Algorithmic resolution of singularities

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Abstract
Although the problem of the existence of a resolution of singularities in characteristic zero was already proved by Hironaka in the 1960s and although algorithmic proofs of it have been given independently by the groups of Bierstone and Milman and of Encinas and Villamayor in the early 1990s, the explicit construction of a resolution of singularities of a given variety is still a very complicated computational task. In this article, we would like to outline the algorithmic approach of Encinas and Villamayor and simultaneously discuss the practical problems connected to the task of implementing the algorithm.

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1 Introduction

The problem of existence and construction of a resolution of singularities is one of the central tasks in algebraic geometry. In its shortest formulation it can be stated as: Given a variety \( X \) over a field \( K \), a resolution of singularities of \( X \) is a proper birational morphism \( \pi : Y \rightarrow X \) such that \( Y \) is a non-singular variety.

Historically, a question of this type has first been considered in the second half of the 19th century – in the context of curves over the field of complex numbers. It was already a very active area of research at that time with a large number of contributions (of varying extent of rigor) and eventually lead to a proof of existence of resolution of singularities in this special situation at the end of the century. Although it does not seem feasible to even give a nearly complete list of important contributions to the field during that period, there are certain names which have to be mentioned in this context, e.g. L. Kronecker, M. Noether and A. Brill ([1], [2], etc.). After the case of curves over the complex numbers had been proved, the task was generalized slightly by passing from curves to surfaces. To this extended task, many contributions were made by the Italian school, among others by O. Chisini, G. Albanese and F. Severi; but many of these treatments lacked the necessary rigor in the proofs. Thus the first mathematically rigorous proof of the existence of resolution of singularities for surfaces over the field \( \mathbb{C} \) was presented by R.J. Walker in 1935 ([13]) by patching the local arguments of the article of H. W. Jung dating from 1908 ([10]) in a suitable way; this article of Jung had locally studied surfaces in 3-dimensional space by means of a projection to the plane, proving that a resolution of singularities exists locally in the given setting.

All of these early contributions to the task of resolving singularities relied on analytic arguments. It was not until the early 1930s that a more algebraic approach to dealing with geometric problems became established which allowed a more systematic treatment. This change of point of view and methods manifests itself in the groundbreaking work of O. Zariski, e.g. in his 1939 proof of the existence of resolution of singularities of surfaces over an algebraically closed field of characteristic zero ([14]) and the proof for the three-dimensional case in 1944 ([15]). It also lead the way to considering the problem in full generality, i.e. without restriction to the dimension of \( X \) or in arbitrary characteristic. In positive characteristic, only partial results are known; the general case is still open ([1], [5]). In characteristic zero, however, the existence of resolution of singularities in the general case has been proved by H. Hironaka in his monumental article in 1964 ([9]). In fact, he was the first one to consider the non-hypersurface case and introduced the concept of a standard basis and a generalization of the order of a hypersurface at a
point as tool for achieving his goal of proving the general case. But his proof is highly non-constructive, which led to an intensive interest in the search for a constructive approach, whether it is the quest for fast algorithms in special cases like the toric one or the task of finding an algorithm for the general case. To the latter problem, important contributions have been made independently by the groups of E. Bierstone and P. Milman and of O. Villamayor and S. Encinas since the late 1980s, which eventually gave rise to implementations in recent years.

In this article, we would like to give a brief overview of the constructive approach of Villamayor and Encinas and of the computational tasks arising when implementing the algorithm. To this end, it is important to understand that this algorithm actually considers the more general set-up, not only constructing a resolution, but a strong factorizing desingularization, from which principalization of ideals and embedded resolution of singularities can be obtained as corollaries – as well as resolution of singularities without reference to an embedding. Since the problem we shall be considering in this article is embedded resolution of singularities, we need to state this task in a more detailed way: Given a subscheme $X$ of a smooth algebraic scheme $W$, the task is to construct a sequence of blowing ups of $W$ at smooth centers such that

- the exceptional divisors in each step are normal crossing,
- the respective centers are normal crossing with them,
- the strict transform of $X$ under the sequence of blowing ups is eventually smooth and normal crossing with the exceptional divisors and
- the blowing ups have only altered $W$ in the points of $\text{Sing}(X)$.

As the resolution process consists of a sequence of blowing ups, the first issue which we consider is the blowing up (in section 2), which is a very well known type of a birational map in algebraic geometry. Therefore the main purpose of this section is to fix notation and explain implementational aspects.

In the following section, the notions of the $b$-singular locus and of basic objects are introduced. These are Encinas’ and Villamayor’s way of describing the collection of data that is used to describe the current situation at each step of the resolution process, including appropriate information on the history of the process.

After treating the special case of monomial basic objects separately in section 4, the algorithm for finding appropriate centers is then described and illustrated by a detailed example in section 5.

The final section 6 then briefly outlines how to use the implemented algorithm for some applications focusing on the problem of how to represent the final result of the resolution algorithm and how to extract information from it. This is again illustrated by means of the example of the preceding section.
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2 Blowing up

As the main goal of this article is to explain how to construct a resolution of singularities algorithmically and how to compute this in practice, we start by explaining the main ingredients to the algorithms. The first one to consider is the blowing up at a given center. After briefly recalling the notion of the blowing up of a variety and listing some properties, we explain how to compute it by means of Gröbner bases techniques. We refer to [7] for more details on blowing ups and to [6] for the computational details.

Definition 2.1. Let $W$ be an algebraic variety, $C \subseteq W$ a closed sub–variety defined by the ideal sheaf $\mathcal{K} \subseteq \mathcal{O}_W$. The blowing up of $W$ with center $C$ is

$$\pi : \widetilde{W} := \text{Proj}(\bigoplus_{n \geq 0} \mathcal{K}^n) \to W.$$ 

Theorem 2.2. (Universal property of blowing up): Let $f : Y \to W$ be a morphism such that $\mathcal{K}\mathcal{O}_Y$ is locally principal. Then there is a unique morphism $g : Y \to \widetilde{W}$ such that $f = \pi \circ g$.

Remark 2.3. The blowing up with center $C$ has the following properties:

1. $\widetilde{W}$ is an algebraic variety.

2. $\pi$ is proper.

3. $\pi$ induces an isomorphism over $W \setminus C$

4. $\mathcal{K}\mathcal{O}_{\widetilde{W}}$ is a locally principal ideal sheaf.

5. If $W$ is projective then $\widetilde{W}$ is projective

In the context of this article, we only consider blowing ups at nonsingular centers, as these are the only ones appearing in the resolution process; more precisely, we will later impose further conditions on the choice of the center leading to the notion of a permissible center:

Definition 2.4. Let $W$ be an algebraic variety.

1. A subscheme $Y \subseteq W$ is called normal crossing at a point $y \in Y$ if $y$ is a regular point of $W$ and there is a regular system of parameters $f_1, \ldots, f_k$ for $y \in W$ such that $Y$ is given by the equation $f_1 \cdots f_k = 0$ on a Zariski neighbourhood of $y$. 
(2) Let a subscheme \( Y \subseteq W \) be normal crossing at all points of \( Y \). Then a regular closed subscheme \( X \subseteq W \) is called a permissible center w.r.t. \( W \) and \( Y \), if \( X \) has normal crossings with \( Y \).

For dealing with explicit examples, it is more convenient to pass to a covering by affine charts. In particular, the calculation of the blowing up can easily be formulated in the affine case in a way which is also accessible to direct implementation:

**Remark 2.5.** Let \( U \subseteq W \) be an affine open subset and and denote \( \Gamma(U, \mathcal{O}_W) \) by \( A \) and \( \Gamma(U, \mathcal{K}) = (f_1, \ldots, f_m) \subseteq A \) by \( K \). Then the blowing up of \( U \) at the center \( C \cap U \) is

\[
\pi^{-1}(U) = \text{Proj}( \bigoplus_{n \geq 0} K^n ) = \bigcup_{i=1}^m \text{Spec} \left[ \frac{A}{f_i} \right].
\]

\( \text{Spec} \left[ \frac{A}{f_i} \right] \) is called the \( i \)-th affine chart of the blowing up.

