can be found in Illian et al. (2008, page 254). We describe how this method can be used in visualising anisotropy, testing isotropy and estimating the directions and the corresponding strengths of deformation. Within this framework, we perform a 2D and 3D simulation study to evaluate the performance of the method for regular geometric anisotropic point patterns. We also compare the method with the methods based on integral transforms of the second order product density and the ellipsoid method introduced in Rajala et al. (2016), which fits ellipses/ellipsoids to the Fry points.

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2. Definition of anisotropy

We denote by \( b(x, r) \) the ball in \( \mathbb{R}^d \), \( d = 2, 3 \), with center \( x \) and radius \( r > 0 \). Furthermore, \( S^{d-1} = \{ x \in \mathbb{R}^d : ||x|| = 1 \} \) is the unit sphere and \( S_+^{d-1} \) the half unit sphere, which consists of the elements of \( S^{d-1} \) with nonnegative last component. We moreover let \( SO_d \) be the special orthogonal group of rotations about the origin.

In the following, we assume that we observe a point pattern \( \mathbf{x} = \{ x_1, \ldots, x_n \} \) with \( n > 2 \) points in a bounded window \( W \subseteq \mathbb{R}^d \) with volume \( |W| \). The point pattern \( \mathbf{x} \) is a realization of the intersection \( X_W := X \cap W \), where \( X \) is a simple (no multiple points) stationary point process.

Typical tasks in anisotropy analysis are

1. testing whether an observed point pattern can be assumed to be isotropic, and
2. characterizing an observed anisotropy, e.g. estimating preferred directions or the strength of anisotropy.

For formalizing the latter, we will assume a special anisotropy mechanism called geometric anisotropy (Møller and Toftaker, 2014). That is, the observed point pattern \( X \) is modelled as

\[
X = RCX_0, \tag{1}
\]

where \( X_0 \) is a stationary and isotropic point process, \( R \) is a rotation matrix and \( C \) is a diagonal matrix. We assume that \( C \) has determinant 1, so that the transform is volume preserving. The matrix \( C \) stretches/compresses \( X_0 \) along the coordinate axes while \( R \) rotates the deformed process \( CX_0 \). The axes of deformation of \( X \) are then the images of the coordinate axes after rotation by \( R \). They form an orthonormal system denoted by \( u_i, i = 1, \ldots, d \). Note that in 2D, \( R \) is completely determined by the rotation angle \( \theta \).

Figure 1 shows two anisotropic point patterns, one regular and one clustered. In this case, \( R \) is the identity matrix and the compression matrix \( C \) is of the form

\[
C = \begin{pmatrix} 1/c & 0 \\ 0 & c \end{pmatrix}
\]

with \( c = 0.5 \). The ground process \( X_0 \) for the regular case is chosen as a Strauss process with 300 points, regularity parameter \( \gamma = 0 \) and interaction radius \( r_0 = 0.04 \). For the clustered case, \( X_0 \) is chosen as a Matérn cluster process with cluster radius 0.03, intensity of the Poisson process of cluster centers equal to 10 and with an average of 40 points per cluster. The deformed processes are observed in the unit square \( W = [0, 1]^2 \). In the clustered case, anisotropy can be detected from the shape and orientation of clusters. In contrast, the plot of the regular point pattern does not immediately reveal any preferred directions.

Figure 1: Example patterns obtained by compressing stationary and isotropic regular Strauss (left) and Matérn cluster (right) patterns by factor 0.5 in the \( y \) direction and stretching them with factor 2 in the \( x \) direction.
3. Second order anisotropy analysis using integral transforms

3.1. Definitions

Second order summary statistics for point processes are based on the second-order factorial moment measure

$$\alpha^{(2)}(A \times B) = \mathbb{E} \left( \sum_{x,y \in X} 1(x \in A)1(y \in B) \right)$$

for Borel sets $A$ and $B$ in $\mathbb{R}^d$. The upper $\neq$ means that only pairs with $x \neq y$ are considered in the sum. We assume that the (second order) product density $\rho^{(2)}$ exists, i.e.

$$\alpha^{(2)}(A \times B) = \int_A \int_B \rho^{(2)}(x,y) dy dx.$$

The value of $\rho^{(2)}(x,y)dx dy$ gives the probability that $X$ has a point in each of the infinitesimally small discs with centers $x$ and $y$ and volumes $dx$ and $dy$, respectively. For stationary $X$, we have that

$$\rho^{(2)}(x,y) = \rho^{(2)}(o,y-x) =: \rho^{(2)}(z),$$

where $z = y - x$.

Under the additional assumption of isotropy, $\rho^{(2)}(z) =: \rho^{(2)}(||z||) \forall z \in \mathbb{R}^d$. (2)

Note that to simplify notation we use $\rho^{(2)}$ for all three cases.

