# 10. Smooth Varieties

Let $a$ be a point on a variety $X$. In the last chapter we have introduced the tangent cone $C_aX$ as a way to study $X$ locally around $a$ (see Construction 9.20). It is a cone whose dimension is the local dimension $\operatorname{codim}_X\{a\}$ (Corollary 9.24), and we can think of it as the cone that best approximates $X$ around $a$. In an affine open chart where $a$ is the origin, we can compute $C_aX$ by choosing an ideal with zero locus $X$ and replacing each polynomial in this ideal by its initial term (Exercise 9.22 (b)).

However, in practice one often wants to approximate a given variety by a linear space rather than by a cone. We will therefore study now to what extent this is possible, and how the result compares to the tangent cones that we already know. Of course, the idea to construct this is just to take the *linear terms* instead of the *initial terms* of the defining polynomials when considering the origin in an affine variety. For simplicity, let us therefore assume for a moment that we have chosen an affine neighborhood of the point $a$ such that $a = 0$ — we will see in Lemma 10.5 that the following construction actually does not depend on this choice.

**Definition 10.1** (Tangent spaces)**.** Let $a$ be a point on a variety $X$. By choosing an affine neighborhood of $a$ we assume that $X \subset \mathbb{A}^n$ and that $a = 0$ is the origin. Then

$$T_aX := V(f_1 : f \in I(X)) \quad \subset \mathbb{A}^n$$

is called the **tangent space** of $X$ at $a$, where $f_1 \in K[x_1,\dots,x_n]$ denotes the linear term of a polynomial $f \in K[x_1,\dots,x_n]$ as in Definition 6.6 (a).

As in the case of tangent cones, we can consider $T_aX$ either as an abstract variety (leaving its dimension as the only invariant since it is a linear space) or as a subspace of $\mathbb{A}^n$.

**Remark 10.2.**

(a) In contrast to the case of tangent cones in Exercise 9.22 (c), it always suffices in Definition 10.1 to take the zero locus only of the linear parts of a set $S$ of generators for $I(X)$: if $f,g \in S$ are polynomials such that $f_1$ and $g_1$ vanish at a point $x \in \mathbb{A}^n$ then

$$(f+g)_1(x) = f_1(x) + g_1(x) = 0$$

$$\text{and} \qquad (hf)_1(x) = h(0)\,f_1(x) + f(0)\,h_1(x) = h(0)\cdot 0 + 0\cdot h_1(x) = 0$$

for an arbitrary polynomial $h \in K[x_1,\dots,x_n]$, and hence $x \in T_aX$.

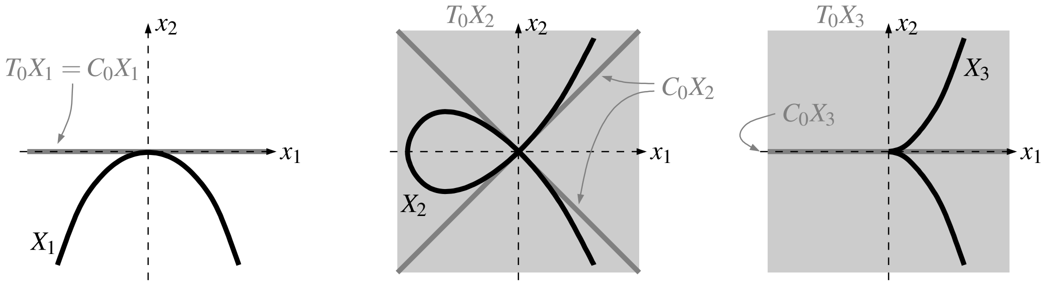
(b) However, again in contrast to the case of tangent cones in Exercise 9.22 it is crucial in Definition 10.1 that we take the radical ideal of $X$ and not just any ideal with zero locus $X$: the ideals $(x)$ and $(x^2)$ in $K[x]$ have the same zero locus $\{0\}$ in $\mathbb{A}^1$, but the zero locus of the linear term of $x$ is the origin again, whereas the zero locus of the linear term of $x^2$ is all of $\mathbb{A}^1$.