To compute the blowing up explicitly, we consider the canonical graded \( A \)-algebra homorphism

\[
\Phi : A[y_1, \ldots, y_m] \longrightarrow \bigoplus_{n \geq 0} K^n t^n \subseteq A[t]
\]

defined by \( \Phi(y_i) = tf_i \). Then \( \bigoplus_{n \geq 0} K^n \) is obviously isomorphic to \( A[y_1, \ldots, y_m]/\ker(\Phi) \) and we can hence describe the situation by means of the embedding \( \pi^{-1}(U) \cong V(\ker(\Phi)) \subseteq \text{Spec}(A) \times \mathbb{P}^{m-1} \).

Now let \( X = V(J) \subseteq W \) be a subvariety defined by the ideal sheaf\(^1\) \( J \subseteq \mathcal{O}_W \). For describing how \( X \) is transformed under the blow-up, the following notions are used:

**Definition 2.6.** (1) The total transform of \( X \), \( \pi^*(X) \), is the subvariety of \( \widehat{W} \) defined by \( \pi^*(J) = J\mathcal{O}_W \).

(2) The strict transform of \( X \), \( \tilde{X} \) is the Zariski closure of \( \pi^{-1}(X \setminus V(\mathcal{K})) \) in \( \widehat{W} \). Its ideal sheaf is \( \tilde{J} := J\mathcal{O}_W : \mathcal{K}\mathcal{O}_W^{\infty} \).

(3) The exceptional hypersurface \( E \) is the reduced subvariety of \( \widehat{W} \) defined by \( \mathcal{K}\mathcal{O}_W \); the corresponding ideal sheaf is denoted by \( I(E) \)

(4) The weak transform of \( X \), \( \overline{X} \), is defined by the ideal sheaf \( \overline{J} \) such that the property

\[
J\mathcal{O}_W = I(E)^{\mathcal{O}} \overline{J} \quad \text{and} \quad I(E) \nmid \overline{J}
\]

holds.

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\(^1\)Here \( V(J) \) is considered with the structure sheaf defined by \( \mathcal{O}_W / J \), i.e. not necessarily reduced.
Remark 2.7. The strict transform $\tilde{X}$ is the blowing up of $X$ in the subvariety defined by $\mathcal{K}\mathcal{O}_X$,

$$\tilde{X} = \text{Proj} \oplus \left( \mathcal{K}\mathcal{O}_X \right)^n.$$ 

Example 2.8. To illustrate the difference between the weak and the strict transform under a blow-up, we will now consider the blow-up of the affine variety defined by the ideal $J = \langle xy, x^3 + y^3 + z^3 \rangle \subset \mathbb{C}[x, y, z] = \mathcal{O}_W$ at the origin, that is at the center $C$ defined by $K = \langle x, y, z \rangle$.

Then $\tilde{W} \subset \mathbb{A}^3_\mathbb{C} \times \mathbb{P}^2_\mathbb{C}$ is the set \{(x, y, z; u : v : w) \in \mathbb{A}^3_\mathbb{C} \times \mathbb{P}^2_\mathbb{C} | uy - xv = uz - xw = vz - yw = 0\} which can be covered by the three affine charts corresponding to the open sets $D(u), D(v)$ and $D(w)$. In each of these, the above equations imply that the affine chart looks again like an $\mathbb{A}^3_\mathbb{C}$.

Chart 1: $u \neq 0$

$\mathcal{I}(H) = \langle x \rangle$

$\pi^*J = \langle x^2v, x^3 + x^3v^3 + x^3w^3 \rangle \subset \mathbb{C}[x, v, w]$ 

$J = \langle v, 1 + w^3 \rangle$ 

$\mathcal{J} = \langle v, x + xv^3 + xw^3 \rangle = \langle v, 1 + w^3 \rangle \cap \langle v, x \rangle$

Chart 2: $v \neq 0$

By symmetry in $x$ and $y$ in the generators of the original ideal, the equations here are the same as in chart 1 after exchanging $x$ by $y$ and $v$ by $u$.

Chart 3: $w \neq 0$

$\mathcal{I}(H) = \langle z \rangle$

$\pi^*J = \langle uvz^2, u^3z^3 + v^3z^3 + z^3 \rangle \subset \mathbb{C}[u, v, z]$ 

$\tilde{J} = \langle uw, u^3 + v^3 + 1 \rangle = \langle u, 1 + v^3 \rangle \cap \langle v, 1 + w^3 \rangle$ 

$\mathcal{J} = \langle uw, u^3z + v^3z + z \rangle = \langle u, 1 + v^3 \rangle \cap \langle u, z \rangle \cap \langle v, z \rangle$

In particular, we see that the strict transform does not contain any components which are contained in the hypersurface $H$, whereas in the weak transform all of those components are present except the hypersurface $H$ itself.

Computational Remark 2.9. In example 2.8, the explicit calculation of the blowing up was rather straightforward due to the simplicity of the generators of the ideal $K$. In general, however, generators for the ideal of the center (in an affine chart) are not of such a simple structure and hence the preimage computation of remark 2.5 for the blowing up, which is a Gröbner basis calculation involving the original variables and additionally the new variable $t$ and one new variable for each generator of $K$, can become quite expensive. In practice, this problem can be tackled in two ways. First of all, blowing up at a center consisting of several disjoint components may be implemented as a single blowing up or alternatively as a sequence of blowing ups, each
of which involving just one of the components, because outside the center the blowing up is an isomorphism as we already mentioned; the latter leads to fewer and simpler generators for the centers and turns out to be faster than the other variant, although it produces more affine charts. The other improvement, that can be implemented, is that instead of using the given generators of the center, it is possible to drop redundant generators before continuing.

After computing the blowing up of the ambient space $W$, determining the total, weak and strict transform of subvarieties does not pose any additional difficulties. The only further issue that has to be considered is that the calculations of the weak and strict transform are carried out by means of iterated ideal quotients which are in turn Gröbner basis calculations. Therefore it is again vital for the efficiency of an implementation that the number of variables is kept as small as possible.

3 The $b$-Singular Locus and Basic Objects

After outlining in the previous section that the blowing up itself is not too difficult to handle algorithmically, we now turn our focus to the heart of the algorithm, the choice of the center. Before we can describe how it is constructed in section 5, we need some preparations including the notions of the $b$-singular locus and of basic objects in this section and the separate treatment of a special case in the following section.

In the simplest case, the case of resolving singularities of plane curves, it is a well known fact that the centers are always finite sets of points and the choice of the respective points is governed by an invariant whose main ingredient is the order of the power series locally generating the ideal of the curve. In the general case, the first important ingredient to the governing invariant is a generalization of this, the order of an ideal at a point:

**Definition 3.1.** Let $W$ be a non-singular algebraic variety, $J \subseteq \mathcal{O}_W$ an ideal sheaf and $w \in W$. The order at $w$ with respect to $J$ is defined as

$$v_J(w) = \sup \{m \mid J_w \subseteq m_{W,w}^m \}.$$

The function $v_J : W \to \mathbb{N}$ is upper semi–continuous.

If $X \subseteq W$ is the subvariety defined by $J$, often the notation $v_X$ is used instead of $v_J$.

