8. Grassmannians

After having introduced (projective) varieties — the main objects of study in algebraic geometry — let us now take a break in our discussion of the general theory to construct an interesting and useful class of examples of projective varieties. The idea behind this construction is simple: since the definition of projective spaces as the sets of 1-dimensional linear subspaces of K^n turned out to be a very useful concept, let us now generalize this and consider instead the sets of k-dimensional linear subspaces of K^n for an arbitrary $k = 0, \dots, n$.

Definition 8.1 (Grassmannians)**.** Let $n \in \mathbb{N}_{>0}$, and let $k \in \mathbb{N}$ with $0 \leq k \leq n$. We denote by $G(k,n)$ the set of all k-dimensional linear subspaces of K^n. It is called the **Grassmannian** of k-planes in K^n.

Remark 8.2. By Example 6.12 (b) and Exercise 6.32 (a), the correspondence of Remark 6.17 shows that k-dimensional linear subspaces of K^n are in natural one-to-one correspondence with $(k-1)$-dimensional linear subspaces of $\mathbb{P}^{n-1}$. We can therefore consider $G(k,n)$ alternatively as the set of such projective linear subspaces. As the dimensions k and n are reduced by 1 in this way, our Grassmannian $G(k,n)$ of Definition 8.1 is sometimes written in the literature as $G(k-1,n-1)$ instead.

Of course, as in the case of projective spaces our goal must again be to make the Grassmannian $G(k,n)$ into a variety — in fact, we will see that it is even a projective variety in a natural way. For this we need the algebraic concept of *alternating tensor products*, a kind of multilinear product on K^n generalizing the well-known *cross product*

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

on K^3 whose coordinates are all the 2×2 minors of the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

If you have seen ordinary tensor products in commutative algebra already [G5, Chapter 5], you probably know that the best way to introduce these products is by a universal property similar to the one for products of varieties in Definition 5.16. Although the same is true for our alternating tensor products, we will follow a faster and more basic approach here, whose main disadvantage is that it is not coordinate-free. Of course, if you happen to know the "better" definition of alternating tensor products using their universal property already, you can use this definition as well and skip the following construction.

Construction 8.3 (Alternating tensor products)**.** Let $(e_1, \dots, e_n)$ denote the standard basis of K^n. For $k \in \mathbb{N}$ we define $\Lambda^k K^n$ to be a K-vector space of dimension $\binom{n}{k}$ with basis vectors formally written as

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \qquad (*)$$

for all multi-indices $(i_1, \dots, i_k)$ of natural numbers with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Note that the set of these strictly increasing multi-indices is in natural bijection with the set of all k-element subsets $\{i_1, \dots, i_k\}$ of $\{1, \dots, n\}$, so that there are in fact exactly $\binom{n}{k}$ of these basis vectors. In particular, $\Lambda^k K^n$ is the zero vector space if $k > n$.

We extend the notation $(*)$ to arbitrary (i. e. not strictly increasing) multi-indices $(i_1, \dots, i_k)$ with $1 \leq i_1, \dots, i_k \leq n$ by setting $e_{i_1} \wedge \cdots \wedge e_{i_k} := 0$ if any two of the $i_1, \dots, i_k$ coincide, and

$$e_{i_1} \wedge \cdots \wedge e_{i_k} := \operatorname{sign} \sigma \cdot e_{i_{\sigma(1)}} \wedge \cdots \wedge e_{i_{\sigma(k)}}$$

if all $i_1,\dots,i_k$ are distinct, and σ is the unique permutation of $\{1,\dots,k\}$ such that $i_{\sigma(1)} < \cdots < i_{\sigma(k)}$. We can then extend this notation multilinearly to a product $(K^n)^k \to \Lambda^k K^n$: for $v_1,\dots,v_k \in K^n$ with basis expansions $v_j = \sum_{i=1}^n a_{j,i}\, e_i$ for some $a_{j,i} \in K$ we define

$$v_1 \wedge \cdots \wedge v_k := \sum_{i_1,\dots,i_k} a_{1,i_1} \cdots a_{k,i_k} \cdot e_{i_1} \wedge \cdots \wedge e_{i_k} \quad \in \Lambda^k K^n.$$