Choose a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, which we call the template function, and define the family of rotated versions of $f$ via \( \{ f_R(\cdot) := f(R^{-1}(\cdot)), R \in SO_d \} \). Additionally, assume that

(P1) $f_R \rho^{(2)} \in L^1(\mathbb{R}^d, \mathbb{R}) \forall R \in SO_d$, and

(P2) $\exists R \in SO_d$ such that $f_R \neq f$.

Due to property (P1) we can define the coefficients

$$S(R) := \int_{\mathbb{R}^d} f_R(z) \rho^{(2)}(z) dz \quad R \in SO_d.$$ (3)

Property (P2) ensures that the coefficients $S(R)$ are informative for a directional analysis. Under the hypothesis of isotropy, the coefficients $S(R)$ do not depend on $R \in SO_d$ for any choice of $f$ and any significant variation of the coefficients for different rotations indicates anisotropy.

We can simplify the analysis by additionally assuming that the function $f$ is invariant with respect to rotations about one particular axis. Without loss of generality, this axis is chosen to be the x-axis $u_x$.

(P3) $f_R = f, \quad \forall R \text{ s.t. } Ru_x = u_x$

In 2D property (P3) is always trivially satisfied. Assuming (P3), it is sufficient to study the coefficients

$$S_u := \int_{\mathbb{R}^d} f_u(z) \rho^{(2)}(z) dz, \quad u \in S^{d-1},$$ (4)

where we define $f_u(\cdot)$ as $f(R^{-1}(\cdot))$ with $Ru_x = u$.

For even template functions $f$ we additionally have

(P4) $f_u = f_{-u} \forall u \in S^{d-1}$

such that the coefficients $S_u$ only have to be computed for $u \in S^{d-1}$ (exploiting the fact that $\rho^{(2)}(z) = \rho^{(2)}(-z)$).

A generalization of the Campbell theorem (Illian et al., 2008, p. 228) states that

$$\mathbb{E} \sum_{x,y \in X} h(x,y) = \int \int h(x,y) \alpha^{(2)}(d(x,y))$$ (5)
for all suitably behaving functions $h$. Hence, an unbiased estimator for $S_u$ is given by

$$
\hat{S}_u := \frac{1}{|A|} \sum_{x \in A, y \in X} f_u(y - x) \quad u \in S^{d-1}
$$

for all bounded Borel-sets $A$. The estimators $\hat{S}_u$ sum up the values of the rotated template function $f_u$ at the pairwise difference vectors of the set

$$
Z_A := \{ y - x : \quad x \neq y, x \in X_A, y \in X \},
$$

which is locally finite due to the boundedness of $A$. We call $Z_A$ the set of Fry points of the stationary process $X$ (Fry, 1979). In practice, sampling of the Fry points will suffer from edge effects since only the differences vectors

$$
Z = \{ y - x : \quad x \neq y, y \in X_W \}
$$

can be observed. In the following, the set $Z$ will also be called the set of Fry points of $X$. The observation window of $Z$ will be denoted by $W^*$. An edge corrected estimator of $S_u$ using translational edge correction weights can be defined by

$$
\hat{S}_u := \sum_{x, y \in X_W} f_u(y - x) \quad |W_y \cap W_x| \quad u \in (S^{d-1})^+.
$$

Again, unbiasedness of the estimator follows from (5). If $X$ is isotropic and if $W^* = b(c, r)$ for some $r \in \mathbb{R}^+$, $c \in \mathbb{R}^d$, the distribution of $Z$ is rotationally symmetric about the origin, i.e.

$$
RZ \overset{d}{=} Z \quad \text{for all } R \in SO_d.
$$

If $W^*$ is not a ball, we can always consider the largest ball which is contained in $W^*$. Consequently, under isotropy, the distribution of the estimators $\hat{S}_u$ does not depend on the direction $u$.

Figure 2: Fry points of the example patterns in Figure 1 restricted to the window $[-0.11,0.11] \times [-0.11,0.11]$. The regular case is shown on the left, the clustered case on the right.

In Figure 2 we show the Fry points corresponding to the example patterns in Figure 1. The distribution of the Fry points is clearly not rotationally invariant in either case, indicating anisotropy. The Fry points of the regular pattern show an elliptical void space centered in the origin with minor axis equal to the axis of compression and major axis equal to the axis of dilation. In the clustered case the Fry points accumulate in the center of the plot, reflecting the shape of the clusters.