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$^2$To be more precise, it is Zariski upper-semicontinuous and infinitessimally upper-semicontinuous; in particular, it can only stay constant or drop upon blowing up, but never increase.
Computing the order \( v_J(w) \) at a point \( w \in W \) can be done using the following construction: Let \( A = k[[x_1, \ldots, x_n]] \) be the power series ring and let \( J = \langle f_1, \ldots, f_r \rangle \subseteq A \) be an ideal. We set

\[
\hat{\Delta}(J) := \langle f_1, \ldots, f_r, \left\{ \frac{\partial f_i}{\partial x_j} \right\}_{i,j} \rangle.
\]

It is not difficult to prove that the definition of \( \hat{\Delta}(J) \) neither depends on the choice of generators of \( J \) nor on the choice of regular parameters of \( A \). Its properties and its relation to the order at a point are outlined by the following propositions:

**Proposition 3.2.** Let \( W \) be a non-singular algebraic variety, \( J \subseteq \mathcal{O}_W \) an ideal sheaf. Then there exists an ideal sheaf \( \Delta(J) \subseteq \mathcal{O}_W \) such that \( \Delta(J)\hat{\mathcal{O}}_{W,w} = \hat{\Delta}(J\hat{\mathcal{O}}_{W,w}) \) for all \( w \in W \).

For a proof we refer to [4]. Defining inductively \( \Delta^0(J) = J \) and \( \Delta^r(J) = \Delta(\Delta^{r-1}(J)) \), it is then easy to see that

**Lemma 3.3.**

1. \( v_J(w) = b > 0 \) if and only if \( v_{\Delta(J)}(w) = b - 1 \).
2. \( v_J(w) \geq b > 0 \) if and only if \( w \in V(\Delta^{b-1}(J)) \).

On the basis of these observations, we can now describe the \( b \)-singular locus of \( J \), i.e. the set of points where the order is at least \( b \):

\[
\text{Sing}_b(J) := \{ w \in W \mid v_J(w) \geq b \} = V(\Delta^{b-1}(J)).
\]

**Computational Remark 3.4.** In practice, it is, of course, not feasible to compute \( \hat{\Delta}(J) \) at each point \( w \in W \) separately. Passing to an affine covering, we can, however, obtain the desired result as follows: Let \( W = V(g_1, \ldots, g_r) \) be a smooth affine algebraic variety and \( J = \langle f_1, \ldots, f_s \rangle \subseteq \mathcal{O}_W \) an ideal.

If \( r = 0 \), i.e. \( W \) is an affine space, the calculation can be performed by directly applying the definition:

\[
\Delta(J) = \langle f_1, \ldots, f_r, \left\{ \frac{\partial f_i}{\partial x_j} \right\}_{i,j} \rangle.
\]

In the general case, the difficulty arises that we have to determine a system which induces a local system of parameters at each point \( w \). To this end, we use the fact that \( W \) is smooth; more precisely, let \( m := n - \dim(W) \) and \( L \) be the set of \( m \times m \) submatrices of the Jacobian matrix of \( (g_1, \ldots, g_r) \) with non-vanishing determinant. For \( M \in L \) let \( r(M) \) (resp. \( c(M) \)) be the set of row indices (resp. column indices) of the Jacobian matrix defining \( M \). Let \( A(M) = (A_{ij}(M)) \) be defined by \( A(M) \cdot M = \det(M) \cdot E_m \). On the open set
defined by \( \det(M) \neq 0 \) in \( W \) we can use \( \{x_i\}_{i \in r(M)} \) as a regular system of parameters. There we can hence compute the respective partial derivatives and define the ideal

\[
\Delta(J, M) := (J + \langle \{ \det(M) \frac{\partial f_i}{\partial x_j} \sum_{k \in r(M) \cap \{ i \}} \frac{\partial g_k}{\partial x_j} A_{kk}(M) \frac{\partial f_i}{\partial x_k} \rangle_{i \leq j \leq s, j \notin r(M)} ) : \det(M)^\infty.
\]

Outside of \( V(\det(M)) \) this ideal coincides with \( \Delta(J) \) as can easily be checked by direct computation; the saturation, on the other hand, enables us to remove all components which are contained in \( V(\det(M)) \). Hence \( \Delta(J) \) can be obtained by computing the intersection of all \( \Delta(J, M) \) where \( M \in L \):

\[
\Delta(J) = \bigcap_{M \in L} \Delta(J, M).
\]

**Remark 3.5.** The notion of the \( b \)-singular locus, as it is defined above, appears in the algorithmic approach of Villamayor and Encinas, which we follow in this article, but not in the approach of Bierstone and Milman. In the latter algorithm, the Hilbert-Samuel function and a slightly different notion of an order are used in its place. More precisely, they use a particular choice of the local system of parameters, imposing the condition that it contains the local generators of all exceptional hypersurfaces meeting this point; instead of considering the ideal generated by the generators of the ideal and their partial derivatives w.r.t. each of the elements of the local system of parameters, they then use \( x_i \frac{\partial f_i}{\partial x_i} \) instead of \( \frac{\partial f_i}{\partial x_i} \) whenever \( x_i \) corresponds to one of the exceptional hypersurfaces.

On the other hand, the \( b \)-singular locus of a variety \( X \) is not the only piece of data that has influence on the resolution process. If exceptional divisors are present, these need to be taken into account in a suitable way as well. To this end, all necessary data is collected into the notion of a basic object:

**Definition 3.6.** Let \( b \) be a positive integer, \( W \) a pure-dimensional smooth algebraic variety of dimension \( d \), \( X \subseteq W \) a subvariety. Let \( E = \{ E_1, \ldots, E_k \} \) be an ordered set of normal crossing hypersurfaces in \( W \), \( E_{bad} \subseteq E \) a subset.\(^3\) The tuple \( B = (W, X, b, E, E_{bad}) \) is called a \((d \text{-dimensional}) \) basic object.

\( B \) is called monomial, if \( I(X) = \prod_{H \in E} I(H)^{a(H)} \).

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\(^3\)The role of the subset \( E_{bad} \) in the resolution process is rather technical. As the construction of the center involves an induction on the dimension of the ambient space and hence the construction of lower dimensional auxiliary objects, \( E_{bad} \) is used to indicate those among the exceptional divisors which need to be taken into account before the subsequent descent in dimension.
Consequently, the task of resolution of singularities needs to be reformulated in terms of resolution of basic objects. To this end, we first need to define how a basic object is transformed under a permissible blow-up:

**Definition 3.7.** Let \( B = (W, X, b, E, E_{\text{bad}}) \) be a basic object, \( J = I(X) \), and \( E = \{ H_1, \ldots, H_r \} \). Let \( Y \subseteq W \) be a smooth closed subvariety which is permissible w.r.t. \( W \) and \( E \) and which satisfies \( Y \subseteq \text{Sing}_b(B) := \text{Sing}_b(X) \). Then the blowing up \( \pi : \tilde{B} \to B \) of the basic object \( B \) at the center \( Y \) is induced by the blowing up \( \pi : \tilde{W} \to W \) of \( W \) at \( Y \) (which gives rise to a new exceptional divisor \( H \)) in the following way:

Because \( Y \subseteq \text{Sing}_b(X) \), we may now consider \( J\mathcal{O}_{\tilde{W}} = I(H)^\# \tilde{J} \) for a suitable ideal \( \tilde{J} \subseteq \mathcal{O}_{\tilde{W}} \), the weak transform of \( J \) under the blowing up. Denoting by \( \{ \tilde{H}_1, \ldots, \tilde{H}_r \} \) the set of strict transforms of the \( H_i \), we define \( \tilde{E} = \{ \tilde{H}_1, \ldots, \tilde{H}_r, H \} \). Then \( \tilde{B} = (\tilde{W}, \tilde{X}, b, \tilde{E}, \tilde{E}_{\text{bad}}) \), where \( \tilde{X} \) is the algebraic variety defined by \( \tilde{J} \). The last piece of data which still needs to be specified is \( \tilde{E}_{\text{bad}} \); we postpone this to the following sections 4 and 5, as it is rather technical and will not be needed any earlier.