More generally, we obtain bilinear and associative products $\Lambda^k K^n \times \Lambda^l K^n \to \Lambda^{k+l} K^n$ by a bilinear extension of

$$(e_{i_1} \wedge \cdots \wedge e_{i_k}) \wedge (e_{j_1} \wedge \cdots \wedge e_{j_l}) := e_{i_1} \wedge \cdots \wedge e_{i_k} \wedge e_{j_1} \wedge \cdots \wedge e_{j_l}.$$

The vector space $\Lambda^k K^n$ is usually called the k-fold **alternating** or **antisymmetric tensor product** of K^n, the elements of $\Lambda^k K^n$ are referred to as **alternating** or **antisymmetric tensors**.

Example 8.4.

(a) By definition we have $\Lambda^0 = K$ and $\Lambda^1 K^n = K^n$; a basis of $\Lambda^1 K^n$ is again $(e_1,\dots,e_n)$. We also have $\Lambda^n K^n \cong K$, with single basis vector $e_1 \wedge \cdots \wedge e_n$.

(b) As in (a), $\Lambda^2 K^2$ is isomorphic to K with basis vector $e_1 \wedge e_2$. For two arbitrary vectors $v = a_1 e_1 + a_2 e_2$ and $w = b_1 e_1 + b_2 e_2$ of K^2 their alternating tensor product is

$$\begin{aligned} v \wedge w &= a_1 b_1\, e_1 \wedge e_1 + a_1 b_2\, e_1 \wedge e_2 + a_2 b_1\, e_2 \wedge e_1 + a_2 b_2\, e_2 \wedge e_2 \\ &= (a_1 b_2 - a_2 b_1)\, e_1 \wedge e_2, \end{aligned}$$

so under the isomorphism $\Lambda^2 K^2 \cong K$ it is just the determinant of the coefficient matrix of v and w.

(c) Similarly, for $v = a_1 e_1 + a_2 e_2 + a_3 e_3$ and $w = b_1 e_1 + b_2 e_2 + b_3 e_3$ in K^3 we have

$$v \wedge w = (a_1 b_2 - b_2 a_1)\, e_1 \wedge e_2 + (a_1 b_3 - b_3 a_1)\, e_1 \wedge e_3 + (a_2 b_3 - a_3 b_2)\, e_2 \wedge e_3 \quad \in \Lambda^2 K^3 \cong K^3,$$

so (up to a simple change of basis) $v \wedge w$ is just the cross product $v \times w$ considered in the introduction to this chapter.

As we will see now, it is in fact a general phenomenon that the coordinates of alternating tensor products can be interpreted as determinants.

Remark 8.5 (Alternating tensor products and determinants)**.** Let $0 \le k \le n$, and let $v_1,\dots,v_k \in K^n$ with basis expansions $v_j = \sum_i a_{j,i} e_i$ for $j = 1,\dots,k$. For a strictly increasing multi-index $(j_1,\dots,j_k)$ let us determine the coefficient of the basis vector $e_{j_1} \wedge \cdots \wedge e_{j_k}$ in the tensor product $v_1 \wedge \cdots \wedge v_k$. As in Construction 8.3 we have

$$v_1 \wedge \cdots \wedge v_k = \sum_{i_1,\dots,i_k} a_{1,i_1} \cdots a_{k,i_k} \cdot e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

Note that the indices $i_1,\dots,i_k$ in the products $e_{i_1} \wedge \cdots \wedge e_{i_k}$ in the terms of this sum are not necessarily in strictly ascending order. So to figure out the coefficient of $e_{j_1} \wedge \cdots \wedge e_{j_k}$ in $v_1 \wedge \cdots \wedge v_k$ we have to sort the indices in each sum first; the resulting coefficient is then

$$\sum \operatorname{sign} \sigma \cdot a_{1,j_{\sigma(1)}} \cdots a_{k,j_{\sigma(k)}},$$

where the sum is taken over all permutations σ. By definition this is exactly the determinant of the maximal quadratic submatrix of the coefficient matrix $(a_{i,j})_{i,j}$ obtained by taking only the columns $j_1,\dots,j_k$. In other words, the coordinates of $v_1 \wedge \cdots \wedge v_k$ are just all the maximal minors of the matrix whose rows are $v_1,\dots,v_k$. So the alternating tensor product can be viewed as a convenient way to encode all these minors in a single object.