We considered the Fry points in the restricted window $[-0.11, 0.11] \times [-0.11, 0.11]$. Assuming ergodicity of $X$, we have that $\rho^{(2)}(z) \to \lambda^2$ as $||z|| \to \infty$ (Illian et al., 2008, page 220). Hence, the Fry points lose anisotropy information as $||z|| \to \infty$. Anisotropy analysis based on the Fry points, therefore, involves the choice of an appropriate scale. For regular point patterns, Fry points at distances close to the interaction radius are most informative (Redenbach et al., 2009; Sormani, 2019). For clustered patterns, distances corresponding to the size of the clusters should be investigated.
3.2. Choice of $f$

The framework formulated in the previous section includes several established summary statistics as special cases. These are summarized in the following. For further details and a practical evaluation of the approaches we refer to Rajala et al. (2018); Sormani (2019).

3.2.1. Directional K-functions

When $f(z) = I_{b(0,r)}(z)$ is the indicator function of the ball $b(0,r)$, we have $S_u(r) = K(r)$ independent of $u$ where $K$ is Ripley’s K-function. Due to the rotational symmetry of the ball, this choice is not helpful for directional analysis as (P2) is not satisfied. Therefore, we will replace the ball by rotation variant sets $B \subset \mathbb{R}^d$. If $B$ is rotationally invariant with respect to the x-axis $u_x$ and point symmetric w.r.t. the origin, then $f(z) = I_B(z)$ satisfies the properties (P3) and (P4). Let $B^r = RB$ with $R$ being a rotation such that $Ru_x = u$. With this choice, Equation (4) reduces to

$$S_u = \int_{B^r} \rho^{(2)}(z) dz \quad u \in \mathbb{S}_+^{d-1},$$

which can be estimated by

$$\hat{S}_u := \sum_{x,y \in X} \frac{I_{B^r}[y-x]}{|W_y \cap W_x|} \quad u \in \mathbb{S}_+^{d-1}.$$  \hspace{1cm} (11)

Several different choices of such $B$ have been discussed in the literature. For instance, $B^r$ can be chosen as a double conical sector with main axis $u$, opening angle $\epsilon$ and radius $r$. In 2D this choice has been introduced in Stoyan (1991) and the 3D version can be found in Redenbach et al. (2009). Often, in analogy to Ripley’s K-function, the coefficients $S_u$ are considered as functions of the radius $r$ which yields the conical K-function. This allows to analyse the behaviour of the point pattern at variable distance. Here, we will fix $r$ at some informative range. The additional parameter $\epsilon$ plays the role of a smoothing parameter, similar to a bandwidth in density estimation, as it determines how far Fry points are allowed to depart from the direction $u$ to be included in the analysis. Another possibility is to choose $B$ as a rectangle (2D) or cylinder (3D) centred in the origin. In these cases, both (P3) and (P4) are satisfied and the functions $f_u$ depend on two additional parameters: the height $h$ and the base radius $r$ of the cylinder. Letting $r$ vary, the cylindrical K-function is obtained, see Møller et al. (2016). Contrary to the conical K-function, $r$ plays the role of a smoothing parameter while the height $h$ measures range. Note, however, that the cylindrical K-function was designed to characterize columnar structures, where $r$ would indeed describe the range (width of the columns). A difference between using cones or cylinders is that the cones in different directions $u$ can be chosen non-overlapping.

A third option is obtained by choosing $B^r$ to be an ellipse/ellipsoid with major axis $u$ yielding the elliptical K-function (Sormani, 2019). In this case, the length of the major axis determines the range, while the length of the minor axis plays the role of the smoothing parameter. In the regular case, the best choice of ellipse is the one fitting the void shape in the center of the Fry points as in Figure 3 (Sormani, 2019). Additional smoothing can be introduced by replacing the indicator of the ellipse with a Gaussian kernel with the same shape (for details see Sormani (2019)).

Figure 3: Some template functions $f$. From left to right: indicator function of a cone, indicator function of an ellipse, and elliptic Gaussian kernel overlaid with the Fry points of the regular example pattern from Figure 1.
In all cases, the coefficients \( \hat{S}_u \) for the regular pattern assume their maximum and minimum value in the direction of compression and dilation, respectively. In the clustered case we have the opposite situation. Consequently, estimators for the directions of deformation are obtained by searching for the minimum/maximum values of \( \hat{S}_u \) over a grid of directions, see Redenbach et al. (2009); Rajala et al. (2018); Sormani (2019).

3.2.2. Wavelet analysis

Another choice of \( f \) is a wavelet function \( \psi_{a,b} \) on \( \mathbb{R}^d \), where \( \psi \) denotes some mother wavelet and

\[
\psi_{a,b}(x) := \frac{1}{\sqrt{a^d}} \psi \left( \frac{x-b}{a} \right) \quad a \in \mathbb{R}^+, b \in \mathbb{R}^d.
\]  

(12)

Here \( a \) is a scaling parameter, while \( b \) is a translation parameter which can be set to zero in the stationary case. The multiplication by \( 1/\sqrt{a^d} \) is added so that

\[
||\psi_{a,b}||_{L^2} = ||\psi||_{L^2} = 1.
\]

Properties (P3) and (P4) are satisfied if the mother wavelet \( \psi \) satisfies them. Assuming that (P3) is satisfied, the choice \( f_u = (\psi_{a,b})_u := \psi_{u,a,b} \) in Equation (4) constitutes a \( d \)-dimensional directional wavelet transform of the product density \( \rho^{(2)} \). Note that for complex valued wavelets the functions \( \psi_{a,b} \) need to be conjugated and that, in order to satisfy property (P2), the mother wavelet \( \psi \) has to be anisotropic. For a practical example using the Morlet mother wavelet see Sormani (2019).