(c) For polynomials vanishing at the origin, a non-vanishing linear term is clearly always initial. Hence by Exercise 9.22 (b) it follows that $C_aX \subset T_aX$, i.e. that the tangent space always contains the tangent cone. In particular, this means by Corollary 9.24 that $\dim T_aX \geq \operatorname{codim}_X\{a\}$.

**Example 10.3.** Consider again the three curves $X_1,X_2,X_3$ of Example 9.21. By taking the initial resp. linear term of the defining polynomials we can compute the tangent cones and spaces of these curves at the origin:

- $X_1 = V(x_2 + x_1^2)$: $C_0X_1 = T_0X_1 = V(x_2)$;
- $X_2 = V(x_2^2 - x_1^2 - x_1^3)$: $C_0X_2 = V(x_2^2 - x_1^2)$, $T_0X_2 = V(0) = \mathbb{A}^2$;
- $X_3 = V(x_2^2 - x_1^3)$: $C_0X_3 = V(x_2^2) = V(x_2)$, $T_0X_3 = V(0) = \mathbb{A}^2$.

The following picture shows these curves together with their tangent cones and spaces. Note that for the curve $X_1$ the tangent cone is already a linear space, and the notions of tangent space and tangent cone agree. In contrast, the tangent cone of $X_2$ at the origin is not linear. By Remark 10.2 (c), the tangent space $T_0X_2$ must be a linear space containing $C_0X_2$, and hence it is necessarily all of $\mathbb{A}^2$. However, the curve $X_3$ shows that the tangent space is not always the linear space spanned by the tangent cone.




$X_1 = V(x_2 + x_1^2)$ $\quad$ $X_2 = V(x_2^2 - x_1^2 - x_1^3)$ $\quad$ $X_3 = V(x_2^2 - x_1^3)$

Before we study the relation between tangent spaces and cones in more detail, let us show first of all that the (dimension of the) tangent space is actually an intrinsic local invariant of a variety around a point, i. e. that it does not depend on a choice of affine open subset or coordinates around the point. We will do this by establishing an alternative description of the tangent space that does not need any such choices. The key observation needed for this is the isomorphism of the following lemma.

**Lemma 10.4.** *Let $X \subset \mathbb{A}^n$ be an affine variety containing the origin. Moreover, let us denote by $M := (\overline{x_1}, \dots, \overline{x_n}) = I(0) \trianglelefteq A(X)$ the ideal of the origin in $X$. Then there is a natural vector space isomorphism*

$$M/M^2 \cong \operatorname{Hom}_K(T_0X, K).$$

*In other words, the tangent space $T_0X$ is naturally the vector space dual to $M/M^2$.*

*Proof.* Consider the $K$-linear map

$$\varphi : M \to \operatorname{Hom}_K(T_0X, K), \;\; \overline{f} \mapsto f_1|_{T_0X}$$

sending the class of a polynomial modulo $I(X)$ to its linear term, regarded as a map restricted to the tangent space. By definition of the tangent space, this map is well-defined. Moreover, note that $\varphi$ is surjective since any linear map on $T_0X$ can be extended to a linear map on $\mathbb{A}^n$. So by the homomorphism theorem it suffices to prove that $\ker \varphi = M^2$:

"$\subset$" Consider the vector subspace $W = \{g_1 : g \in I(X)\}$ of $K[x_1, \dots, x_n]$, and let $k$ be its dimension. Then its zero locus $T_0X$ has dimension $n - k$, and hence the space of linear forms vanishing on $T_0X$ has dimension $k$ again. As it clearly contains $W$, we conclude that $W$ must be equal to the space of linear forms vanishing on $T_0X$.