As a consequence, alternating tensor products can be used to encode the linear dependence and linear spans of vectors in a very elegant way.

Lemma 8.6. *Let $v_1,\dots,v_k \in K^n$ for some $k \le n$. Then $v_1 \wedge \cdots \wedge v_k = 0$ if and only if $v_1,\dots,v_k$ are linearly dependent.*

Proof. By Remark 8.5, we have $v_1 \wedge \cdots \wedge v_k = 0$ if and only if all maximal minors of the matrix with rows $v_1, \dots, v_k$ are zero. But this is the case if and only if this matrix does not have full rank [G2, Exercise 18.25], i. e. if and only if $v_1, \dots, v_k$ are linearly dependent. □

Remark 8.7.

(a) By construction, the alternating tensor product is antisymmetric in the sense that for all $v_1, \dots, v_k \in K^n$ and all permutations σ we have

$$v_1 \wedge \cdots \wedge v_k = \operatorname{sign} \sigma \cdot v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)}.$$

Moreover, Lemma 8.6 tells us that $v_1 \wedge \cdots \wedge v_k = 0$ if two of the vectors $v_1, \dots, v_k$ coincide.

(b) We have constructed the alternating tensor product using a fixed basis $e_1, \dots, e_n$ of K^n. However, if $v_1, \dots, v_n$ is an arbitrary basis of K^n it is easy to see that the alternating tensors $v_{i_1} \wedge \cdots \wedge v_{i_k}$ for strictly increasing multi-indices $(i_1, \dots, i_k)$ form a basis of $\Lambda^k K^n$ as well: there are $\binom{n}{k}$ of these vectors, and they generate $\Lambda^k K^n$ since every standard unit vector e_i is a linear combination of $v_1, \dots, v_n$, and hence every k-fold alternating product $e_{i_1} \wedge \cdots \wedge e_{i_k}$ is a linear combination of k-fold alternating products of $v_1, \dots, v_n$ — which can be expressed by (a) in terms of such products with strictly increasing indices.

Lemma 8.8. *Let $v_1, \dots, v_k \in K^n$ and $w_1, \dots, w_k \in K^n$ both be linearly independent. Then $v_1 \wedge \cdots \wedge v_k$ and $w_1 \wedge \cdots \wedge w_k$ are linearly dependent in $\Lambda^k K^n$ if and only if $\operatorname{Lin}(v_1, \dots, v_k) = \operatorname{Lin}(w_1, \dots, w_k)$.*

Proof. As we have assumed both $v_1, \dots, v_k$ and $w_1, \dots, w_k$ to be linearly independent, we know by Lemma 8.6 that $v_1 \wedge \cdots \wedge v_k$ and $w_1 \wedge \cdots \wedge w_k$ are both non-zero.

"⇒" Assume that $\operatorname{Lin}(v_1, \dots, v_k) \neq \operatorname{Lin}(w_1, \dots, w_k)$, so without loss of generality that $w_1 \notin \operatorname{Lin}(v_1, \dots, v_k)$. Then $w_1, v_1, \dots, v_k$ are linearly independent, and thus $w_1 \wedge v_1 \wedge \cdots \wedge v_k \neq 0$ by Lemma 8.6. But by assumption we know that $v_1 \wedge \cdots \wedge v_k = \lambda\, w_1 \wedge \cdots \wedge w_k$ for some $\lambda \in K$, and hence

$$0 \neq w_1 \wedge v_1 \wedge \cdots \wedge v_k = \lambda\, w_1 \wedge w_1 \wedge \cdots \wedge w_k$$

in contradiction to Remark 8.7 (a).

