Signature Rewriting in Gröbner Basis Computation

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June 29, 2013

- **Signature-based algorithms**
  - The basic idea
  - Generic signature-based criteria
- **Rewritable signature criterion in detail**

Let $I = \langle f_1, \ldots, f_m \rangle$.
Idea: Give each $f \in I$ a bit more structure:

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4. **A minimal signature** of p exists due to $\prec$.

Notations concerning signatures

Let $\alpha \in R^m$, then α contains all data we need:

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Moreover, we agree on the following:

- For $\alpha, \beta \in R^m$, let $\alpha \simeq \beta$ if $\alpha = s\beta$ for some $s \in K$. In the same sense we define $\overline{\alpha} \simeq \overline{\beta}$ if $\overline{\alpha} = t\overline{\beta}$ for some $t \in K$.
- $\mathcal{G}$ always denotes a finite subset of R^m such that for all $\alpha, \beta \in \mathcal{G}$ with $\mathfrak{s}(\alpha) \simeq \mathfrak{s}(\beta)$ it holds that $\alpha = \beta$.
- $\alpha \in R^m$ is called a syzygy if $\overline{\alpha} = 0$.

Reductions concerning signatures

Let $\alpha \in R^m$, and let t be a term of $\overline{\alpha}$. We can **$\mathfrak{s}$-reduce** t by $\beta \in R^m$ if

- $\exists$ a term b such that $\text{lt}\left(\overline{b\beta}\right) = t$ and
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1. If $\operatorname{lt}\left(\overline{b\beta}\right) \simeq \operatorname{lt}\left(\overline{\alpha}\right)$ $\Longrightarrow$ **top $\mathfrak{s}$-reduction step**;
   otherwise $\Longrightarrow$ **tail $\mathfrak{s}$-reduction step**.
2. If $\mathfrak{s}\left(b\beta\right) \simeq \mathfrak{s}\left(\alpha\right)$ $\Longrightarrow$ **singular $\mathfrak{s}$-reduction step**;
   otherwise $\Longrightarrow$ **regular $\mathfrak{s}$-reduction step**.

- $\mathfrak{s}$-reductions are always performed w.r.t. a finite basis $\mathcal{G} \subset R^m$.
- $\mathcal{G}$ is a **signature Gröbner basis in signature** T if all $\alpha \in R^m$ with $\mathfrak{s}(\alpha) = T$ $\mathfrak{s}$-reduce to zero w.r.t $\mathcal{G}$.
- $\mathcal{G}$ is a **signature Gröbner basis** if it is a signature Gröbner basis in all signatures.
- If $\mathcal{G}$ is a signature Gröbner basis, then $\{\overline{\alpha} \mid \alpha \in \mathcal{G}\}$ is a Gröbner basis for $\langle f_1, \ldots, f_m \rangle$.

Note

In the following we do not need much of the details of signature-based Gröbner basis algorithms, just one property:

The pair set is ordered by increasing signatures.

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Sketch of proof

1. There exists a syzygy $\beta \in R^m$ such that $\mathrm{lt}(\beta) = \mathfrak{s}(\alpha)$.
   $\Rightarrow$ We can represent $\overline{\alpha}$ with a lower signature.
2. Pairs are handled by increasing signatures.
   $\Rightarrow$ All relations of lower signature are already taken care of.

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$\mathfrak{s}(\alpha) = \mathfrak{s}(\beta)$? $\Rightarrow$ Remove either α or β.

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Sketch of proof

1. $\mathfrak{s}(\alpha - \beta) \prec \mathfrak{s}(\alpha), \mathfrak{s}(\beta)$.
2. Pairs are handled by increasing signatures.
   $\Rightarrow$ All necessary computations of lower signature have already taken place.
   $\Rightarrow$ We can represent β by

$$\beta = \alpha + \text{ elements of lower signature.}$$

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- A **rewrite order** $\lhd$ is a total order on $\mathcal{G}$ such that $\mathfrak{s}(\alpha) \mid \mathfrak{s}(\beta) \Rightarrow \alpha \lhd \beta$. (Exists due to $\mathfrak{s}(\alpha) \simeq \mathfrak{s}(\beta) \Rightarrow \alpha = \beta$.)

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- A multiple of a basis element $t\alpha$ is called **rewritable** if α is not the canonical rewriter in $\mathfrak{s}(t\alpha)$.

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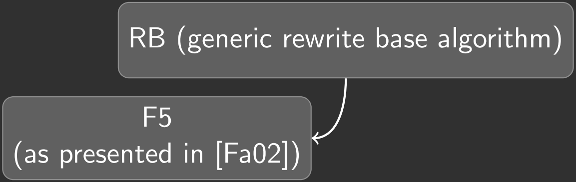
Lemma

If $\mathcal{G}$ is a rewrite basis up to signature T then $\mathcal{G}$ is also a signature Gröbner basis up to T.

