Midway upon the Journey

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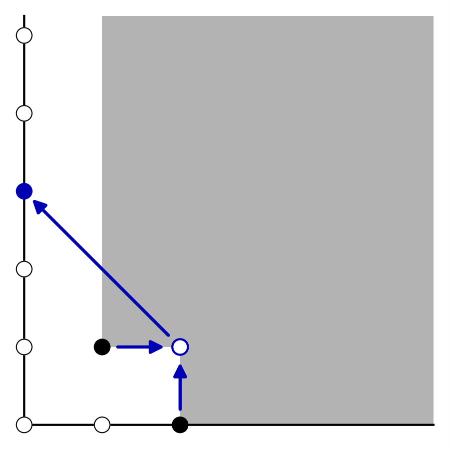
ACA 2014

Overview

§1. The journey

Buchberger's algorithm



Expand ideal of **leading terms** ⇝ Gröbner basis

Question

How do we determine a “leading” term?

$$x^2 + y^3 - 4, \quad xy - 1$$

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How do we determine a "leading" term?

$$x^2 + y^3 - 4, \quad \mathbf{xy} - 1$$

- not -4, -1

How do we determine a “leading” term?

$$\mathbf{x^2} + y^3 - 4, \quad \mathbf{xy} - 1$$

- **lex**icographic (lex)

Question

How do we determine a “leading” term?

$$x^2 + \mathbf{y}^3 - 4, \quad xy - 1$$

- graded **rev**erse **lex**icographic (grevlex)

Question

How do we determine a "leading" term?

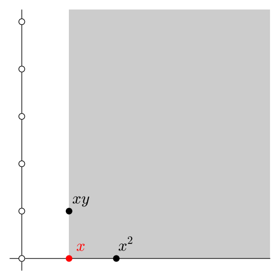
$$x^2 + y^3 - 4, \quad xy - 1$$

- weighted **deg**ree orderings (wdeg)

Comparison

lex

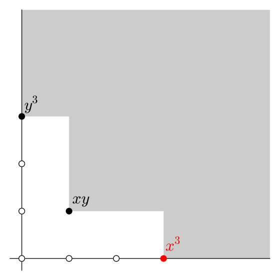
$x^2 + y^3 - 4$ $\quad\xrightarrow{y}\quad$ $x + y^4 - 4y$ $\quad\xleftarrow{x}\quad$ $xy - 1$



Comparison

grevlex

$x^2 + y^3 - 4$ $\quad\xrightarrow{x}\quad$ $x^3 + y^2 - 4x$ $\quad\xleftarrow{x}\quad$ $xy - 1$



“Long-term” behavior can vary

$$x^2 + y^3 - 4,\ xy - 1$$

	lex	grevlex
#s-polys	3	2
#polys	2	3

Another example

$$\left\{\begin{array}{c} x_1 + x_2 + \cdots + x_7, \\ x_1x_2 + x_2x_3 + \cdots + x_1x_1, \\ x_1x_2x_3 + x_2x_3x_4 + \cdots + x_7x_1x_2, \\ \vdots \\ x_1x_2x_3x_4x_5x_6x_7 - h^7 \end{array}\right\}$$

	lex	grevlex	wdeg
#-spolys	5458	2192	372
#polys	985	443	101
time† (sec)	641.699	1.553	0.482

†2.5 GHz Intel Core i5, OSX 10.10.4, custom SINGULAR

wdeg uses (5769, 5768, 4588, 2902, 5726, 4904, 1, 3453)

Can we find a “good” ordering *during* computation?

Question

Can we find a "good" ordering *during* computation?

Dynamic Gröbner basis computation

Can we find a "good" ordering *during* computation?

Dynamic Gröbner basis computation

- smaller basis $\Longrightarrow$ easier to work with
- less computation $\Longrightarrow$ quicker result?

§2. The forest dark

Previous work

- MR88 Mora and Robbiano (1988)
- GS93 Gritzmann and Sturmfels (1993)
- C93 Caboara (1993)
- GP O. Golubitsky (preprint)
- CP14 Caboara and Perry (2014)

ω: weight vector

$$x^{\mathbf{a}} > y^{\mathbf{b}} \quad \Longleftrightarrow \quad \omega \cdot \mathbf{a} > \omega \cdot \mathbf{b}$$

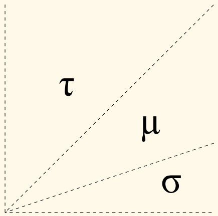
ω: weight vector

lex: $\mathbf{x^2} + y^3 - 4 \quad \Longrightarrow \quad \omega = (1,0)$

ω: weight vector

grevlex: $x^2 + \mathbf{y}^3 - 4 \quad \Longleftarrow \quad \omega = (1, 1)$

open polyhedral cone



Corner vectors identify potential leading terms (CP14)