"⇐" If $v_1, \dots, v_k$ and $w_1, \dots, w_k$ span the same subspace of K^n then the basis $w_1, \dots, w_k$ of this subspace can be obtained from $v_1, \dots, v_k$ by a finite sequence of basis exchange operations $v_i \to v_i + \lambda\, v_j$ and $v_i \to \lambda\, v_i$ for $\lambda \in K$ and $i \neq j$. But both these operations change the alternating product of the vectors at most by a multiplicative scalar, since

$$v_1 \wedge \cdots \wedge v_{i-1} \wedge (v_i + \lambda\, v_j) \wedge v_{i+1} \wedge \cdots \wedge v_n = v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_n$$

and

$$v_1 \wedge \cdots \wedge (\lambda\, v_i) \wedge \cdots \wedge v_n = \lambda\, v_1 \wedge \cdots \wedge v_n$$

by multilinearity and Remark 8.7 (a). □

We can now use our results to realize the Grassmannian $G(k,n)$ as a subset of a projective space.

Construction 8.9 (Plücker embedding)**.** Let $0 \leq k \leq n$, and consider the map $f : G(k,n) \to \mathbb{P}^{\binom{n}{k}-1}$ given by sending a linear subspace $\operatorname{Lin}(v_1, \dots, v_k) \in G(k,n)$ to the class of $v_1 \wedge \cdots \wedge v_k \in \Lambda^k K^n \cong K^{\binom{n}{k}}$ in $\mathbb{P}^{\binom{n}{k}-1}$. Note that this is well-defined: $v_1 \wedge \cdots \wedge v_k$ is non-zero by Lemma 8.6, and representing the same subspace by a different basis does not change the resulting point in $\mathbb{P}^{\binom{n}{k}-1}$ by the part "⇐" of Lemma 8.8. Moreover, the map f is injective by the part "⇒" of Lemma 8.8. We call it the **Plücker embedding** of $G(k,n)$; for a k-dimensional linear subspace $L \in G(k,n)$ the (homogeneous) coordinates of $f(L)$ in $\mathbb{P}^{\binom{n}{k}-1}$ are the **Plücker coordinates** of L. By Remark 8.5, they are just all the maximal minors of the matrix whose rows are $v_1, \dots, v_k$.

In the following, we will always consider $G(k,n)$ as a subset of $\mathbb{P}^{\binom{n}{k}-1}$ using this Plücker embedding.

Example 8.10.

(a) The Plücker embedding of $G(1,n)$ simply maps a linear subspace $\operatorname{Lin}(a_1 e_1 + \cdots + a_n e_n)$ to the point $(a_1 : \cdots : a_n) \in \mathbb{P}^{\binom{n}{1}-1} = \mathbb{P}^{n-1}$. Hence $G(1,n) = \mathbb{P}^{n-1}$ as expected.

(b) Consider the 2-dimensional subspace $L = \operatorname{Lin}(e_1+e_2, e_1+e_3) \in G(2,3)$ of K^3. As

$$(e_1+e_2)\wedge(e_1+e_3) = -e_1\wedge e_2 + e_1\wedge e_3 + e_2\wedge e_3,$$

the coefficients $(-1:1:1)$ of this vector are the Plücker coordinates of L in $\mathbb{P}^{\binom{3}{2}-1} = \mathbb{P}^2$. Alternatively, these are the three maximal minors of the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

whose rows are the given spanning vectors e_1+e_2 and e_1+e_3 of L. Note that a change of these spanning vectors will just perform row operations on this matrix, which changes the maximal minors at most by a common constant factor. This shows again in this example that the homogeneous Plücker coordinates of L are well-defined. 13

So far we have embedded the Grassmannian $G(k,n)$ into a projective space, but we still have to see that it is a closed subset, i. e. a projective variety. So by Construction 8.9 we have to find suitable equations describing the alternating tensors in $\Lambda^k K^n$ that can be written as a so-called *pure tensor*, i. e. as $v_1\wedge\cdots\wedge v_k$ for some $v_1,\dots,v_k \in K^n$ — and not just as a linear combination of such expressions. The key lemma to achieve this is the following.

Lemma 8.11. *For a fixed non-zero $\omega \in \Lambda^k K^n$ with $k<n$ consider the K-linear map*

$$f : K^n \to \Lambda^{k+1}K^n, \quad v \mapsto v\wedge\omega.$$

Then $\operatorname{rk} f \geq n-k$, with equality holding if and only if $\omega = v_1\wedge\cdots\wedge v_k$ for some $v_1,\dots,v_k\in K^n$.