So if $\overline{f} \in \ker \varphi$, i. e. the linear term of $f$ vanishes on $T_0X$, we know that there is a polynomial $g \in I(X)$ with $g_1 = f_1$. But then $f - g$ has no constant or linear term, and hence $\overline{f} = \overline{f - g} \in M^2$.

"$\supset$" If $\overline{f}, \overline{g} \in M$ then $(fg)_1 = f(0)\,g_1 + g(0)\,f_1 = 0 \cdot g_1 + 0 \cdot f_1 = 0$, and hence $\varphi(\overline{fg}) = 0$. □

In order to make Lemma 10.4 into an intrinsic description of the tangent space we need to transfer it from the affine coordinate ring $A(X)$ (which for a general variety would require the choice of an affine coordinate chart that sends the given point $a$ to the origin) to the local ring $\mathscr{O}_{X,a}$ (which is independent of any choices). To do this, recall by Lemma 3.21 that with the notations from above we have $\mathscr{O}_{X,a} \cong S^{-1}A(X)$, where $S = A(X) \backslash M = \{f \in A(X) : f(a) \neq 0\}$ is the multiplicatively closed subset of polynomial functions that are non-zero at the point $a$. In this ring

$$S^{-1}M = \left\{ \frac{g}{f} : g, f \in A(X) \text{ with } g(a) = 0 \text{ and } f(a) \neq 0 \right\}$$

is just the maximal ideal $I_a$ of all local functions vanishing at $a$ as in Definition 3.22. Using these constructions we obtain the following result.

**Lemma 10.5.** *With notations as above we have*

$$M/M^2 \cong (S^{-1}M)/(S^{-1}M)^2.$$

*In particular, if $a$ is a point on a variety $X$ and $I_a = \{\varphi \in \mathscr{O}_{X,a} : \varphi(a) = 0\}$ is the maximal ideal of local functions in $\mathscr{O}_{X,a}$ vanishing at $a$, then $T_aX$ is naturally isomorphic to the vector space dual to $I_a/I_a^2$, and thus independent of any choices.*

*Proof.* This time we consider the vector space homomorphism

$$\varphi : M \to (S^{-1}M)/(S^{-1}M)^2, \;\; g \mapsto \overline{\left(\frac{g}{1}\right)}$$

where the bar denotes classes modulo $(S^{-1}M)^2$. In order to deduce the lemma from the homomorphism theorem we have to show the following three statements:

- $\varphi$ is surjective: Let $\frac{g}{f} \in S^{-1}M$. Then $\frac{g}{f(0)} \in M$ is an inverse image of the class of this fraction under $\varphi$ since

$$\frac{g}{f} - \frac{g}{f(0)} = \frac{g}{f} \cdot \frac{f(0)-f}{f(0)} \quad \in (S^{-1}M)^2$$

  (note that $f(0)-f$ lies in $M$ as it does not contain a constant term).
- $\ker\varphi \subset M^2$: Let $g \in \ker\varphi$, i. e. $\frac{g}{1} \in (S^{-1}M)^2$. This means that

$$\frac{g}{1} = \sum_i \frac{h_i k_i}{f_i} \qquad (*)$$

  for a finite sum with elements $h_i, k_i \in M$ and $f_i \in S$. By bringing this to a common denominator we can assume that all $f_i$ are equal, say to $f$. The equation $(*)$ in $\mathscr{O}_{X,a}$ then means $\tilde{f}(fg - \sum_i h_i k_i) = 0$ in $A(X)$ for some $\tilde{f} \in S$ by Construction 3.12. This implies $\tilde{f}fg \in M^2$. But $((\tilde{f}f)(0) - \tilde{f}f)\,g \in M^2$ as well, and hence $(\tilde{f}f)(0)\,g \in M^2$, which implies $g \in M^2$ since $(\tilde{f}f)(0) \in K^*$.
- $M^2 \subset \ker\varphi$ is trivial. □

**Exercise 10.6.** Let $f : X \to Y$ be a morphism of varieties, and let $a \in X$. Show that $f$ induces a linear map $T_aX \to T_{f(a)}Y$ between tangent spaces.