Improving the rewritable signature criterion

RB (generic rewrite base algorithm)

Improving the rewritable signature criterion



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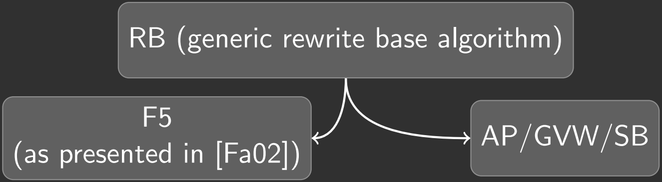

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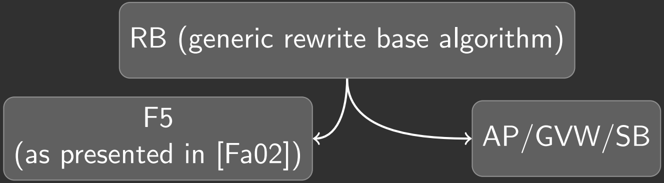

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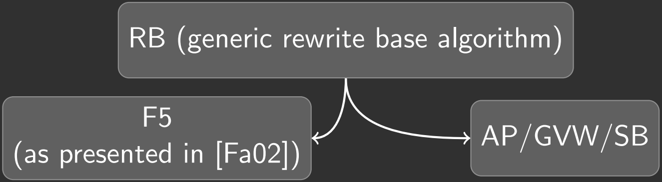

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$$\alpha \lhd \beta$$
$$\Longleftrightarrow$$
$$(i < j) \text{ or } (i = j \text{ and } a < b)$$

Once an s-polynomial in a given signature T is computed all others are rewritable.

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F5 (as presented in [Fa02])

AP/GVW/SB

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$$\alpha \lhd \beta \iff \frac{\mathfrak{s}(\alpha)}{\mathrm{lt}(\overline{\alpha})} \prec \frac{\mathfrak{s}(\beta)}{\mathrm{lt}(\overline{\beta})}$$

For any signature T define
$$M_T = \{t\alpha \mid \alpha \in \mathcal{G}, \mathfrak{s}(t\alpha) = T\}$$

Choose $t\alpha$ such that $\mathrm{lt}(\overline{t\alpha})$ is minimal.

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Difference: There might be no such s-polynomial

Example for differences in the rewritable signature criterion

Let K be the finite field with 13 elements and let $R := K[x, y, z, t]$. Let $<$ be the graded reverse lexicographic monomial ordering. Consider the three input elements

$$g_1 := -2y^3 - x^2z - 2x^2t - 3y^2t, \quad g_2 := 3xyz + 2xyt,$$
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$\alpha_i \in \mathcal{G}$	reduced from	$\operatorname{lt}(\overline{\alpha_i})$	$\mathfrak{s}(\alpha_i)$
α_1	$\mathbf{e}_1$	y^3	$\mathbf{e}_1$
α_2	$\mathbf{e}_2$	xyz	$\mathbf{e}_2$
α_3	$y^2\alpha_2 - xz\alpha_1 = \mathcal{S}(\alpha_2, \alpha_1)$	x^3z^2	$y^2\mathbf{e}_2$
α_4	$\mathbf{e}_3$	yz^2	$\mathbf{e}_3$
α_5	$x\alpha_4 - z\alpha_2 = \mathcal{S}(\alpha_4, \alpha_2)$	xz^3	$x\mathbf{e}_3$
α_6	$y^2\alpha_4 - z^2\alpha_1 = \mathcal{S}(\alpha_4, \alpha_1)$	x^2z^3	$y^2\mathbf{e}_3$
α_7	$y\alpha_5 - z^2\alpha_2 = \mathcal{S}(\alpha_5, \alpha_2)$	x^2y^2t	$xy\mathbf{e}_3$
α_8	$x\alpha_5 - \alpha_6 = \mathcal{S}(\alpha_5, \alpha_6)$	z^5	$x^2\mathbf{e}_3$
α_9	$x\alpha_6 - z\alpha_3 = \mathcal{S}(\alpha_6, \alpha_3)$	x^4zt	$xy^2\mathbf{e}_3$
α_{10}	$y\alpha_8 - z^3\alpha_4 = \mathcal{S}(\alpha_8, \alpha_4)$	x^3y^2t	$x^2y\mathbf{e}_3$
α_{11}	$x^3\alpha_4 - y\alpha_3 = \mathcal{S}(\alpha_4, \alpha_3)$	x^4yt	$x^3\mathbf{e}_3$
α_{12}	$z\alpha_{11} - x^3\alpha_2 = \mathcal{S}(\alpha_{11}, \alpha_2)$	x^3zt^3	$x^3z\mathbf{e}_3$
α_{13}	$y\alpha_{10} - x^3\alpha_1 = \mathcal{S}(\alpha_{10}, \alpha_1)$	x^5zt	$x^2y^2\mathbf{e}_3$
α_{14}	$x\alpha_{12} - \alpha_9 = \mathcal{S}(\alpha_{12}, \alpha_9)$	x^4t^4	$x^4z\mathbf{e}_3$

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