Example 8.12. Let $k=2$ and $n=4$.

(a) For $\omega = e_1\wedge e_2$ the map f of Lemma 8.11 is given by

$$\begin{aligned} f(a_1e_1+a_2e_2+a_3e_3+a_4e_4) &= (a_1e_1+a_2e_2+a_3e_3+a_4e_4)\wedge e_1\wedge e_2 \\ &= a_3 e_1\wedge e_2\wedge e_3 + a_4 e_1\wedge e_2\wedge e_4, \end{aligned}$$

for $a_1,a_2,a_3,a_4\in K$, and thus has rank $\operatorname{rk} f = 2 = n-k$ in accordance with the statement of the lemma.

(b) For $\omega = e_1\wedge e_2 + e_3\wedge e_4$ we get

$$\begin{aligned} &f(a_1e_1+a_2e_2+a_3e_3+a_4e_4) \\ &\quad = (a_1e_1+a_2e_2+a_3e_3+a_4e_4)\wedge(e_1\wedge e_2+e_3\wedge e_4) \\ &\quad = a_1e_1\wedge e_3\wedge e_4 + a_2e_2\wedge e_3\wedge e_4 + a_3e_1\wedge e_2\wedge e_3 + a_4e_1\wedge e_2\wedge e_4 \end{aligned}$$

instead, so that $\operatorname{rk} f = 4$. Hence Lemma 8.11 tells us that there is no way to write ω as a pure tensor $v_1\wedge v_2$ for some vectors $v_1,v_2\in K^4$.

Proof of Lemma 8.11. Let $v_1,\dots,v_r$ be a basis of $\ker f$ (with $r = n - \operatorname{rk} f$), and extend it to a basis $v_1,\dots,v_n$ of K^n. By Remark 8.7 (b) the alternating tensors $v_{i_1}\wedge\cdots\wedge v_{i_k}$ with $1\leq i_1<\cdots<i_k\leq n$ then form a basis of $\Lambda^k K^n$, and so we can write

$$\omega = \sum_{i_1<\cdots<i_k} a_{i_1,\dots,i_k}\, v_{i_1}\wedge\cdots\wedge v_{i_k}$$

for suitable coefficients $a_{i_1,\dots,i_k}\in K$. Now for $i=1,\dots,r$ we know that $v_i\in\ker f$, and thus

$$0 = v_i\wedge\omega = \sum_{i_1<\cdots<i_k} a_{i_1,\dots,i_k}\, v_i\wedge v_{i_1}\wedge\cdots\wedge v_{i_k}. \qquad (*)$$

Note that $v_i\wedge v_{i_1}\wedge\cdots\wedge v_{i_k} = 0$ if $i\in\{i_1,\dots,i_k\}$, and in the other cases these products are (up to sign) different basis vectors of $\Lambda^{k+1}K^n$. So the equation $(*)$ tells us that we must have $a_{i_1,\dots,i_k}=0$ whenever $i\notin\{i_1,\dots,i_k\}$. As this holds for all $i=1,\dots,r$ we conclude that the coefficient $a_{i_1,\dots,i_k}=0$ can only be non-zero if $\{1,\dots,r\}\subset\{i_1,\dots,i_k\}$.

But at least one of these coefficients has to be non-zero since $\omega \neq 0$ by assumption. This obviously requires that $r \leq k$, i. e. that $\operatorname{rk} f = n-r \geq n-k$. Moreover, if we have equality then only the coefficient $a_{1,\dots,k}$ can be non-zero, which means that ω is a scalar multiple of $v_1 \wedge \cdots \wedge v_k$.