Note that this approach differs from the directional wavelet transforms used in the context of anisotropy analysis of spatial point processes so far (D’Ercole and Mateu, 2013a,b, 2014; Mateu and Nicolis, 2015). In these papers, stationarity of the point process is not assumed. The wavelet transform is applied to the analysis of spatial point processes so far (D’Ercole and Mateu, 2013a,b, 2014; Mateu and Nicolis, 2015).

For a practical example using the Morlet mother wavelet see Sormani (2019).

3.2.3. Spectral analysis

Mugglestone and Renshaw (1996) propose a method for anisotropy analysis based on the Fourier transform

\[
F(\omega) := \int_{\mathbb{R}^d} \kappa(z) \exp(-i\omega^T z) dz, \quad \omega \in \mathbb{R}^d
\]  

(16)

of the complete covariance function

\[
\kappa(z) := \lambda \delta(z) + \rho^{(2)}(z) - \lambda^2, \quad z \in \mathbb{R}^d,
\]

where \( \delta(\cdot) \) is the Dirac delta function.

Defining \( u := \omega/||\omega|| \) and \( r = ||\omega|| \), Equation (16) can be interpreted as a variant of (4) with \( f_u(z) = e^{-iru^T z} \) depending on the scale parameter \( r \). The product density \( \rho^{(2)} \) is replaced by the complete covariance function \( \kappa \) to assure integrability (property (P1)). In general, the coefficients \( S_u \) could also be defined based on \( \kappa \) rather than \( \rho^{(2)} \). However, in this case it is more difficult to define unbiased estimators when observing \( X \) in a finite window. Estimation of \( F(\omega) \) is typically based on the periodogram, see Rajala et al. (2018) for a summary.
3.3. Comments

Analysis using the coefficients $S_u$ is quite flexible since we can choose the shape of $f$ and the analysed directions $u$ depending on prior information on the available data (see e.g. Redenbach et al. (2009) and Möller et al. (2016)). However, it has some drawbacks.

First of all, the functions $f_u$ depend on at least two parameters, a range and a smoothing parameter. Though informative ranges can be known or estimated from the data, the choice of the smoothing parameter is not so obvious, see Sormani (2019) for a simulation study addressing this question. Additionally, one needs to choose a discrete grid of directions $U$. The denser the grid, the more precise are the results at the price of increased computational cost.

Another drawback concerns the estimation of preferred directions in 3D in the context of geometric anisotropy. As outlined above, the coefficients $S_u$ assume their maximum and minimum value, respectively, in the direction of (the strongest) compression and (the strongest) dilation. In the model stated above, the directions of transformation form an orthonormal system. In contrast, the estimated maximum and minimum directions need not be orthogonal to each other. While in 2D it is easy to orthogonalize the solutions, this is not so obvious in 3D.

A test for isotropy can be based on test statistics contrasting the coefficients $S_u$ for different choices of $u$. The hypothesis of isotropy is rejected if the values of this statistic are too large. However, the distribution of the estimators $\hat{S}_u$ is in general not known under the hypothesis of isotropy. Hence, the critical values of the test cannot be determined analytically. If a null model of isotropy is known, a solution is to estimate the distribution of $\hat{S}_u$ via Monte Carlo simulations. A model free alternative is to use the reconstruction algorithm of Tschöchel and Stoyan (2006) to simulate isotropic patterns with some predefined isotropic summary statistics close to the ones of the observed pattern. If replicated data are available, one can also use the approach from Redenbach et al. (2009).

4. Projection method

In what follows we introduce a new method which solves most of the problems listed in Section 3.3: it is parametrised by ranges only, leads to orthogonal estimates of the preferred directions in the context of geometric anisotropy, and allows for a straightforward definition of an isotropy test. The idea is to consider projections of the Fry points in a certain range to the unit sphere. Let

$$ P_{r_1,r_2}(Z) := \left\{ \frac{z}{||z||} : z \in Z \text{ s.t. } r_1 \leq ||z|| \leq r_2 \right\} \subset S^{d-1} \text{ where } 0 \leq r_1 < r_2. \quad (17) $$

Since the Fry points are symmetric with respect to the origin, it is enough to consider

$$ P_{r_1,r_2}^+(Z) := P_{r_1,r_2}(Z) \cap S^d_+. \quad (18) $$

In practice, the range parameters $0 \leq r_1 < r_2$ have to be chosen depending on the specific application. In particular, one has to guarantee that $P_{r_1,r_2}(Z)$ contains enough points.