We have now constructed two objects associated to the local structure of a variety $X$ at a point $a \in X$:

- the *tangent cone* $C_aX$, which is a cone of dimension $\operatorname{codim}_X\{a\}$, but in general not a linear space; and
- the *tangent space* $T_aX$, which is a linear space, but whose dimension might be bigger than $\operatorname{codim}_X\{a\}$.

Of course, we should give special attention to the case when these two notions agree, i. e. when $X$ can be approximated around $a$ by a linear space whose dimension is the local dimension of $X$ at $a$.

**Definition 10.7** (Smooth and singular varieties)**.** Let $X$ be a variety.

(a) A point $a \in X$ is called **smooth**, **regular**, or **non-singular** if $T_aX = C_aX$. Otherwise it is called a **singular** point of $X$.

(b) If $X$ has a singular point we say that $X$ is singular. Otherwise $X$ is called smooth, regular, or non-singular.

**Example 10.8.** Of the three curves of Example 10.3, exactly the first one is smooth at the origin. As in our original motivation for the definition of tangent spaces, this is just the statement that $X_1$ can be approximated around the origin by a straight line — in contrast to $X_2$ and $X_3$, which have a "multiple point" resp. a "corner" there. A more precise geometric interpretation of smoothness can be obtained by comparing our algebraic situation with the Implicit Function Theorem from analysis, see Remark 10.14.

**Lemma 10.9.** *Let $X$ be a variety, and let $a \in X$ be a point. The following statements are equivalent:*

(a) *The point $a$ is smooth on $X$.*

(b) $\dim T_a X = \operatorname{codim}_X \{a\}$.

(c) $\dim T_a X \leq \operatorname{codim}_X \{a\}$.

*Proof.* The implication (a) ⇒ (b) follows immediately from Corollary 9.24, and (b) ⇒ (c) is trivial. To prove (c) ⇒ (a), note first that (c) together with Remark 10.2 (c) implies $\dim T_a X = \operatorname{codim}_X \{a\}$. But again by Remark 10.2 (c), the tangent space $T_a X$ contains the tangent cone $C_a X$, which is of the same dimension by Corollary 9.24. As $T_a X$ is irreducible (since it is a linear space), this is only possible if $T_a X = C_a X$, i. e. if $a$ is a smooth point of $X$. □ 17

**Remark 10.10** (Smoothness in commutative algebra)**.** Let $a$ be a point on a variety $X$.

(a) Let $I_a \trianglelefteq \mathscr{O}_{X,a}$ be the maximal ideal of local functions vanishing at $a$ as in Definition 3.22. Combining Lemma 10.5 with Lemma 10.9 we see that $a$ is a smooth point of $X$ if and only if the vector space dimension of $I_a/I_a^2$ is equal to the local dimension $\operatorname{codim}_X \{a\}$ of $X$ at $a$. This is a property of the local ring $\mathscr{O}_{X,a}$ alone, and one can therefore study it with methods from commutative algebra. A ring with these properties is usually called a *regular local ring* [G5, Definition 11.38], which is also the reason for the name "regular point" in Definition 10.7 (a).

(b) It is a result of commutative algebra that a regular local ring as in (a) is always an integral domain [G5, Proposition 11.40]. Translating this into geometry as in Proposition 2.9, this yields the intuitively obvious statement that a variety is *locally irreducible* at every smooth point $a$, i. e. that $X$ has only one irreducible component meeting $a$. Equivalently, any point on a variety at which two irreducible components meet is necessarily a singular point.