**Definition 3.8.** A resolution of the singularities of a basic object \( B \) is a sequence of blowing ups

\[
B(n) \xrightarrow{\pi_n} B(n - 1) \to \cdots \to B(1) \xrightarrow{\pi_1} B,
\]

where the \( B(i) = (W(i), X(i), b, E(i), E_{\text{bad}}(i)) \) are basic objects and

\[
\pi_i : W(i) \to W(i - 1)
\]

blowing ups at permissible centers \( Y_i \subseteq \text{Sing}_b(X(i - 1)) \), such that

(a) \( \text{Sing}_b(X(n), b) = \emptyset \)

(b) \( W(n) \setminus \bigcup_{H \in E(n)} H \cong W \setminus \text{Sing}_b(X) \)

(c) \( X(n) \) has normal crossings with \( E(n) \).

**Example 3.9.** Consider the basic object \( B = (\mathbb{C}^2, V(x^3 - y^5), 3, \emptyset, \emptyset) \) and let \( \pi : W(1) \to \mathbb{C}^2 = W \) be the blowing up of \( \mathbb{C}^2 \) at 0. Denoting by \( X(1) \) the strict transform of \( X = V(x^3 - y^5) \) and by \( H \) the exceptional divisor, we obtain the following resolution of singularities of \( B \):

\[
B(1) = (W(1), X(1), 3, \{ H \}, \{ H \}) \to B.
\]

Here it is important to observe that \( \text{Sing}_3(B(1)) = \emptyset \) and \( B(1) \) is resolved, although \( X(1) \) still has a singularity at the origin which it is a cusp, i.e. of order 2.
The key point to the whole resolution process, the choice of the suitable centers, is governed by an invariant

\[ f_{(w,x,b,E,\text{bad})} : X \rightarrow \mathcal{I} \]

assigning to each point of \( X \) a value in a totally ordered set \( \mathcal{I} \). As the maximal locus of this invariant is determining the upcoming center and as the decrease of its maximal value under blowing up is the measure for the progress in the resolution process, the invariant has to be Zariski upper semicontinuous as well as infinitessimally upper semicontinuous. In the general case, the construction of this invariant it rather complicated and involves an iterated descent in dimension each giving rise to a new auxiliary basic object. The details of this construction are outlined in section 5, following the algorithmic approach of Villamayor and Encinas. One special case, however, has to be treated separately, the monomial case, because in this situation the general invariant does not provide a suitable approach.

**Remark 3.10.** In the algorithmic approach of Bierstone and Milman resp. the one of Wlodarczyk a similar collection of data is used, which Wlodarczyk calls a marked ideal; it contains information on the ambient space, the ideal itself, the exceptional divisors and an integer used in a way similar to the \( b \) in a basic object. Although the Bierstone-Milman approach uses the strict instead of the weak transform, the way in which the invariant is constructed (including the descent in dimension) also follows the same type of main ideas which will be outlined for Villamayor’s construction in the subsequent sections. This reflects the fact that all of these approaches have their roots in the non-constructive proof of Hironaka; the subtle but important differences arise from the different approaches to filling in the constructive details.

As the main goal of this article is to explain a construction of resolution of singularities and the practical problems arising in its implementation, the simultaneous treatment of the approaches of Villamayor and Encinas and of Bierstone and Milman is beyond the scope of this article and we hence focus on one of the two, the one of Villamayor and Encinas.

### 4 The Monomial Case

Before turning to the general case, we still need to deal with one special situation separately, the case of a monomial basic object:

Let \( B = (W, X, b, E, \text{bad}) \) be a monomial basic object where \( E = \{H_1, \ldots, H_r\}, I(X) = \prod_{i=1}^r I(H_i)^{a_i} \). We define

\[ f_B : X \rightarrow \mathbb{Z}^{#E+2} \]

\[ x \mapsto (-p(x), w(x), i(x)) \]
where \( p(x), w(x) \) and \( i(x) \) are defined by

\[
\begin{align*}
p(x) &= \min \left\{ q \mid x \in H_{i_1} \cap \ldots \cap H_{i_q} \text{ and } \sum_{j=1}^q a_{ij} \geq b \right\} \\
w(x) &= \max \left\{ \sum_{j=1}^{p(x)} a_{ij} \mid x \in H_{i_1} \cap \ldots \cap H_{i_{p(x)}}, \sum_{j=1}^{p(x)} a_{ij} \geq b \right\} \\
i(x) &= \max \left\{ (i_1, \ldots, i_{p(x)}) \mid x \in H_{i_1} \cap \ldots \cap H_{i_{p(x)}}, \sum_{j=1}^{p(x)} a_{ij} \geq b \right\}
\end{align*}
\]

**Remark 4.1.** At this point, it is important to observe that in the monomial case the subset \( E_{bad} \) of \( E \) is not considered in any way.

**Example 4.2.** As an example for the monomial case, let us consider the problem of resolving the basic object

\[ B = (\mathbb{C}^2, V(x^2 y^2), 2, \{V(x), V(y)\}, \emptyset). \]

By direct calculation, we can check that along the exceptional divisors the value of \( p \) is \(-1\) and the one of \( w \) is \( 2 \) at all points. Therefore the choice of the upcoming center has to be made on the basis of the last entry \( i \) which has the value \( 2 \) for points on \( V(y) \) and \( 1 \) on \( V(x) \). This leads to the center \( V(y) \) with invariant \((-1, 2, (2))\).

After one blowing up at this center, the transformed object is

\[ \tilde{B} = (\mathbb{C}^2, V(x^2), 2, \{V(x), 0, V(y)\}, \emptyset). \]

By the same direct calculation as before, the subsequent center is now \( V(x) \) with invariant value \((-1, 2, (1))\). After this second blowing up, the object is clearly resolved.

## 5 The Tower of a Basic Object

In the general case, the governing invariant of the resolution algorithm is constructed inductively by means of a descent in dimension. Therefore, we will first define the respective ‘fragment’ of the invariant corresponding to a basic object and then explain how an auxiliary basic object is constructed whose ambient space is of smaller dimension. Iterating this process, we obtain a tower of basic objects and then construct the invariant by concatenating the ‘fragments’ of the invariant corresponding to the objects in the tower.

Given such a tower of basic objects, we then need to define how the tower is
transformed under a blow-up and how the invariant is constructed for the
transformed tower.

Construction 5.1. (Building the Tower)
To define the 'fragments' of the invariant, let $B = (W, X, b, E, E_{bad})$ be a basic
object and define

$$f_B : X \to \mathbb{Z}^2 \quad \text{by} \quad f_B(x) = (v_X(x), n_x(E))$$

where $n_x(E) = \# \{ H \in E_{bad} \mid x \in H \}$.

For a given basic object $B = (W, X, b, E, E_{bad})$, which we want to resolve,
we now construct locally in the neighbourhood of every point $w \in W$ a tower
of lower dimensional basic objects. If the dimension of $W$ is one or if $B$ is
monomial then the tower is $T(B) = \{ B \}$.
Otherwise, let $Y = \{ x \in X \mid (v_X(x), n_x(E)) \text{ maximal} \}$ and consider two
cases separately: If $\dim(Y) = \dim(W) - 1$, denote the reduced variety asso-
ciated to the top-dimensional part of $Y$ by $Y_{eq}$ and define the tower as
$T(B) = \{ B, B_{aux} \}$ where $B_{aux}$ is the auxilliary basic object of the form
$(Y_{eq}, Y_{eq}, 1, 0, 0)$.$^4$ If this is not the case, set $E_{bad} = \{ H_1, \ldots, H_s \}$
and define $X' \subseteq W$ by

$$b' = \max\{ v_X(x) \mid x \in X \}$$

$$m = \max\{ n_x(E) \mid x \in X, v_X(x) = b' \}$$

$$I(X') = I(X) + \left( \prod_{i_1 < \cdots < i_m} \sum_{j=1}^{m} I(H_{i_j}) \right)^{b'}$$

$$E' = E \setminus E_{bad}$$

Choose $U \subseteq W$ open and a smooth hypersurface $Z \subseteq U$ (a hypersurface of
maximal contact) such that

- $Z \supseteq U \cap \{ x \in X \mid (v_X(x), n_x(E)) = (b', m) \}$
- $Z$ intersects all $H \in E'$ transversally.