The projection method can be used to visualize anisotropy, test isotropy, and estimate the direction and strength of compression/stretching.

4.1. Visualizing anisotropy in 3D

Kernel estimates of the density of the projections $P_{r_1,r_2}(Z)$ on the sphere can be used for visual inspection. In Figure 4, we can see the estimates for three 3D Strauss patterns with $R = 0.06$ and 1000 points in the unit window after a geometric anisotropic transform. In two of the cases (left and middle) we consider the Fry points in the window $W^* = [-0.11,0.11]^3$, while in one (right) the window is $W^* = [-0.15,0.15]^3$. We use the transformation model

$$ C = \begin{pmatrix} c_x := \frac{c_2}{\sqrt{c_1}} & 0 & 0 \\ 0 & c_y := c_1 & 0 \\ 0 & 0 & c_z := \frac{1}{\sqrt{c_1 c_2}} \end{pmatrix}. \quad (19) $$

The pattern on the left is isotropic, i.e. $c_x = c_y = c_z = 1$. The pattern in the middle is resulting from a spheroidal transform with $c_1 = 0.5$ and $c_2 = 1$ which gives $c_x = c_z \approx 1.4$ and $c_y = 0.5$. The pattern
on the right corresponds to the transform with $c_1 = 0.5$ and $c_2 = 0.7$ which gives $c_x \approx 0.9$, $c_y = 0.5$, $c_z \approx 2$. As kernel function in the density estimation we use the density of the Fisher distribution with scale parameter equal to 10. Since in the regular case informative distances are just above the interaction radius, see e.g. Redenbach et al. (2009), and since we do not have noise, we fix $r_1 = 0$ and $r_2 = 0.07$. In the isotropic case (left), the distribution of the projections is the uniform distribution on the sphere. In the case of the spheroidal transform (middle), the direction of compression $y$ can be found by looking at the direction where the density assumes its maximum value. Density values are constant along the great circle formed by intersecting the sphere with the $xz$ plane. The plot of the non-spheroidal transform (right) is similar to the plot in the middle. However, the density takes smaller values in the direction of elongation $z$ than in the direction of weaker compression $x$.

Figure 4: Heat map of the kernel estimated intensity of the projections $P_{r_1 r_2}(Z)$ of three Strauss patterns subjected to geometric anisotropy: isotropic (left), compressed in $y$ direction (middle), strongly compressed in $y$ direction and weakly compressed in $x$ direction (right). Red corresponds to high and yellow to low values of the density.

4.2. Testing isotropy

Under the hypothesis of isotropy, the projections $P_{r_1 r_2}(Z)$ are uniformly distributed on $S^{d-1}$ for all $r_1 < r_2$. To test isotropy, we can therefore simply test uniformity of the projections. Several tests for uniformity on the sphere are available in the literature. Typically, they require the formulation of a specific alternative hypothesis. Assuming geometric anisotropy and regularity, a suitable alternative distribution is the angular central Gaussian (ACG) distribution (Mardia and Jupp, 2009, page 182). The score test of uniformity against a general ACG distribution is also known as Bingham test (Mardia and Jupp, 2009, page 232). It is based on the test statistic

$$S = \frac{d(d+2)}{2} m \left( \text{tr}\left( \frac{1}{m} O \right)^2 - \frac{1}{d} \right),$$

where $O = \sum_{i=1}^{m} x_i x_i^T$ is the orientation matrix and $x_i$ denotes the unit vector obtained by projecting the $i$-th Fry point. Note that the Bingham test assumes independence of the $x_i$’s, an assumption that is not fulfilled by the projections $P_{r_1 r_2}(Z)$. Nevertheless, we will apply the test.

Under the hypothesis of uniformity, $S$ is asymptotically $\chi^2_{(d-1)(d+2)/2}$ distributed with an error of $O(m^{-1})$. The approximation can be improved by using the modified Bingham statistic

$$\tilde{S} = S \left\{ 1 - \frac{1}{m} [B_0 + B_1 S + B_2 S^2] \right\}$$

where

$$B_0 = \frac{2d^2 + 3d + 4}{6(d + 4)}$$
$$B_1 = \frac{-(4d^2 + 3d - 4)}{3(d + 4)(d^2 + d + 2)}$$
$$B_2 = \frac{4(d^2 - 4)}{3(d + 4)(d^2 + d + 2)(d^2 + d + 6)}.$$ 

Then $\tilde{S}$ is asymptotically $\chi^2_{(d-1)(d+2)/2}$ distributed with an error of $O(m^{-2})$. The power of the test will depend on the type of point process, the strength of compression, and the choice of the radii $r_1$ and $r_2$. 