The good thing about smoothness is that it is very easy to check using (formal) partial derivatives:

**Proposition 10.11 (Affine Jacobi criterion).** *Let $X \subset \mathbb{A}^n$ be an affine variety with ideal $I(X) = (f_1, \dots, f_r)$, and let $a \in X$ be a point. Then $X$ is smooth at $a$ if and only if the rank of the $r \times n$* ***Jacobian matrix***

$$\left( \frac{\partial f_i}{\partial x_j}(a) \right)_{i,j}$$

*is at least $n - \operatorname{codim}_X \{a\}$.*

*Proof.* Let $x = (x_1, \dots, x_n)$ be the coordinates of $\mathbb{A}^n$, and let $y := x - a$ be the shifted coordinates in which the point $a$ becomes the origin. By a formal Taylor expansion, the linear term of the polynomial $f_i$ in these coordinates $y$ is $\sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(a) \cdot y_j$. Hence the tangent space $T_a X$ is by Definition 10.1 and Remark 10.2 (a) the zero locus of these linear terms, i. e. the kernel of the Jacobian matrix $J = \left( \frac{\partial f_i}{\partial x_j}(a) \right)_{i,j}$. So by Lemma 10.9 the point $a$ is smooth if and only if $\dim \ker J \leq \operatorname{codim}_X \{a\}$, which is equivalent to $\operatorname{rk} J \geq n - \operatorname{codim}_X \{a\}$. □

To check smoothness for a point on a projective variety, we can of course restrict to an affine open subset of the point. However, the following exercise shows that there is also a projective version of the Jacobi criterion that does not need these affine patches and works directly with the homogeneous coordinates instead.

**Exercise 10.12.**

(a) Show that
$$\sum_{i=0}^{n} x_i \cdot \frac{\partial f}{\partial x_i} = d \cdot f$$
for any homogeneous polynomial $f \in K[x_0,\dots,x_n]$ of degree $d$.

(b) (**Projective Jacobi criterion**) Let $X \subset \mathbb{P}^n$ be a projective variety with homogeneous ideal $I(X) = (f_1,\dots,f_r)$, and let $a \in X$. Prove that $X$ is smooth at $a$ if and only if the rank of the $r \times (n+1)$ Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{i,j}$ is at least $n - \operatorname{codim}_X\{a\}$.

In this criterion, note that the entries $\frac{\partial f_i}{\partial x_j}(a)$ of the Jacobian matrix are not well-defined: multiplying the coordinates of $a$ by a scalar $\lambda \in K^*$ will multiply $\frac{\partial f_i}{\partial x_j}(a)$ by $\lambda^{d_i - 1}$, where $d_i$ is the degree of $f_i$. However, these are just row transformations of the Jacobian matrix, which do not affect its rank. Hence the condition in the projective Jacobi criterion is well-defined.

**Remark 10.13** (Variants of the Jacobi criterion)**.** There are a few ways to extend the Jacobi criterion even further. For simplicity, we will discuss this here in the case of an affine variety $X$ as in Proposition 10.11, but it is easy to see that the corresponding statements hold in the projective setting of Exercise 10.12 (b) as well.

(a) If $X$ is irreducible then $\operatorname{codim}_X\{a\} = \dim X$ for all $a \in X$ by Proposition 2.25 (b). So in this case $a$ is a smooth point of $X$ if and only if the rank of the Jacobian matrix is at least $n - \dim X = \operatorname{codim} X$.

(b) Let $f_1,\dots,f_r \in K[x_1,\dots,x_n]$ be polynomials such that $X = V(f_1,\dots,f_r)$, but that do not necessarily generate the ideal of $X$ (as required in Proposition 10.11). Then the Jacobi criterion still holds in one direction: assume that the rank of the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{i,j}$ is at least $n - \operatorname{codim}_X\{a\}$. The proof of the affine Jacobi criterion then shows that the zero locus of all linear terms of the elements of $(f_1,\dots,f_r)$ has dimension at most $\operatorname{codim}_X\{a\}$. The same then necessarily holds for the zero locus of all linear terms of the elements of $\sqrt{(f_1,\dots,f_r)} = I(X)$ (which might only be smaller). By Proposition 10.11 this means that $a$ is a smooth point of $X$.