Computing orderings

pre-2014: ***simplex***

- pros
  - fast + iterative
  - huge literature
  - many implementations
- cons
  - integer solving *bottleneck*
  - solution $\rightsquigarrow$ *one* ray

Computing orderings

current: ***double description method***

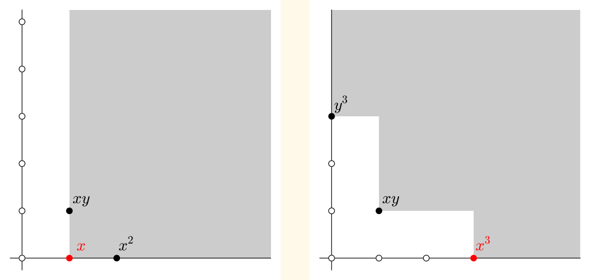
- pros
    - iterative + not too slow
    - literature exists
    - solving $\rightsquigarrow$ *all* rays
    - integer solving *fast* (mostly)
- cons
    - "edge detection" can be slow

SINGULAR

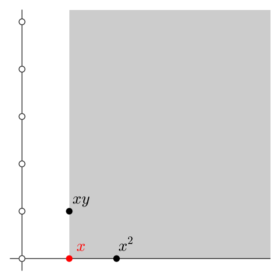
- C++
- works for many systems
- optimization, debuggin, research
- available online: https://github.com/johnperry-math/Sources.git

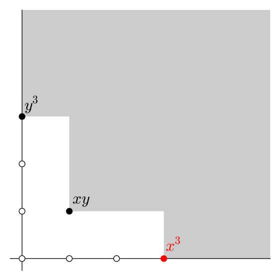
§3. The straight path

A “good” ordering?



A "good" ordering?





rightmost diagram nicer

- finite # residues
- no redundant basis elements (yet)

Idea #1: Hilbert combinatorics

Definition ($HF(d)$)

(intuitive) number of dots outside gray region
(precise) linear dimension of residues of degree d

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Definition (Hilbert series)

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(intuitive) number of dots outside gray region
(precise) linear dimension of residues of degree d

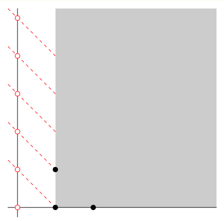
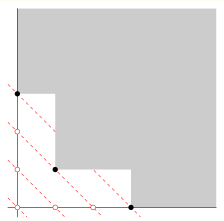
Definition (Hilbert series)

$$\sum_{d=0}^{\infty} HF(d)\, t^d$$

Definition ($HP(d)$)

Eventual value of Hilbert function (after **degree of regularity**)

Example

		
$HF(d)$	$(1, 1, 1, \ldots)$	$(1, 2, 2, 0, 0, 0, \ldots)$
$HP(d)$	1	0
$HS(t)$	$1 + 1t + \cdots = -\frac{1}{t-1}$	$1 + 2t + 2t^2 + 0t^3 + \cdots$

Hilbert heuristic (C93, GS93)

Approximate ideal of leading terms

Long-term test

Hilbert polynomial minimized at disagreement of *largest* degree

$$1 + 2t + t^3 + \mathbf{3t^4} > 7 + 3t + \mathbf{2t^4}$$

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Approximate ideal of leading terms

Long-term test

Hilbert polynomial minimized at disagreement of *largest* degree

$$1 + 2t + t^3 + \mathbf{3t^4} > 7 + 3t + \mathbf{2t^4}$$

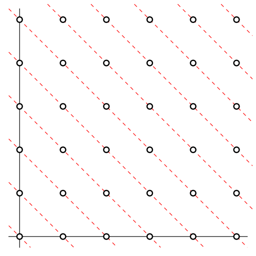
Short-term test

Hilbert function minimized at disagreement of *smallest* degree

$$(1, \mathbf{1}, 1, \ldots) < (1, \mathbf{2}, 2, 0, 0, 0, \ldots)$$

Remarks

Accurate measure, *but*...



$$\omega = (1,1)$$

$$(1,2,3,4,\ldots)$$

Remarks

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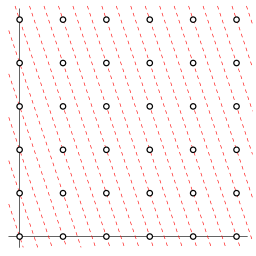
- new order? nonstandard deg



$\omega = (3,1)$

Remarks

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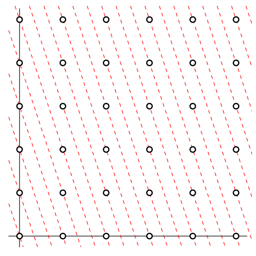
- graded *HF* better?



$$\omega = (3, 1)$$

$$(1, 1, 1, 2, 2, 2, \ldots)$$

Remarks

Accurate measure, *but...*

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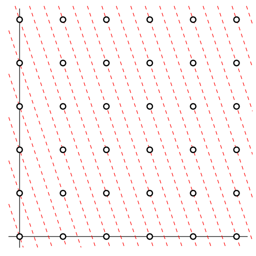
- no graded *HP*!