Conversely, if $\omega = w_1 \wedge \cdots \wedge w_k$ for some (necessarily linearly independent) $w_1,\dots,w_k \in K^n$ then $w_1,\dots,w_k \in \ker f$. Hence in this case $\dim \ker f \geq k$, i. e. $\operatorname{rk} f \leq n-k$, and together with the above result $\operatorname{rk} f \geq n-k$ we have equality. □

Corollary 8.13 ($G(k,n)$ as a projective variety)**.** *With the Plücker embedding of Construction 8.9, the Grassmannian $G(k,n)$ is a closed subset of $\mathbb{P}^{\binom{n}{k}-1}$. In particular, it is a projective variety.*

Proof. As $G(n,n)$ is just a single point (and hence clearly a variety) we may assume that $k < n$. Then by construction a point $\omega \in \mathbb{P}^{\binom{n}{k}-1}$ lies in $G(k,n)$ if and only if it is the class of a pure tensor $v_1 \wedge \cdots \wedge v_k$. Lemma 8.11 shows that this is the case if and only if the rank of the linear map $f: K^n \to \Lambda^{k+1}K^n$, $v \mapsto v \wedge \omega$ is $n-k$. As we also know that the rank of this map is always at least $n-k$, this condition can be checked by the vanishing of all $(n-k+1)\times(n-k+1)$ minors of the matrix corresponding to f [G2, Exercise 18.25]. But these minors are polynomials in the entries of this matrix, and thus in the coordinates of ω. Hence we see that the condition for ω to be in $G(k,n)$ is closed. □

Example 8.14. By the proof of Corollary 8.13, the Grassmannian $G(2,4)$ is given by the vanishing of all sixteen 3×3 minors of a 4×4 matrix corresponding to a linear map $K^4 \to \Lambda^3 K^4$, i. e. it is a subset of $\mathbb{P}^{\binom{4}{2}-1} = \mathbb{P}^5$ given by 16 cubic equations.

As you might expect, this is by no means the simplest set of equations describing $G(2,4)$ — in fact, we will see in Exercise 8.19 (a) that a single quadratic equation suffices to cut out $G(2,4)$ from $\mathbb{P}^5$. Our proof of Corollary 8.13 is just the easiest way to show that $G(k,n)$ is a variety; it is not suitable in practice to find a nice description of $G(k,n)$ as a zero locus of simple equations.

However, there is another useful description of the Grassmannian in terms of affine patches, as we will see now. This will then also allow us to easily read off the dimension of $G(k,n)$ — which would be very hard to compute from its equations as in Corollary 8.13.

Construction 8.15 (Affine cover of the Grassmannian)**.** Let $U_0 \subset G(k,n) \subset \mathbb{P}^{\binom{n}{k}-1}$ be the affine open subset where the $e_1 \wedge \cdots \wedge e_k$-coordinate is non-zero. Then by Remark 8.5 a linear subspace $L = \operatorname{Lin}(v_1,\dots,v_k) \in G(k,n)$ is in U_0 if and only if the $k\times n$ matrix A with rows $v_1,\dots,v_k$ is of the form $A = (B|C)$ for an invertible $k\times k$ matrix B and an arbitrary $k\times(n-k)$ matrix C. This in turn is the case if and only if A is equivalent by row transformations, i. e. by a change of basis for L, to a matrix of the form $(E_k|D)$, where E_k denotes the $k\times k$ unit matrix and $D \in \operatorname{Mat}(k\times(n-k),K)$: namely by multiplying A with B^{-1} from the left to obtain $(E_k|D)$ with $D = B^{-1}C$. Note that this is in fact the only choice for D, so that we get a bijection

$$f: \mathbb{A}^{k(n-k)} = \operatorname{Mat}(k\times(n-k),K) \to U_0,$$
$$D \mapsto \text{the linear subspace spanned by the rows of } (E_k|D).$$

As the Plücker coordinates of this subspace, i. e. the maximal minors of $(E_k|D)$, are clearly polynomial functions in the entries of D, we see that f is a morphism. Conversely, the (i,j)-entry of D can be reconstructed (up to sign) from $f(D)$ as the maximal minor of $(E_k|D)$ where we take all columns of E_k except the i-th, together with the j-th column of D. Hence f^{-1} is a morphism as well, showing that f is an isomorphism and thus $U_0 \cong \mathbb{A}^{k(n-k)}$ is an affine *space* (and not just an affine *variety*, which is already clear from Proposition 7.2).