$^4$Here, it is important to note that this auxilliary basic object is only introduced to also
include this special case into the general way of expressing the construction of the center.
The top-dimensional components of the locus of maximal invariant are hypersurfaces in $W$
in this case; it can be checked directly (see [4]) that $Y_{eq}$ consists of smooth hypersurfaces
which do not intersect. These form the upcoming center which is needed in the algorithm
to allow it to proceed in the usual way afterwards.

$^5$If the tower is being constructed for the very first time, the set $E_{bad}$ needs to be
initialized and is set to contain all elements of $E$. If a tower has to be computed during
the resolution process, the appropriate changes to the set $E_{bad}$ are explained at respective
steps in the algorithm.
• $E' \cap Z := \{H \cap Z \mid H \in E'\}$ have normal crossings.

For every $w \in W$, such an open subset $U$ and a hypersurface $Z$ satisfying the above conditions exists (see [4]). To simplify the following notations, we assume $U = W$.\footnote{Actually, the fact that these constructions need to be performed in an open subset leads to the notion of a general basic object, a generalization of basic objects. But this is a rather technical concept which we would lead beyond the scope of this short article. In particular, it is not contributing to the general ideas behind the resolution process.} The coefficient ideal of $X$ is defined (locally) as

$$\text{Coeff}_Z(I(X')) = \sum_{i=0}^{b'-1} (\Delta^i(I(X')) \mathcal{O}_Z)^{\frac{b'}{b}}.$$

Let $C \subseteq Z$ be defined by $\text{Coeff}_Z(I(X')) \subseteq \mathcal{O}_Z$, then the first auxiliary object is $B_Z := (Z, C, b', E, \emptyset)$. Note that $\text{Sing}_{b'}(X') = \text{Sing}_{b'}(C)$.

At the beginning of the resolution process, we define the tower of the basic object $B$ inductively by

$$T_0(B) = \{B\} \cup T_0(B_Z),$$

where the lower index 0 indicates that we have not performed any blowing up yet. Let $T_0(B) = \{B^{[0]}, \ldots, B^{[e]}\}$ be the tower in the neighbourhood of a point $w \in W$. The invariant vector at $w$ is then constructed as

$$\text{inv}^{(0)}(w) = (f_{B^{[0]}}, \ldots, f_{B^{[e]}}(w)).$$

**Computational Remark 5.2.** In practice, it is clearly not feasible to calculate the value of the invariant at each point $w \in W$ separately by a local construction. But this is not necessary for determining the center anyway, since the center is given by the locus of maximal value. It is therefore sufficient to calculate only the maximal value and its locus – of course iteratively passing from left to right through the invariant as the comparison is done lexicographically. The calculation of the $b'$-singular locus follows along the lines of construction 3.4; the calculation of the locus of maximal $n_x(E)$ (inside the $b'$-singular locus) is only a task of combinatorial nature, which can easily be implemented.

The crucial point in the construction is the descent in dimension, which forces the calculation to pass from the affine charts to open covers thereof in a similar way as in the calculation of the $\Delta(I(X))$. More precisely, $b'$ denotes the maximal order and, hence, we know that the ideal $\Delta^{b'-1}(I(X))$ describing $\text{Sing}_{b'}(X)$ is itself of order at most 1 at all points of $W$; on an affine chart, we may now choose a system of generators for $\Delta^{b'-1}(I(X))$ such that for a subset $f_1, \ldots, f_s$ thereof the intersection of the singular loci of the corresponding
hypersurfaces is empty and the conditions on the intersection properties of each of these hypersurfaces $V(f_i)$ with the exceptional divisors are satisfied outside $\text{Sing}(V(f_i))$. As the open cover, we then choose the complements of the singular loci or, in practice, the complements of hypersurfaces generated by (an appropriate subset of the) partial derivatives of the $f_i$; the hypersurfaces $Z$ on each of these open sets are chosen to be the respective $V(f_i)$.

The remaining part of the calculation of the $\text{Coeff}$-ideal does not cause any further difficulties. But we cannot recombine the local results to obtain an auxiliary basic object on $W$ or on the affine chart, because the choice of the hypersurface $Z$ clearly affects the auxiliary basic object. Nevertheless, the value of the invariant and hence the maximal locus are independent of the choice of $Z$ and we can hence recombine the pieces of the maximal locus to obtain the center.

**Construction 5.3. (Transformation of the Tower)**

After describing how a tower of basic objects is created from a given basic object, the next task is to consider how such a tower is transformed under a blowing up. (At this point it is important to note that the algorithm of Villamayor does not recompute the tower of basic objects from the transformed object, but transforms the whole tower instead – obtaining the new value of the invariant from the transformed tower.)

To this end, let $B = (W, X, b, E, E_{bad})$ be a basic object, say at the $i$-th step of the resolution process, $T_i(B) = \{B^{[0]}, B^{[1]}, \ldots, B^{[k]}\}$ the corresponding tower of basic objects, where $B^{[0]} = B$, and $Y = \{x \in X \mid \text{inv}^{(i)}(x) \text{ is maximal} \}$ the center computed by means of this tower, which is shown to be permissible for $B$ in [4]. In particular, it consists of non-singular components which do not intersect.

Let $\tilde{B}^{[i]}$ be the blowing up of $B^{[i]}$ at the center $Y$ and denote the collection of the transformed basic objects by $T_i(B) = \{\tilde{B}^{[0]}, \ldots, \tilde{B}^{[k]}\}$. Let $\text{inv}^{(i)}_{[j]} = \max\{f_{B^{[j]}(x)}\}$ be the maximal value of the $j$-th fragment of the invariant corresponding to the $j$-th auxiliary object before blowing up. Now, let $k$ be minimal such that the locus of the fixed invariant value $\text{inv}^{(i)}_{[k]}$ is empty$^7$ for $\tilde{B}^{[k]}$. This can occur in four situations:

(a) the ideal of the total transform of the second entry of the basic object

is a product of exceptional divisors, i.e. the new object is monomial.

---

$^7$As the invariant drops at each blowing up (by construction) and as the lowest dimension of an ambient space in the tower is in general 1, there is always such a $k$ for which the locus of value $\text{inv}^{(i)}_{[k]}$ is empty. In the special case that the tower does not reach 1 as the lowest dimension of the ambient space, the previous auxiliary object of lowest dimension is either monomial or is of the type $B_{aux}$. In the first case, the transformed auxiliary object is again monomial; in the other case, $B_{aux}$ becomes empty after one blowing up by construction.
(b) the $b$-singular locus for the corresponding auxilliary object is empty

(c) the $b'$-singular locus for the corresponding auxilliary object is empty, but the $b$-singular locus is non-empty

(d) the $b'$-singular locus is non-empty, but the maximal number of exceptional divisors from $E_{bad}$ simultaneously meeting a point thereof has dropped.