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  - quasi-polynomial (periodic)
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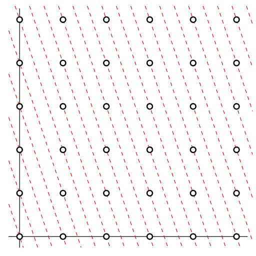

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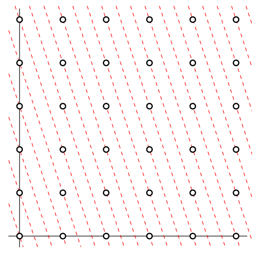


$\omega = (3, 1)$

$(1, 1, 1, 2, 2, 2, \ldots)$

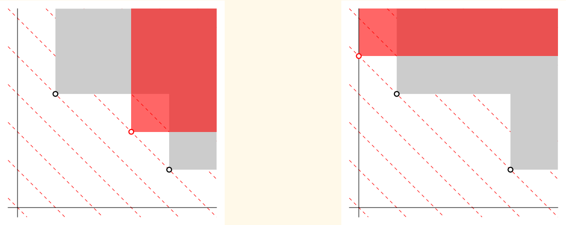
Alternate Hilbert Heuristic

1. minimize *standard* Hilbert polynomial
2. break ties w/*graded* Hilbert function

Degree heuristic

Lower- degree terms an indirect Hilbert heuristic. . . ?

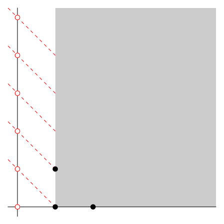
(computation *much* faster!)



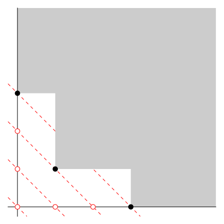
$\deg\left(x^3y^2\right) = 5$ $\deg y^4 = 3$

Degree heuristic

Not always!



$\deg(x) = 1$



$\deg x^3 = 3$

Critical pair heuristic (Eder)

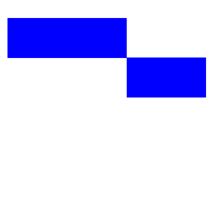
Maximize work at low degree

- count deg, # of new pairs
- minimize degree of new pairs
- break ties by maximal number at low degree

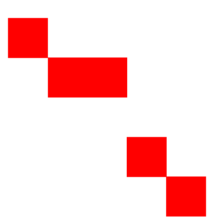
Minimum difference in degree

Good computational properties of homogeneous polynomials

- $\max(\deg t) - \min(\deg t)\ \forall t \in g \ldots ?$



homogeneous-ish

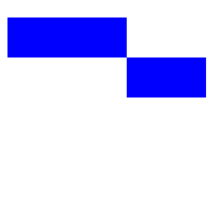


not so much!

Largest component of maximal degree (Faugère)

Good computational properties of regular systems

- polys have large component of maximal degree



small component

large component

Difficulty with last two

Min difference degree, Largest maximal component:
Require weight vector for each term: expensive

- workaround solve integer program for each term
- "simplex oracle" mitigates cost?
- not yet implemented

§4. Example systems

Benchmark systems

| basis |

	std()	Hilbert	degree	crit pairs
Caboara 1	239	34	114	142
Caboara 2	455	19	24	26
Cyc-7 hom	443	101	108	110
Cyc-8 hom	1182	513	422	595

Benchmark systems

#spolys

	std()	Hilbert	degree	crit pairs
Caboara 1	888	125	383	486
Caboara 2	8826	29	36	36
Cyc-7 hom	2192	372	397	402
Cyc-8 hom	7018	2625	2069	3080

Benchmark systems

time (sec)

	std()	Hilbert	degree	crit pairs
Caboara 1	.032	.225	.734	.658
Caboara 2	.527	.047	.060	.064
Cyc-7 hom	1.518	1.605	2.177	1.825
Cyc-8 hom	50.308	235	492	247

Sparse polynomials

|basis|

	std()	Hilbert	degree	crit pairs
7 x, 5 f	251	289	**BUG!**	386
6 x, 4 f	38	12	14	40
10 x, 8 f (bin)	61	16	7	17
6 x, 10 f (bin)	3	5	5	5
6 x, 10 f (trin)	12	12	12	14
5 x, 9 f (bin)	27	24	28	41
8 x, 5 f (bin)	50	37	37	105

