Signature-based Gröbner basis algorithms in SINGULAR

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- An ideal I in R is an additive subgroup of R such that for $f \in I$, $g \in R$ it holds that $fg \in I$.
- $G = \{g_1, \ldots, g_s\} \subset R$ is a Gröbner basis of $I = \langle f_1, \ldots, f_m \rangle$ w.r.t. $<$

$$:\Longleftrightarrow$$

$$L_<(G) = L_<(I)$$

$$\Longleftrightarrow$$

For all $f, g \in G$ $\mathrm{spol}(f, g)$ reduces to zero w.r.t. G.

- **The basic problem**
- **Generic signature-based algorithms**
  - The basic idea
  - Generic signature-based Gröbner basis algorithm
  - Signature-based criteria
- **Implementations and recent work**
  - Efficient variants
  - Timings
  - Recent work

Example

Let $I = \langle g_1, g_2 \rangle \in \mathbb{Q}[x, y, z]$ be given where $\mathbf{g_1} = \mathbf{xy} - \mathbf{z^2}$, $\mathbf{g_2} = \mathbf{y^2} - \mathbf{z^2}$, and let $<$ be the graded reverse lexicographical ordering.

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$$\begin{aligned}\mathrm{spol}(g_2, g_1) &= xg_2 - yg_1 = \mathbf{xy^2} - xz^2 - \mathbf{xy^2} + yz^2 \\ &= -xz^2 + yz^2,\end{aligned}$$

so it reduces w.r.t. G to $\mathbf{g_3} = \mathbf{xz^2} - \mathbf{yz^2}$.

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$\Rightarrow$ **How can we discard such zero reductions in advance?**

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4. **A minimal signature** of p exists due to $\prec$.

Our example – now with signatures and $\prec_{\text{pot}}$

We have already computed the following data:

$$\begin{aligned} g_1 &= xy - z^2, \operatorname{sig}(g_1) = e_1, \\ g_2 &= y^2 - z^2, \operatorname{sig}(g_2) = e_2, \\ g_3 &= \operatorname{spol}(g_2, g_1) = xg_2 - yg_1 \\ \Rightarrow \operatorname{sig}(g_3) &= x \operatorname{sig}(g_2) = xe_2. \end{aligned}$$

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$\Rightarrow$ **We know that $\mathrm{spol}(g_3, g_1)$ will reduce to zero w.r.t. G.**

Why do we know this?

The general idea is to check the signatures of the generated s-polynomials.

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Our task

We need to take care of the correctness of the signatures throughout the computations.

Generic signature-based Gröbner basis algorithm

Input: Ideal $I = \langle f_1, \ldots, f_m \rangle$
Output: Gröbner Basis $\text{poly}(G)$ for I

1. $G \leftarrow \emptyset$
2. $G \leftarrow G \cup \{(e_i, f_i)\}$ for all $i \in \{1, \ldots, m\}$
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      (iii) If $\text{poly}(h) \xrightarrow{G} \text{poly}(r) \neq 0$

      $P \leftarrow P \cup \{(r, g) \mid g \in G\}$
      $G \leftarrow G \cup \{r\}$
5. Return $\text{poly}(G)$.

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      (i) $h \leftarrow \text{spol}(f, g)$
      (ii) If $\text{poly}(h) \xrightarrow{G} 0 \Leftarrow$ **signature-safe**
      (iii) If $\text{poly}(h) \xrightarrow{G} \text{poly}(r) \neq 0 \Leftarrow$ **signature-safe**
      & $\nexists g \in G$ such that $m\,\text{sig}(g) = \text{sig}(r)$ and
      $m\,\text{lm}(\text{poly}(g)) = \text{lm}(\text{poly}(r))$
      $P \leftarrow P \cup \{(r, g) \mid g \in G\}$
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signature-increasing: $\mathrm{sig}(p - cmq) = m\,\mathrm{sig}(q)$
signature-decreasing: $\mathrm{sig}(p - cmq) \prec \mathrm{sig}(p), m\,\mathrm{sig}(q)$

Termination

- If $\operatorname{sig}(r) = m \operatorname{sig}(g)$ and $\operatorname{lm}(\operatorname{poly}(r)) = m \operatorname{lm}(\operatorname{poly}(g))$ is not added to G.
- Each new element in G enlarges $\langle(\operatorname{sig}(r), \operatorname{lm}(\operatorname{poly}(r)))\rangle$.