In the first case, the object in question is monomial. In case (b), the respective object is resolved and the maximal locus of the invariant truncated before $inv_{i}^{(i)}$ is a permissible center. In case (c), we add all exceptional divisors of $\tilde{B}^{[k]}$ to the corresponding set $E_{bad}$ and define a new tower$^{8}$ by

$$T_{i+1}(B) := \{ \tilde{B}^{[0]}, \ldots, \tilde{B}^{[k-1]} \} \cup T_{0}(\tilde{B}^{[k]}).$$

In the last case, the set $E_{bad}$ of $\tilde{B}^{[k]}$ remains unchanged, we compute a descent in dimension with the new (lower) $m$ to obtain an auxilliary object $B_{(i+1)}^{[k+1]}$ whose set $E_{bad}$ coincides with the set $E'$ of the descent. Hence the new tower in this case is defined by

$$T_{i+1}(B) := \{ \tilde{B}^{[0]}, \ldots, \tilde{B}^{[k]} \} \cup T_{0}(B^{[k+1]}_{(i+1)}).$$

Remark 5.4. The construction of the center ensures that all exceptional divisors which have been born after the corresponding fragment of the invariant last dropped are normal crossing with the centers arising from the maximal locus of the invariant. As soon as this fragment of the invariant drops, the subsequent auxilliary objects are recomputed and hence there is no way to predict the intersection properties of the exceptional divisors with the center arising from the new tower. Consequently, the subset $E_{bad} \subseteq E$ is used to mark those exceptional divisors which might cause problems concerning the normal crossing condition for the center; the value of $E_{bad}$ for a given object in the tower is altered precisely at the moments when the subsequent parts of the tower need to be recomputed.

Example 5.5. To illustrate the rather technical construction, we consider a very simple example for which all calculations can still be done by hand:

$$X = V(z^2 - x^2 y^2) \subseteq \mathbb{C}^3 = W$$

(0) Initialization step: Construction of center of first blowing up

---

$^{8}$By construction, all new auxilliary objects below $B^{[k]}$ in the tower $T_{0}(\tilde{B}^{[k]})$ are, hence, created with $E = \emptyset$. 
Construction of the basic object $B$

- $B = (\mathbb{C}^3, V(z^2 - x^2y^2), 2, \emptyset, \emptyset) =: B^{[0]}$

maximal locus of $\text{inv}^{[0]}_1$

- Computation of the maximal order:
  \[
  \Delta(z^2 - x^2y^2) = \langle z, xy^2, x^2y \rangle \\
  \Delta^2(z^2 - x^2y^2) = \langle 1 \rangle
  \]
  Hence the maximal order $\nu$ is 2 and the $\nu$-singular locus is $\text{Sing}_2(B) = V(z, xy)$.

- Computation of maximal $n_x(E)$ unnecessary, because $E = \emptyset$

first descent in dimension

- Choice of hypersurface of maximal contact:
  We choose $Z_0 = V(z)$ which clearly satisfies $\langle z \rangle \subseteq \Delta(z^2 - x^2y^2)$. As the set of exceptional divisors is empty, there are no further conditions to be checked.

- Construction of the first auxiliary object:
  $\text{Coeff}_{Z_0}(z^2 - x^2y^2) = \langle x^2y^2, (xy^2)^2, (x^2y)^2 \rangle$
  \[
  = \langle x^2y^2 \rangle
  \]
  $B_{Z_0} = (\mathbb{C}^2, V(x^2y^2), 2, \emptyset, \emptyset) =: B^{[1]}$

maximal locus of $\text{inv}^{[0]}_2$

- Computation of the maximal order:
  \[
  \Delta(x^2y^2) = \langle x^2y, xy^2 \rangle \\
  \Delta^2(x^2y^2) = \langle x^2, xy, y^2 \rangle \\
  \Delta^3(x^2y^2) = \langle x, y \rangle \\
  \Delta^4(x^2y^2) = \langle 1 \rangle
  \]
  Hence the maximal order $\nu$ is 4 and the $\nu$-singular locus is $\text{Sing}_4(B^{[1]}) = V(x, y)$.

- Computation of maximal $n_x(E)$ unnecessary, because $E = \emptyset$
second descent in dimension

- Choice of hypersurface of maximal contact:
  We choose $Z_1 = V(x)$ which satisfies $\langle x \rangle \subseteq \Delta^3(x^2y^2)$. The other conditions hold trivially.

- Construction of the second auxiliary object:
  $\text{Coeff}_{Z_1}(x^2y^2) = \langle y^2 \frac{dx}{x}, y^2 \frac{dt}{t} \rangle$
  $B_{Z_1}^{[1]} = (\mathbb{C}, V(y^{24}), 24, \emptyset, \emptyset) =: B^{[2]}$

maximal locus of $\text{inv}_{[2]}^{[0]}$

- Maximal order: 24
- maximal $n_x(E)$: 0

Hence the tower of the original basic object is

$$T_0(B) = \{ B^{[0]}, B^{[1]}, B^{[2]} \}$$

leading to the invariant values

$$\text{inv}_{[0]}^{[0]}(w) = \{ (2, 0; 4, 0; 24, 0) \quad w = (0, 0, 0) \}
\quad w = (0, y, 0), y \neq 0 \}

which in turn imply that the first center is $(0, 0, 0)$.

1. **First Blowing Up** and Transformation of the Tower

By blowing up this center, we obtain three charts:

It can easily be checked by direct computation that the first one is already resolved and that the two remaining ones are showing the same objects up to renaming of variables due to the symmetry of the original situation. Hence, we only consider one of the latter two in detail: the chart defined by $x = uy$, $z = wy$. As transformed basic objects, we obtain:

$\tilde{B}^{[0]} = (\mathbb{C}^2, V(u^2 - u^2 y^2), 2, \{ V(y) \}, \emptyset)$

$\tilde{B}^{[1]} = (\mathbb{C}^2, V(u^2), 2, \{ V(y) \}, \emptyset)$

$\tilde{B}^{[2]} = (\mathbb{C}^2, \emptyset, 24, \{ V(y) \}, \emptyset)$.

Obviously, $\text{Sing}_2(\tilde{B}^{[0]}) = \text{Sing}_2(u^2 - u^2 y^2) \neq \emptyset$, but $\text{Sing}_0(\tilde{B}^{[1]}) = \text{Sing}_0(u^2) = \emptyset$. Thus we can set $\tilde{B}_1^{[0]} = \tilde{B}^{[0]}$, but we have to recompute $\tilde{B}_1^{[1]}$. As $\text{Sing}_2(u^2) \neq \emptyset$, we only need to correct $E_{bad}$ in $\tilde{B}^{[1]}$ setting it to $E_{bad} = \{ V(y) \}$ before assigning this basic object to $B_1^{[1]}$.

---

9 Testing whether the $n_x$ have dropped is meaningless here, because they are non-negative integers and had value zero in the previous step.
Figure 2: These three images illustrate the three charts arising from the first blowing up, which introduced new variables $u, v, w$ for the $\mathbb{P}^2$. Fixing $u, v$ and $w$ as names for the new variables introduced in this blowing up, the left image corresponds to the chart $w \neq 0$, that is $x = uz$ and $y = vz$, the ideal of the transformed surface (lighter grey) is generated by $1 - u^2v^2$ in this chart, the one of the exceptional divisor (darker grey to black) by $z$. The image in the center illustrates the chart $v \neq 0$, i.e. $x = wy$, $z = wy$; the ideal of the transformed surface is generated by $z^2 - x^2y^2$, the one of the exceptional divisor by $y$. The last image illustrates the third chart, which basically coincides with the second one up to exchange of the roles of $x$ and $y$ and of $u$ and $v$.

Additionally, we have to recompute the tower starting at $B_1^{[2]}$: According to the formulae, we obtain $\ell' = 2, m = 1$ and $I(X') = \langle u^2, y^2 \rangle$ from the basic object $B_1^{[1]}$ implying that $B_2^{[2]}$ is assigned the value $(\mathbb{C}, V(u^2), 2, 0, 0)$. Therefore the new tower is

$$T_1(B) = \{ B_1^{[0]}, B_1^{[1]}, B_1^{[2]} \}$$

leading to the invariant values

$$\text{inv}^{[1]}(w) = \begin{cases} (2, 0; 2, 1; 2, 0) & w = (0, 0, 0) \\ (2, 0; 2, 2; 0, 0, 0) & w = (0, y, 0), y \neq 0 \end{cases}$$

which in turn imply that the second center is $(0, 0, 0)$.\(^\text{10}\)

(2) **Second Blowing Up** and corresponding transformations of the towers.