#spolys

	std()	Hilbert	degree	crit pairs
$7\,x$, $5\,f$	1304	1538	**BUG**	2048
$6\,x$, $4\,f$	121	28	32	124
$10\,x$, $8\,f$ (bin)	21	35	10	40
$6\,x$, $10\,f$ (bin)	75	5	5	5
$6\,x$, $10\,f$ (trin)	20	20	20	24
$5\,x$, $9\,f$ (bin)	78	76	83	111
$8\,x$, $5\,f$ (bin)	182	122	141	424

Remarks on the tables

Good performance in binomial systems

- esp. “Caboara 2” (integer programming)

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- easy optimization made Hilbert timings competitive on Cyclic-7h
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time (sec)

	grevlex	Hilbert	wdeg
Cyc-7h	1.518	1.605	0.482
Cyc-8h	50.308	235	43.8

Complexity

Computing Hilbert polynomial, function?

- profiling: not an issue!
- faster algorithms exist: “Frobby” (Roune)

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DDM v. Simplex?

- wdeg **requires integer solution**
- simplex integer solvers inefficient
    - branch and cut, . . .
- DDM’s integer solutions free!

§5. Into the inferno

Signature-based dynamic algorithm

Mere blueprints…

Pros

- usual pros of signatures
- avoid zero reductions?
  - avoid *most* of them?
  - a lot of time spent computing 0 in dynamic algorithm

Cons

- inequalities need to preserve signatures
  - including reduction!
- can't use standard, graded Hilbert functions

Problem for Hilbert function?

Signature-based reduction sometimes incomplete

Problem for Hilbert function?

Signature-based reduction sometimes incomplete

$$r = (\mathbf{xy^2}, x^5 + \cdots) \quad g = (\mathbf{x^2}, x^4 + \cdots)$$

$$x \cdot \mathbf{x^2} > \mathbf{xy^2} \quad \Longrightarrow \quad \text{cannot reduce } r \text{ by } xg$$

Problem for Hilbert function?

Signature-based reduction sometimes incomplete

$$r = (\mathbf{xy}^2, x^5 + \cdots) \quad g = (\mathbf{x}^2, x^4 + \cdots)$$

$$x \cdot \mathbf{x}^2 > \mathbf{xy}^2 \implies \text{cannot reduce } r \text{ by } xg$$

- add r, *ideal does not expand!* $(x^4 \mid x^5)$

∴ $HF(d)$ does not change!

Workaround

$\mathbb{X}_n \times \mathbb{X}_n \longrightarrow \mathbb{X}_{2n}$:

$$f = \left(\mathbf{x}\mathbf{y}^2, x^5 + \cdots\right) \longmapsto \mathbf{u}\mathbf{v}^2 x^5$$
$$g = \left(\mathbf{x}^2, x^4 + \cdots\right) \longmapsto \mathbf{u}^2 x^4$$

Workaround

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$$f = (\mathbf{xy}^2, x^5 + \cdots) \longmapsto \mathbf{uv}^2 x^5$$
$$g = (\mathbf{x}^2, x^4 + \cdots) \longmapsto \mathbf{u}^2 x^4$$

Gives Hilbert heuristic in $\mathbb{X}_{2n}$!

- can reject[†] polynomials that don't expand $\mathbb{X}_{2n}$ ideal
- *necessary* polynomials expand ideal

[†]Arri & Perry 2011, Eder & Perry 2011

§6. Conclusion

Future work

Near term

- bug fixing, optimization
- full tests of heuristics on many ideals

Mid term

- signature-based dynamic algorithm

Long term

- parallelization of DDM
- F4-style implementation?
    - refinement $\leftrightsquigarrow$ swap columns

Thank you!

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Centre for Computer Algebra

University of Southern Mississippi

Галина Валеревна Заморий

Further reading

- Mora and Robbiano, "The Gröbner fan of an ideal." *JSC* 6 (1988) 183–208.
- Grtizmann and Sturmfels, "Minkowski Addition of Polytopes: …" *SIAM J. Disc. Math* 6 (1993) 246–269.
- Caboara, "A Dynamic Algorithm for Gröbner basis computation." *ISSAC '93*, ACM Press, 275–283.
- O. Golubitsky, "Converging term order sequences and the dynamic Buchberger algorithm." Preprint (private communication).
- Caboara and Perry, "Reducing the size and number of linear programs in a dynamic Gröbner basis algorithm." *AAECC* 25 (2014) 99–117.
- Carla Mascia, *On the Hilbert quasi-polynomials of non-standard graded rings*, Master's Thesis (2014), Università degli Studi di Pisa.