The converse is in general false, as we have already seen in Remark 10.2 (b).

(c) Again let $f_1,\dots,f_r \in K[x_1,\dots,x_n]$ be polynomials with $X = V(f_1,\dots,f_r)$. This time assume that the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{i,j}$ has maximal row rank, i.e. that its rank is equal to $r$. As every irreducible component of $X$ has dimension at least $n-r$ by Proposition 2.25 (c) we know moreover that $\operatorname{codim}_X\{a\} \geq n-r$. Hence the rank of the Jacobian matrix is $r \geq n - \operatorname{codim}_X\{a\}$, so $X$ is smooth at $a$ by (b).

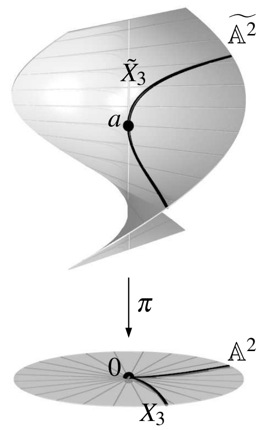
**Remark 10.14** (Relation to the Implicit Function Theorem)**.** The version of the Jacobi criterion of Remark 10.13 (c) is closely related to the *Implicit Function Theorem* from analysis. Given real polynomials $f_1,\dots,f_r \in \mathbb{R}[x_1,\dots,x_n]$ (or more generally continuously differentiable functions on an open subset of $\mathbb{R}^n$) and a point $a$ in their common zero locus $X = V(f_1,\dots,f_r)$ such that the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{i,j}$ has rank $r$, this theorem states roughly that $X$ is locally around $a$ the graph of a continuously differentiable function [G2, Proposition 27.9] — so that in particular it does not have any "corners". It can be shown that the same result holds over the complex numbers as well. So in the case $K = \mathbb{C}$ the statement of Remark 10.13 (c) that $X$ is smooth at $a$ can also be interpreted in this geometric way.

Note however that there is no algebraic analogue of the Implicit Function Theorem itself: for example, the polynomial equation $f(x_1,x_2) := x_2 - x_1^2 = 0$ cannot be solved for $x_1$ by a *regular* function locally around the point $(1,1)$, although $\frac{\partial f}{\partial x_1}(1,1) = -2 \neq 0$ — it can only be solved by a continuously differentiable function $x_1 = \sqrt{x_2}$.

**Example 10.15.** Consider again the curve $X_3 = V(x_2^2 - x_1^3) \subset \mathbb{A}^2_{\mathbb{C}}$ of Examples 9.21 and 10.3. The Jacobian matrix of the single polynomial $f = x_2^2 - x_1^3$ is

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -3x_1^2 & 2x_2 \end{pmatrix},$$

so it has rank (at least) $2 - \dim X = 1$ exactly if $(x_1, x_2) \in \mathbb{A}^2 \backslash \{0\}$. Hence the Jacobi criterion as in Remark 10.13 (a) does not only reprove our observation from Example 10.3 that the origin is a singular point of $X_3$, but also shows simultaneously that all other points of $X_3$ are smooth.




In the picture on the right we have also drawn the blow-up $\tilde{X}_3$ of $X_3$ at its singular point again. We have seen already that its exceptional set consists of only one point $a \in \tilde{X}_3$. Let us now check that this is a smooth point of $\tilde{X}_3$ — as we would expect from the picture.