Again, the first chart can easily be checked to be resolved by direct computation. The other two need to be considered separately:

\^\text{10}Although we obtain a center which is just the coordinate origin in the other chart as well due to the symmetry in $x$ and $y$ of the original equation, these two points do not coincide. More precisely, one of the two is $((0, 0, 0); (1 : 0 : 0)) \subset \mathbb{A}^3 \times \mathbb{P}^2$, the other one is $((0, 0, 0); (0 : 1 : 0)) \subset \mathbb{A}^3 \times \mathbb{P}^2$. 
Figure 3: These three images illustrate the three charts after the second blowing up. Again the transformed surface is drawn in lighter grey, the exceptional divisors in darker grey and black. The one on the left corresponds to the resolved object, the one in the center to case (2a) and the one on the left to case (2b).

(2a) Chart \( u = yr, \ w = yt \)

As transformed basic objects, we obtain
\[
\tilde{B}_1^{[0]} = (\mathbb{C}^3, V(t^2 - r^2y^2), 2, \{\emptyset, V(y)\}, \emptyset)
\]
\[
\tilde{B}_1^{[1]} = (\mathbb{C}^2, V(r^2), 2, \{\emptyset, V(y)\}, \emptyset)
\]
\[
\tilde{B}_1^{[2]} = (\mathbb{C}^1, V(r^2), 2, \{V(y)\}, \emptyset)
\]
Clearly, \( \text{Sing}_2(t^2 - r^2y^2) \neq \emptyset \) and \( \text{Sing}_2(r^2) \neq \emptyset \), but the locus of invariant value \( \text{inv}_{[1]}^{[1]} \) is empty. Therefore, we can use \( \tilde{B}_1^{[0]} \) and \( \tilde{B}_1^{[1]} \) as \( B_2^{[0]} \) and \( B_2^{[1]} \) in the new tower (without adding any further exceptional divisors to \( E_{\text{bad}}(B_2^{[1]}) \)) and we have to recompute \( B_2^{[2]} \) obtaining the auxiliary object \( (\mathbb{C}^1, V(0), 2, \{V(y)\}, \{V(y)\}) \) which is already resolved. Hence the upcoming center is determined by the maximal locus of \( \{\text{inv}_{[0]}^{[2]}, \text{inv}_{[1]}^{[2]}\} \) which is \( V(t, r) \).

As the subsequent calculations in this branch are very similar to this one respectively those in the branch (2b), we do not discuss this branch of the resolution any further in this example.

(2b) Chart \( y = us, \ w = ut \)

As transformed basic objects, we obtain
\[
\tilde{B}_1^{[0]} = (\mathbb{C}^3, V(t^2 - u^2s^2), 2, \{V(s), V(u)\}, \emptyset)
\]
\[
\tilde{B}_1^{[1]} = (\mathbb{C}^2, 0, 2, \{V(s), V(u)\}, \{V(s)\})
\]
\[
\tilde{B}_1^{[2]} \text{ is irrelevant due to the structure of } \tilde{B}_1^{[1]}.
\]
Because the second entry of \( \tilde{B}_1^{[1]} \) is the empty set, we know that the ideal of the total transform of the second entry of \( B_1^{[1]} \) is a product of exceptional hypersurfaces and that we are in the monomial case. Using \( \tilde{B}_1^{[0]} \) as \( B_2^{[0]} \), we recompute the auxiliary basic object for entering the algorithm of the monomial case:
Here we still have $b' = 2$, $m = 0$, $I(X') = \langle t^2 - u^2 s^2 \rangle$ and $E' = \{V(s), V(u)\}$, which leads to the auxiliary object $B_2^{[1]} = (\mathbb{C}^2, V(u^2 s^2), 2, \{V(s), V(u)\})$ which has already been dealt with in example 4.2 and hence to the tower

$$T_2(B) = \{B_2^{[0]}, B_2^{[1]}\}.$$

From the calculations of example 4.2, we obtain the center $V(u)$ for the auxiliary basic object $B_2^{[1]}$ and therefore the new center $V(u, t)$ corresponding to the invariant value $(2, 0; -1, 1, 2, (2))$.

(3) **Third Blowing Up** in the (2b) branch

To simplify notation, we rename the variables to be $x, y, z$ again. Then the tower before the blowing up is $T_2(B) = \{B_2^{[0]}, B_2^{[1]}\}$ where

$B_2^{[0]} = (\mathbb{C}^3, V(z^2 - x^2 y^2), 2, \{V(y), V(x)\}, \emptyset)$

$B_2^{[1]} = (\mathbb{C}^3, V(x^2 y^2), 2, \{V(y), V(x)\}, \emptyset)$.

Figure 4: Illustration of the two charts arising from the blowing up at the center determined in case (2b).

Only the object in the second chart, defined by $z = ux$ is not resolved yet. For this one the transformed objects are

$\tilde{B}_2^{[0]} = (\mathbb{C}^3, V(u^2 - y^2), 2, \{V(y), V(1), V(x)\}, \emptyset)$

$\tilde{B}_2^{[1]} = (\mathbb{C}^3, V(y^3), 2, \{V(y), V(1), V(x)\} \{V(x)\})$

The 2-singular locus of the first one is still non-empty and the second one is still monomial. Hence the transformed tower is now

$$T_3(B) = \{\tilde{B}_2^{[0]}, \tilde{B}_2^{[1]}\}.$$

As we are still in the monomial case for the first auxiliary object, we obtain the upcoming center from example 4.2: $V(z, y)$ with corresponding invariant $(2, 0; -1, 2, (1))$

---

\footnote{As $E_{v_{33}}$ is irrelevant for the calculations in the monomial case, we have omitted it here.}
(4) Fourth Blowing Up

Figure 5: Illustration of the two charts arising from the blowing up at the center determined after the third blowing up.

The objects in both charts are resolved. For simplicity, we only consider the chart defined by $u = ty$. Here the transformed objects after the blowing up are:

$$\tilde{B}_3^{[0]} = (C^3, B(t^2 - 1), 2, \{V(1), V(1), V(x), V(y)\}, \{V(y)\})$$

$$\tilde{B}_3^{[1]} = (C^3, \emptyset, 2, \{V(1), V(1), V(x), V(y)\}, \{V(y)\})$$

and $\text{Sing}_2(\tilde{B}_3^{[0]}) = \emptyset$ which is exactly what we wanted to achieve.

Even in such a small example as this one, it is sometimes difficult to keep track of the parent-child relationships of the various charts. Therefore it is often useful to illustrate these in terms of a tree of charts:

**Computational Remark 5.6.** From the implementational point of view, there are three aspects of the resolution process which greatly affect the efficiency of the resulting program: first of all the construction of the tower which was discussed in 5.1, secondly the transformation of the towers, i.e. the blowing up (see 2.9) and last but not least to combinatorial complexity due to the use of charts – arising from blowing up and from passing to open covers – which, of course, have non-empty intersections in general. This last issue turns out to be the crucial point in the overall performance of the algorithm: It is simply the number of redundant blowing ups that often makes the algorithm of S. Encinas and O. Villamayor painfully slow in examples of practical relevance. To tackle this problem, we need to analyze the overall strategy of the choice of the centers. Clearly, the fundamental idea behind the construction of the centers is the need to choose them as large as possible while they still have to be subject to the conditions of permissibility.\(^\ddagger\)

---

\(^\ddagger\)In the algorithm of E. Bierstone and P. Milman, the choice of centers follows the same fundamental idea, but as the construction of the invariant differs from the other algorithm (e.g. using the Hilbert-Samuel function) and the strict transform is used instead of the
Figure 6: Tree of Charts for the example $V(z^2-x^2y^2)$: Boxes with solid border represent final charts; boxes with dashed border are charts, which have been discussed explicitly, whereas boxes with dotted border represent charts which have not been discussed in detail. In the whole resolution process 8 different exceptional divisors appeared.