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4.3. Estimating the compression direction

As can be seen in Figure 4, the distribution of the points of $P_{r_1, r_2}(Z)$ carries information on the directions of transformation under the assumption of geometric anisotropy. In the regular case, the intensity has the maximum concentration of points along the axis of the strongest compression, the minimum concentration along the axis of dilation and weaker concentration along the axis of the weaker compression. As discussed in Fisher et al. (1993, page 162), the axes of deformation can be estimated by computing the eigenvectors of the orientation matrix $O$. In the regular case, the eigenvector corresponding to the smallest eigenvalue gives the axis of the strongest dilation, while the eigenvector corresponding to the largest eigenvalue gives the axis of the strongest compression. In the case of a spheroidal transformation (two axes of equal compression or equal dilation), two eigenvalues of the orientation matrix $O$ take similar values and we only need to consider the remaining eigenvalue and the corresponding eigenvector. In the case of isotropy, all eigenvalues have similar values. Edge effects due to the observation of $X$ in a finite window $W$ can be taken into account by weighting each Fry point with its translational edge corrected weight. The modification for the clustered case is straightforward. Note that this procedure will immediately result in an orthonormal system of compression directions.

An alternative approach for estimating the directions of transformation could be to fit an angular central Gaussian (ACG) distribution to the projections. The directions of transformation are then obtained from the eigenvectors of the estimated matrix parameter $\Sigma$. According to Tyler (1987); Franke et al. (2016), this approach will give similar results as the one presented above but with higher computational cost.

4.4. Estimating strength of compression

For estimating the compression matrix $C$ we assume that the rotation matrix $R$ is known or has already been estimated. Then we can rotate the observed process $X$ obtaining the process $Y := R^{-1}X \sim CX_0$ whose deformation axes are the coordinate axes. Using a grid $C_k$, $k = 1, \ldots, l$, of possible guesses of $C$ we invert the compression to obtain $Y_k := C_k^{-1}Y$. The matrix $C$ is then estimated by $\hat{C}_k$, such that $Y_k$ is the "most isotropic" process among the $Y_k$'s. To find $\hat{k}$ we compute the projections $P_k := P_{r_1, r_2}(Z_k)$ for an appropriate choice of $r_1$ and $r_2$, where the $Z_k$'s are the Fry points of $Y_k$. Let $t_k$ be the score of the Bingham test for testing uniformity of the projection $P_k$. We then define

$$\hat{k} := \arg \min_k (t_k).$$

Note that the estimation of the strengths of compression can be refined iteratively by increasing the resolution of the grid around the solutions of the previous iteration.

5. Simulation study

In this section, we perform a simulation study to see how the projection method works in practise. We consider realizations of regular Strauss processes with a fixed interaction radius, $r_0 = 0.04$ in 2D and $R = 0.06$ in 3D, average number of points equal to 300 in 2D and 1000 in 3D and regularity parameter $\gamma$ chosen as either 0, 0.1, or 0.3 both in 2D and in 3D. The observation window is the unit square/cube. The 2D pattern is transformed by using the matrix $C = \text{diag}(1/c, c)$ where $0 \leq c \leq 1$ is the strength of compression, and rotated by the angle $\theta = \pm \pi/4$. The 3D pattern is transformed by using the matrix $C$ given in Equation (19), assuming that $0 < c_1 \leq \frac{c_2}{\sqrt{3}} < 1$, and without any rotation ($R$ is the identity matrix). Consequently, the $y$-axis is the axis $u_3$ with the strongest compression, the $x$-axis corresponds to the axis $u_2$, and the $z$-axis to the axis $u_1$ with dilation. The strengths of compression $c_1$ and $c_2$ are chosen as

1. $c_1 = 1$, 0.9, 0.8, 0.7, 0.6, 0.5 and $c_2 = 1$ (spheroidal transform)
2. $c_1 = 0.5$ and $c_2 = 0.7$ (non-spheroidal transform).

In 2D $c$ is chosen as $c_1$ in 1. above. For each set of parameters, we simulate 100 patterns.
5.1. Testing isotropy

For the isotropy tests based on the Bingham test, we choose \( r_1 = 0 \) and let \( r_2 \) vary in a grid from 0.045 to 0.06 in steps of 0.005 in 2D and in a grid from 0.06 to 0.08 in steps of 0.005 in 3D. In both cases, these values are slightly larger than the interaction radius of the point process. The significance level is set to \( \alpha = 0.05 \).

The powers for the different choices of regularity parameter \( \gamma \) and strength of compression are shown in Figure 5 for the 3D spheroidal case. The plot for 2D looks very similar and is omitted here. As expected, the power of the test decreases as \( \gamma \) increases, as we have less regularity, and increases as \( c_1 \) decreases. For \( c_1 = 0.9 \), the power is close to 1 only when the smallest radius, just above the interaction radius \( r_0 \), is used and then it drops down rapidly. For values of \( c_1 \leq 0.7 \) the power is always close to 1.