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Correctness

- All possible s-polynomials are taken care of:
  signature-increasing reduction $\Rightarrow$ new pair in the next step.
- All elements r with $\text{poly}(r) \neq 0$ are added to G besides those fulfilling $\text{sig}(r) = m\,\text{sig}(g)$ and $\text{lm}\,(\text{poly}(r)) = m\,\text{lm}\,(\text{poly}(g))$.

Non-minimal signature (NM)

sig(h) not minimal for h? $\Rightarrow$ Remove h.

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Sketch of proof

1. There exists a syzygy $s \in R^m$ such that $\text{lm}(s) = \text{sig}(h)$.
   $\Rightarrow$ We can represent h with a lower signature.
2. Pairs are handled by increasing signatures.
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Our example with $\prec_{\text{pot}}$ revisited

$\text{sig}\left(\text{spol}(g_3, g_1)\right) = xye_2$

$$\left.\begin{array}{l} g_1 = xy - z^2 \\ g_2 = y^2 - z^2 \end{array}\right\} \Rightarrow \text{psyz}(g_2, g_1) = g_1 e_2 - g_2 e_1 = xye_2 + \ldots$$

Rewritable signature (RW)

$\text{sig}(g) = \text{sig}(h)$? $\Rightarrow$ Remove either g or h.

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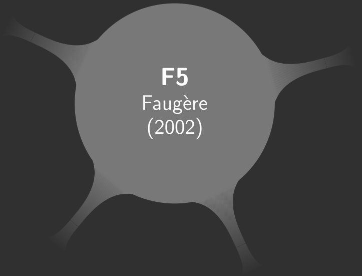
Sketch of proof

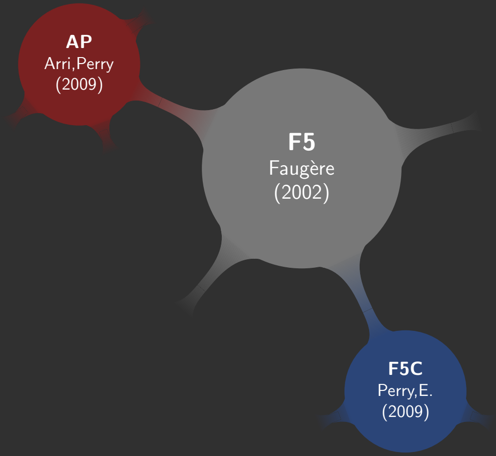
1. $\text{sig}(g - h) \prec \text{sig}(g), \text{sig}(h)$.
2. Pairs are handled by increasing signatures.
   $\Rightarrow$ All necessary computations of lower signature have already taken place.
   $\Rightarrow$ We can represent h by

$$h = g + \text{ elements of lower signature.}$$

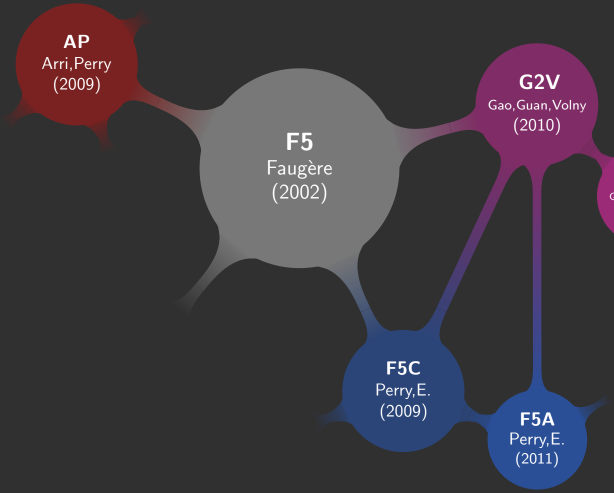
□


F5
Faugère
(2002)