An obvious way to avoid unnecessary calculations is the use of symmetries in the system of equations – as we did in the above example when determining the center for the second blowing up. To achieve an even better improvement, following this idea, it is usually helpful to try to detect and preserve these symmetries throughout the process as far as possible.

Additionally, there is a special situation in which it is often worth applying a simple heuristic: If the singular locus of the original object happens to be a permissible center, this can be used as the very first center. In the case of the Whitney umbrella, for instance, this make the difference between a tree leading to twelve final charts and one consisting of just one blowing up giving rise to 2 final charts.

Apart from the previously mentioned changes which do not alter the course of the algorithm, it is also possible to carefully change the choice of centers by means of a backtracking approach. More precisely, the algorithm of Encinas-Villamayor uses weak transforms, but for the final result of embedded resolution of singularities we are only interested in strict transforms. Thus a natural idea would be to pass from weak to strict transforms after each blow-

weak one, the number of blowing ups in this algorithm tends to be lower. However, the computation of the invariant poses problems of a different kind, like the computation of the maximal locus of the Hilbert-Samuel function. Moreover, as it is still necessary to pass to charts, the problem of redundant blowing ups is also present here.
ing up and compute a new tower corresponding to the new main basic object, keeping only the partition of the set of exceptional divisors into the various sets $E_{bad}$ unless the order of the respective auxiliary object drops (compared to the one appearing before passing to the strict transform). Unfortunately, passing from weak to strict transform the invariant of the first auxiliary object can even go up. Therefore it is not always possible to pass from weak to strict transform, but it can be applied as a kind of heuristic, falling back to the original algorithm of Encinas-Villamayor whenever a step is detected where the invariant increased. In this case, of course, the algorithm has to follow the steps of the unchanged one until the maximal value of a fragment of the invariant corresponding to a higher dimensional object in the tower drops; only then the heuristic can be applied again. This backtracking approach can also be refined by slightly changing the construction of certain auxiliary objects, but this is beyond the scope of this article.

6 Some Remarks on Applications

When considering practical applications of the resolution of singularities, e.g. the calculation of an invariant like the topological zeta function, an obstacle, which is common to all these tasks, is the fact that the final result is represented by means of charts. This makes it possible – even highly probable – that the same point or subvariety may be present in several charts which, in turn, implies that rather simple tasks, like e.g. counting intersection points of two curves, cannot be performed in a direct way. Instead we need to a way to identify the same point in different charts by moving through the whole tree of charts in an appropriate way.

As blowing ups are isomorphisms away from the center, the process of successively blowing down and then blowing up again does not cause any problems for points which do not lie on an exceptional divisor at all or only lie on exceptional divisors, which already exist in the chart at which the history of the considered charts branched. If, however, the point lies on an exceptional divisor which arises later, then blowing down beyond the moment of birth of this divisor will inevitably lead to incorrect results, because this blow up map is not an isomorphism. To avoid this problem, we need to represent the point on the exceptional divisor as the locus of intersection of the exceptional divisor with an auxiliary variety which is not contained in the exceptional divisor. More formally speaking, we use the following simple fact from commutative algebra:

Let $I \subset K[x_1, \ldots, x_n]$ be a prime ideal, $J \subset K[x_1, \ldots, x_n]$ another ideal such that $I + J$ is equidimensional and $ht(I) = ht(I + J) - r$ for some integer
0 < r < n. Then there exist polynomials \( p_1, \ldots, p_r \in I + J \) and a polynomial \( f \in K[x_1, \ldots, x_n] \) such that

\[
\sqrt{I + J} = \sqrt{(I + (p_1, \ldots, p_r)) : f}.
\]

In our situation, the ideal \( I \) is, of course, the ideal of the intersection of the exceptional divisors in which the point or subvariety \( V(J) \) is contained. As any sufficiently general set of polynomials \( p_1, \ldots, p_r \in J \setminus (I \cap J) \) leading to the correct height of \( I + (p_1, \ldots, p_r) \) will do and as the only truly restricting condition on \( f \) is that it has to exclude all extra components of \( I + (p_1, \ldots, p_r) \), we also have enough freedom of choice of the \( p_1, \ldots, p_r, f \) to achieve that none of them is contained in any further exceptional divisor that might be in our way when blowing down.

Having solved the problem of identifying points which exist in more than one chart, we can now consider a very simple application: we determine which exceptional divisor in one chart coincides with which one in another chart. The method to do this is quite simple: we simply compare the centers leading to these exceptional divisors. To this end, we start at the root of the tree of charts of the resolution and work our way up to the final charts. The criteria for identifying the centers are quite simple: first of all, the centers can not be the same, if the corresponding values of the governing function do not agree, secondly, the centers cannot be the same if the exceptional divisors in which they are contained are not the same and, in the last step, the remaining candidates are compared explicitly by mapping them through the resolution tree as described above. In the example of the last section, this corresponding calculations are the following:

**Example 6.1.** (Example 5.5 revisited)
In our example, identifying the exceptional divisor which arose from the first blowing up can be done by simply considering the tree. As we previously mentioned, the subsequent 0-dimensional centers in the charts 2 and 3 do not coincide and hence the identification of the exceptional divisors \( E_2 \) and \( E_3 \) does not involve any further calculations. For the remaining exceptional divisors, however, we cannot avoid passing through the tree. As all of these calculations are rather similar, we restrict our considerations to the comparison of the exceptional divisors which arise from the blowing ups (2a) and (2b) and the respective subsequent blowing ups to illustrate the main ideas of the identification process:

(a) Comparison of Centers in (2a) and (2b)

The ideal of the center (2a) in the respective chart is \( \langle u, t \rangle \) as we had previously computed; the one of (2b) is \( \langle t, r \rangle \) in another chart. Both of
these charts arose from the same blowing up. Therefore we can look at
the respective centers as subsets of $\mathbb{A}^3 \times \mathbb{P}^2$: the first one is $V(u, w, r, t)$,
whereas the second one is $V(u, y, w, t)$. These are two different lines
meeting in the point $V(u, y, w, r, t)$. Hence the two exceptional divisors
arising from the blowing ups at these centers cannot be the same.

(b) Comparison of Center in (2b) and subsequent Center in Branch (2a)
The ideal of the subsequent (and last) center in the branch (2a) is
$V(t, y, a) \subset \mathbb{A}^3 \times \mathbb{P}^1$, using variable names $t, r, y$ for $\mathbb{A}^3$ and $a, b$ for $\mathbb{P}^1$.
Luckily, this is not contained in the newborn exceptional divisor $V(r)$,
which allows us to blow it down directly to obtain $V(u, y, w, t)$ in the
parent, coinciding with the center (2b).

(c) Comparison of Center in (2a) and subsequent Center in Branch (2b)
In this case, the subsequent (and last) center in the branch (2b) is
$V(t, s, c) \subset \mathbb{A}^3 \times \mathbb{P}^1$, using variable names $t, u, s$ for $\mathbb{A}^3$ and $c, d$ for $\mathbb{P}^1$.
Again this is not contained in the newborn exceptional divisor $V(u)$.
Hence blowing down yields $V(y, w, s, t)$ which clearly does not coincide
with the center of (2a).
Here we could also have proceeded by the argument that the newborn
exceptional divisor in the (2b) branch also appears later on in the (2a)
branch and hence any later divisor in the (2b) branch cannot coincide
with the earliest exceptional divisor arising in the (2a) branch.

References