We additionally note that for larger values of \( r_2 \) and \( c \) or \( c_1 \) taking values between 0.8 and 0.9, increasing the lower radius \( r_1 \) increased the power slightly (results not shown).

Figure 5: Power of the test based on the projection method in 3D depending on \( \gamma \), \( r_2 \) and \( c_1 \). Note that the curves representing \( c_1 \) values 0.7, 0.6, and 0.5 are almost on top of each other in the left and middle plots.

5.2. Estimating \( R \)

For estimating \( R \), we choose \( r_1 = 0 \) and let \( r_2 \) vary in a grid from 0.04 to 0.08 in steps of 0.005 in 2D and in a grid from 0.06 to 0.12 in steps of 0.005 in 3D. We do not correct for edge effects since for the chosen values of \( r_1 \) and \( r_2 \) the translational edge correction weights are in general negligible.

In Figure 6, we show the boxplots of the estimates of \( \theta = 3\pi/4 \) with different values of \( \gamma \) and \( r_2 \) in the 2D case for \( c = 0.9 \), 0.7, and 0.5. The estimates are unbiased (boxplots centered at the correct value) and the variance tends to increase with increasing \( \gamma \) and \( c \). The variance is smallest when \( r_2 \) is slightly bigger than the interaction radius \( r_0 \), namely when \( r_2 = 0.045 \). In the case \( c = 0.5 \), the choice of \( r_2 \) does not seem to be crucial.

The 3D results in the spheroidal case, where only the axis of compression \( u_3 \) needs to be estimated, are shown in Figure 7 and the results in the non-spheroidal case in Figure 8. Different values of \( r_2 \) give similar results, only the variance of the estimates is affected. Therefore, only the results for \( r_2 = 0.07 \), which results in the smallest variance, are shown. Note that both in 2D and in 3D, the best results are obtained by using \( r_2 \approx 1.1r_0 \). The projections of the estimated compression directions \( \hat{u}_3 \) on the \( xz \) plane in the spheroidal case are plotted in Figure 7 and the projections of \( \hat{u}_3, \hat{u}_2 \) and \( \hat{u}_1 \) on the \( xz \), \( yz \) and \( xy \) planes, respectively, in the non-spheroidal case in Figure 8. Also in 3D, we obtain unbiased estimates for the directions (projections centered around the origin) and the variance of the estimates increases with increasing \( \gamma \) and \( c_1 \). However, in the non-spheroidal case, the projections of the estimates of \( u_3 \) and \( u_2 \) on the \( xz \) and \( xy \) planes, respectively, show a larger variance in the \( x \) direction than in the \( z \) and \( y \) directions. This is expected e.g. by looking at the right plot in Figure 4.

When working with real data, the interaction radius \( r_0 \) has to be estimated. We recommend trying several values of \( r_2 \) starting from 1.1\( r_0 \) while assureing that the projections contain enough points.
Figure 6: Boxplots of the estimates of $\theta$ with $c$ equal to 0.9 (left), 0.7 (middle), and 0.5 (right). The colors indicate different values of $\gamma$ and the x-axis corresponds to different values of $r_2$. The true direction of compression $\theta = \pi/4$ is highlighted with a green horizontal line.

Figure 7: Spheroidal transform in 3D. Plot of the projections of the estimates of the direction of compression $\hat{\gamma}_3$ on the $xz$ plane for $r_2 = 0.07$. Each plot corresponds to a specific value of $c_1$. Different colors indicate different values of $\gamma$.

Figure 8: General transform in 3D. Projections of the estimates of the axes of deformation $\hat{a}_3$, $\hat{a}_2$, and $\hat{a}_1$ on the $xz$, $yz$, and $xy$ planes when $r_2 = 0.07$. Different colors indicate different values of $\gamma$.

5.3. Estimating C

We assume that the true rotation matrix $R$ is known and fix $r_1$ and $r_2$ as in the isotropy test. We choose a grid from 0.4 to 1.05 in steps of 0.01 both for $c$ in 2D and $c_1$ in 3D in the spheroidal case. For $c_2$ the grid starts at 0.7 instead of 0.4. For the general transform in 3D, we choose a grid from 0.3 to 0.7
in steps of 0.01 for \( c_1 \) and from 0.5 to 0.9 in steps of 0.05 for \( c_2 \). Note that the chosen grids contain the true values of the strengths of compression. In a real application, this would not necessarily be the case. The estimates can be iteratively improved by increasing the resolution of the grid around the estimate of the previous step.