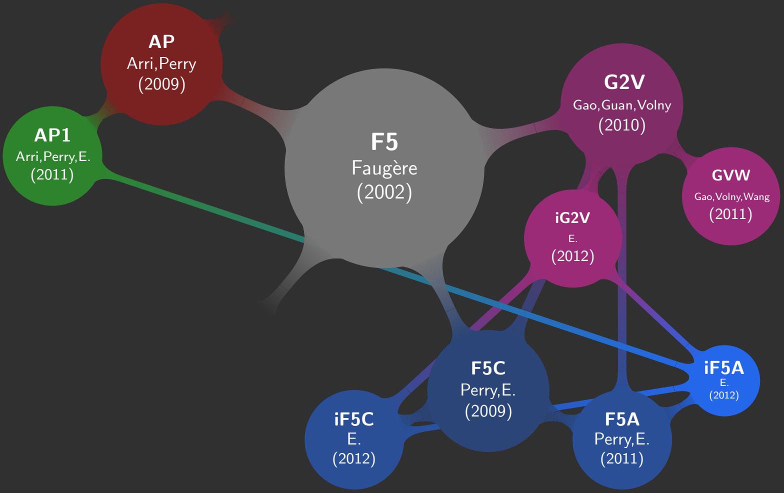
Efficient variants



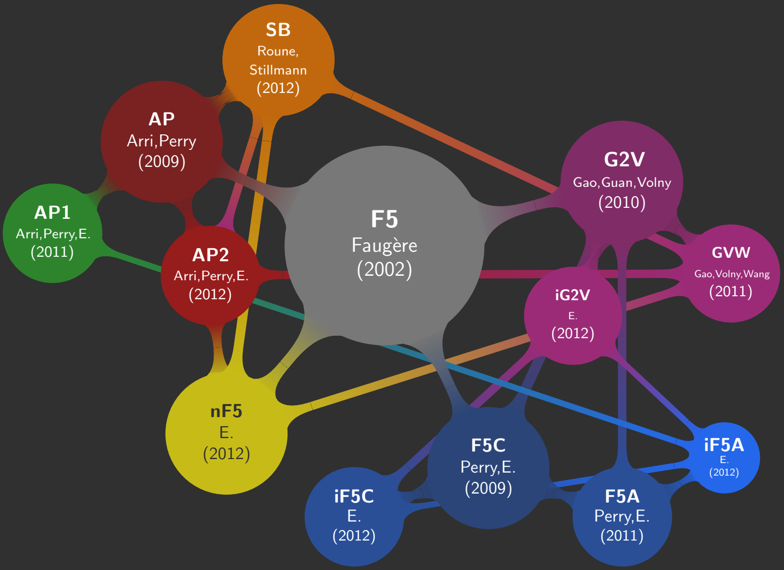
Efficient variants



Efficient variants

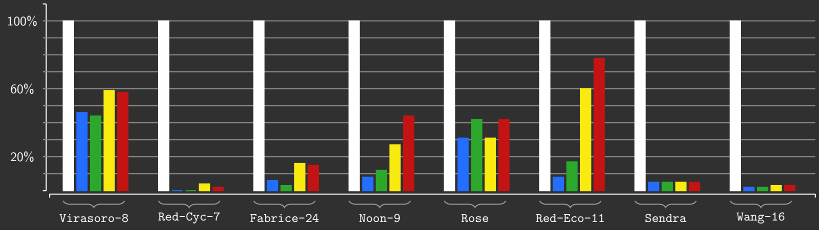


Efficient variants

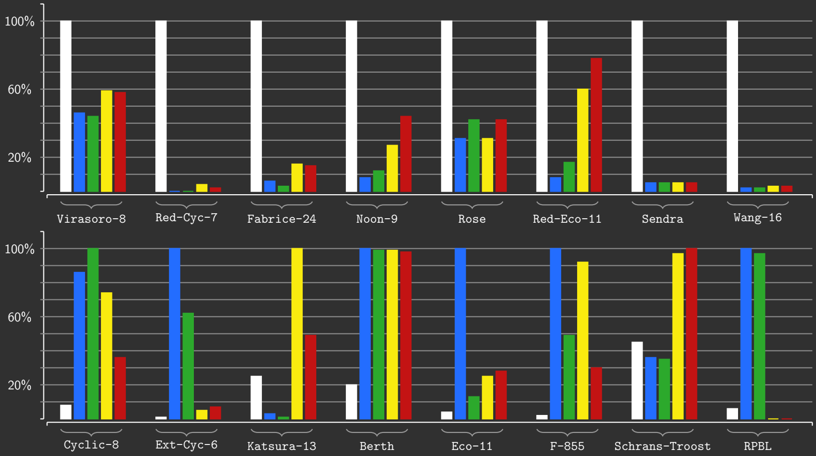


Efficient variants

Timings


100%
60%
20%
Virasoro-8
Red-Cyc-7
Fabrice-24
Noon-9
Rose
Red-Eco-11
Sendra
Wang-16

Timings



100%
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100%
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20%
Cyclic-8
Ext-Cyc-6
Katsura-13
Berth
Eco-11
F-855
Schrans-Troost
RPBL

- **Heuristics:**
  orderings on signatures; orderings for critical pairs (sugar degree), reducers
- **F4:**
  linear algebra for reduction purposes
- **Parallelisation:**
  modular methods, parallel criteria checks
- **Computation of syzygies:**
  implementation
- **Generalization of signature-based criteria:**
  more terms per signature, relaxing criteria for combination with Buchberger's criteria

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