In the coordinates $((x_1, x_2), (y_1 : y_2))$ of $\tilde{X}_3 \subset \widetilde{\mathbb{A}^2} \subset \mathbb{A}^2 \times \mathbb{P}^1$, the point $a$ is given as $((0,0),(1:0))$. So around $a$ we can use the affine open chart $U_1 = \{((x_1, x_2),(y_1 : y_2)) : y_1 \neq 0\}$ with affine coordinates $x_1$ and $y_2$ as in Example 9.15. By Exercise 9.22 (a), the blow-up $\tilde{X}_3$ is given in these coordinates by

$$\frac{(x_1 y_2)^2 - x_1^3}{x_1^2} = 0, \quad \text{i.e. } g(x_1, y_2) := y_2^2 - x_1 = 0.$$

As the Jacobian matrix

$$\begin{pmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial y_2} \end{pmatrix} = \begin{pmatrix} -1 & 2y_2 \end{pmatrix}$$

of this polynomial has rank 1 at every point, the Jacobi criterion tells us that $\tilde{X}_3$ is smooth. In fact, from the defining equation $y_2^2 - x_1 = 0$ we see that on the open subset $U_1$ the curve $\tilde{X}_3$ is just the "standard parabola" tangent to the exceptional set of $\widetilde{\mathbb{A}^2}$ (which is given on $U_1$ by the equation $x_1 = 0$ by the proof of Proposition 9.23).

It is actually a general statement that blowing up makes singular points "nicer", and that successive blow-ups will eventually make all singular points smooth. This process is called *resolution of singularities*. We will not discuss this here in detail, but the following exercise shows an example of this process.

**Exercise 10.16.** For $k \in \mathbb{N}$ let $X_k$ be the affine curve $X_k := V(x_2^2 - x_1^{2k+1}) \subset \mathbb{A}^2$. Show that $X_k$ is not isomorphic to $X_l$ if $k \neq l$.

(Hint: Consider the blow-up of $X_k$ at the origin.)

**Exercise 10.17.** Let $X \subset \mathbb{P}^3$ be the degree-3 Veronese embedding of $\mathbb{P}^1$ as in Exercise 7.32. Of course, $X$ must be smooth since it is isomorphic to $\mathbb{P}^1$. Verify this directly using the projective Jacobi criterion of Exercise 10.12 (b).

**Corollary 10.18.** *The set of singular points of a variety is closed.*

*Proof.* It suffices to prove the statement in the case of an affine variety $X$. We show that the subset $U \subset X$ of smooth points is open. So let $a \in U$. By possibly restricting to a smaller affine subset, we may assume by Remark 10.10 (b) that $X$ is irreducible. Then by Remark 10.13 (a) we know that $U$ is exactly the set of points at which the rank of the Jacobian matrix of generators of $I(X)$ is at least $\operatorname{codim} X$. As this is an open condition (given by the non-vanishing of at least one minor of size $\operatorname{codim} X$), the result follows. □

As the set of smooth points of a variety is open in the Zariski topology by Corollary 10.18, it is very "big" — unless it is empty, of course. Let us quickly study whether this might happen.

**Remark 10.19** (Generic smoothness)**.** Let $f \in K[x_1,\dots,x_n]$ be a non-constant irreducible polynomial, and let $X = V(f) \subset \mathbb{A}^n$. We claim that $X$ has a smooth point, so that the set of smooth points of $X$ is a non-empty open subset by Corollary 10.18, and thus dense by Remark 2.18.

Assume the contrary, i. e. that all points of $X$ are singular. Then by Remark 10.13 (a) the Jacobian matrix of $f$ must have rank 0 at every point, which means that $\frac{\partial f}{\partial x_i}(a) = 0$ for all $a \in X$ and $i = 1,\dots,n$. Hence $\frac{\partial f}{\partial x_i} \in I(V(f)) = (f)$ by the Nullstellensatz. But since $f$ is irreducible and the polynomial $\frac{\partial f}{\partial x_i}$ has smaller degree than $f$ this is only possible if $\frac{\partial f}{\partial x_i} = 0$ for all $i$.