In Figure 9, we show the boxplots of the estimates of \( c \) in 2D depending on \( \gamma \) and \( r_2 \) for \( c = 0.9, 0.7, \) and 0.5. In Figure 10, we show the same boxplots for the estimates of \( c_1 \) and \( c_2 \) for the spheroidal case and in Figure 11, for the non spheroidal case.

Figure 9: Boxplots of the estimates of \( c \) in 2D. The x-axis corresponds to different values of \( r_2 \).

Figure 10: Boxplots of the estimates of \( c_1 \) and \( c_2 \) in 3D in the spheroidal transform. The x-axis corresponds to different values of \( r_2 \).
The projection method has in general a tendency to overestimate the strengths of compression, i.e. to underestimate $c_1$, $c_1$, and $c_2$, especially when $r_2$ is large and the compression strong. However, underestimation of $c_2$ is not obvious in the spheroidal transform case. As expected, the variance of the estimates increases as $\gamma$ increases and the compression becomes weaker. The smallest variance is obtained with values of $r_2$ which are directly above the interaction radius.

5.4. Comparison to other second-order methods

As mentioned earlier, we proposed the projection method to overcome some of the drawbacks of the methods that are based on integral transforms of the second-order product density. Here, we compare its performance with the performance of the integral transform method when estimating the rotation matrix $R$ in 2D. The template function $f$ in the integral transform method is either the indicator function of a cone, the indicator function of an ellipse, an elliptic Gaussian kernel, or a Morlet wavelet function as defined in Sormani (2019). In addition, we include another second-order method, the ellipse fitting method from Rajala et al. (2016) which is based on a least-squares fit of ellipsoids (ellipses in 2D) to the contours of directed cumulative counts from the center of the Fry point pattern. In 3D, we only compare the projection method and the ellipsoid method since they automatically yield an orthonormal system of directions. In all cases, we fix the parameters of the methods according to the recommendations in Sormani (2019) assuming that the interaction radius is known.

All methods included in the comparison result in unbiased estimates of the directions of transformation, see Sormani (2019) and Rajala et al. (2016). The variance of the estimates of $\theta$ for the different methods in 2D is shown in Figure 12. We can see that as $c$ increases, the projection method and the integral transform using an elliptical shape are most stable. For the 3D case, boxplots for the angle distances between the estimated and the true axes of deformation are shown in Figure 13 for the spheroidal case and in Figure 14 for the general transform. The ellipsoid method seems to perform slightly better than the projection method both in the spheroidal and in the general case. An advantage of the projection method is, however, that only the range parameters have to be chosen. There is no need to choose contour levels, grid of directions and smoothing parameters as in the ellipsoid method.
Figure 12: Empirical variance of the estimates of $\theta$ in 2D with the best choice of the parameters for the considered methods. The three plots correspond to different values of $\gamma$ and the $x$-axis to different values of compression.

Figure 13: Boxplots of the angle distances between $u_3$ and $\hat{u}_3$ in the projection and ellipsoid method for the best choice of the parameters in the case of the spheroidal transform with $c_2 = 1$. Different plots correspond to different values of $\gamma$.

Figure 14: Boxplots of the angle distances between $u_i$ and $\hat{u}_i$ for $i = 1 \ldots 3$ in the projection and ellipsoid method for the best choice of the parameters in the case of a general transform.

6. Conclusions

Many of the second-order methods for directional analysis of stationary point processes are integral transforms of the second-order product density and, as shown here, can be formulated under a unified framework. These methods are quite flexible and work well but they also have some drawbacks. First,
they depend on some tuning parameters which are not always easy to choose. Second, these methods do not directly lead to orthogonal estimates of the axes of deformation in the case of geometric anisotropy. Third, replicated data are needed in order to construct a general isotropy test which does not require any assumptions of the underlying isotropic null model. To overcome some of these issues, we introduced a method based on projecting the Fry points of a point pattern on the unit sphere. We evaluated the performance of the proposed method in the case of geometrically anisotropic regular point patterns in a simulation study in 2D and in 3D. In 2D, we compared the method to second order methods based on integral transforms discussed in Section 3 and to the ellipsoid method introduced in Rajala et al. (2016). For each method, we chose the optimal parameters according to the recommendations in Sormani (2019). The performance of all methods included in the study was similar. In 3D, the projection method was compared only to the ellipsoid method since these two methods, unlike the others, give orthogonal estimates of the axes of deformation. The ellipsoid method performs slightly better than the projection method. However, the projection method has the advantage that only some range parameters, which can be known or easily estimated, have to be chosen. In practice, we suggest to apply either the projection or the ellipsoid method (or both) to estimate preferred directions due to their simplicity. In addition, one can compare the coefficients $S_u$ in the estimated preferred directions for some suitable choice of template function, letting e.g. the range parameters vary. For testing isotropy when the underlying isotropic model is not known, we recommend to use the projection method.

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