In the case $\operatorname{char} K = 0$ this is already a contradiction to $f$ being non-constant. If $\operatorname{char} K = p$ is positive, then $f$ must be a polynomial in $x_1^p,\dots,x_n^p$, and so

$$f = \sum_{i_1,\dots,i_n} a_{i_1,\dots,i_n} x_1^{pi_1} \cdot \dots \cdot x_n^{pi_n} = \left( \sum_{i_1,\dots,i_n} b_{i_1,\dots,i_n} x_1^{i_1} \cdot \dots \cdot x_n^{i_n} \right)^p,$$

for $p$-th roots $b_{i_1,\dots,i_n}$ of $a_{i_1,\dots,i_n}$. This is a contradiction since $f$ was assumed to be irreducible.

In fact, one can show this "*generic smoothness*" statement for any variety $X$: the set of smooth points of $X$ is dense in $X$. A proof of this result can be found in [H, Theorem I.5.3].

**Example 10.20** (Fermat hypersurfaces)**.** For given $n,d \in \mathbb{N}_{>0}$ consider the *Fermat hypersurface*

$$X := V_p(x_0^d + \dots + x_n^d) \quad \subset \mathbb{P}^n.$$

We want to show that $X$ is smooth for all choices of $n$, $d$, and $K$. For this we use the Jacobian matrix $(dx_0^{d-1} \;\cdots\; dx_n^{d-1})$ of the given polynomial:

(a) If $\operatorname{char} K \nmid d$ the Jacobian matrix has rank 1 at every point, so $X$ is smooth by Exercise 10.12 (b).

(b) If $p = \operatorname{char} K \mid d$ we can write $d = k\,p^r$ for some $r \in \mathbb{N}_{>0}$ and $p \nmid k$. Since

$$x_0^d + \dots + x_n^d = (x_0^k + \dots + x_n^k)^{p^r},$$

we see again that $X = V_p(x_0^k + \dots + x_n^k)$ is smooth by (a).

**Exercise 10.21.** Let $X$ be a projective variety of dimension $n$. Prove:

(a) There is an injective morphism $X \to \mathbb{P}^{2n+1}$.

(b) There is in general no such morphism that is an isomorphism onto its image.

**Exercise 10.22.** Let $n \geq 2$. Prove:

(a) Every smooth hypersurface in $\mathbb{P}^n$ is irreducible.

(b) A general hypersurface in $\mathbb{P}^n_{\mathbb{C}}$ is smooth (and thus by (a) irreducible). More precisely, for a given $d \in \mathbb{N}_{>0}$ the vector space $\mathbb{C}[x_0,\dots,x_n]_d$ has dimension $\binom{n+d}{n}$, and so the space of all homogeneous degree-$d$ polynomials in $x_0,\dots,x_n$ modulo scalars can be identified with the projective space $\mathbb{P}^{\binom{n+d}{n}-1}$. Show that the subset of this projective space of all (classes of) polynomials $f$ such that $f$ is irreducible and $V_p(f)$ is smooth is dense and open.

**Exercise 10.23** (Dual curves)**.** Assume that $\operatorname{char} K \neq 2$, and let $f \in K[x_0,x_1,x_2]$ be a homogeneous polynomial whose partial derivatives $\frac{\partial f}{\partial x_i}$ for $i = 0,1,2$ do not vanish simultaneously at any point of $X = V_p(f) \subset \mathbb{P}^2$. Then the image of the morphism

$$F : X \to \mathbb{P}^2, \;\; a \mapsto \left( \frac{\partial f}{\partial x_0}(a) : \frac{\partial f}{\partial x_1}(a) : \frac{\partial f}{\partial x_2}(a) \right)$$

is called the *dual curve* to $X$.

(a) Find a geometric description of $F$. What does it mean geometrically if $F(a) = F(b)$ for two distinct points $a,b \in X$?

(b) If $X$ is a conic, prove that its dual $F(X)$ is also a conic.

(c) For any five lines in $\mathbb{P}^2$ in general position (what does this mean?) show that there is a unique conic in $\mathbb{P}^2$ that is tangent to all of them.