Of course, this argument holds in the same way for all other affine patches where one of the Plücker coordinates is non-zero. Hence we conclude:

Corollary 8.16. *$G(k,n)$ is an irreducible variety of dimension $k(n-k)$.*

Proof. We have just seen in Construction 8.15 that $G(k,n)$ has an open cover by affine spaces $\mathbb{A}^{k(n-k)}$. As any two of these patches have a non-empty intersection (it is in fact easy to write down a $k \times n$ matrix such that any two given maximal minors are non-zero), the result follows from Exercises 2.20 (b) and 2.33 (a). □

Remark 8.17. The argument of Construction 8.15 also shows an alternative description of the Grassmannian: it is the space of all full-rank $k \times n$ matrices modulo row transformations. As we know that every such matrix is equivalent modulo row transformations to a unique matrix in reduced row echelon form, we can also think of $G(k,n)$ as the set of full-rank $k \times n$ matrices in such a form. For example, in the case $k = 1$ and $n = 2$ (when $G(1,2) = \mathbb{P}^1$ by Example 8.10 (a)) the full-rank 1×2 matrices in reduced row echelon form are

$$\begin{aligned} &\begin{pmatrix} 1 & * \end{pmatrix} \quad \text{corresponding to } \mathbb{A}^1 \subset \mathbb{P}^1 \\ \text{and} \quad &\begin{pmatrix} 0 & 1 \end{pmatrix} \quad \text{corresponding to } \infty \in \mathbb{P}^1 \end{aligned}$$

as in the homogeneous coordinates of $\mathbb{P}^1$.

The affine cover of Construction 8.15 can also be used to show the following symmetry property of the Grassmannians.

Proposition 8.18. *For all* $0 \le k \le n$ *we have* $G(k,n) \cong G(n-k,n)$.

Proof. There is an obvious well-defined set-theoretic bijection $f : G(k,n) \to G(n-k,n)$ that sends a k-dimensional linear subspace L of K^n to its "orthogonal" complement

$$L^\perp = \{x \in K^n : \langle x,y \rangle = 0 \text{ for all } y \in L\},$$

where $\langle x,y \rangle = \sum_{i=1}^n x_i y_i$ denotes the standard bilinear form. It remains to be shown that f (and analogously f^{-1}) is a morphism. By Lemma 4.6, we can do this on the affine coordinates of Construction 8.15. So let $L \in G(k,n)$ be described as the subspace spanned by the rows of a matrix $(E_k \,|\, D)$, where the entries of $D \in \operatorname{Mat}(k \times (n-k), K)$ are the affine coordinates of L. As

$$(E_k \,|\, D) \cdot \begin{pmatrix} -D \\ E_{n-k} \end{pmatrix} = 0,$$

we see that $L^\perp$ is the subspace spanned by the rows of $(-D^T \,|\, E_{n-k})$. But the maximal minors of this matrix, i. e. the Plücker coordinates of $L^\perp$, are clearly polynomials in the entries of D, and thus we conclude that f is a morphism. □

Exercise 8.19. Let $G(2,4) \subset \mathbb{P}^5$ be the Grassmannian of lines in $\mathbb{P}^3$ (or of 2-dimensional linear subspaces of K^4). We denote the homogeneous Plücker coordinates of $G(2,4)$ in $\mathbb{P}^5$ by $x_{i,j}$ for $1 \le i < j \le 4$. Show:

(a) $G(2,4) = V(x_{1,2}x_{3,4} - x_{1,3}x_{2,4} + x_{1,4}x_{2,3})$.

(b) Let $L \subset \mathbb{P}^3$ be an arbitrary line. Show that the set of lines in $\mathbb{P}^3$ that intersect L, considered as a subset of $G(2,4) \subset \mathbb{P}^5$, is the zero locus of a homogeneous linear polynomial.

How many lines in $\mathbb{P}^3$ would you expect to intersect four general given lines?

Exercise 8.20. Show that the following sets are projective varieties:

(a) the *incidence correspondence*

$$\{(L,a) \in G(k,n) \times \mathbb{P}^{n-1} : L \subset \mathbb{P}^{n-1} \text{ a } (k-1)\text{-dimensional linear subspace and } a \in L\};$$

(b) the *join* of two disjoint varieties $X, Y \subset \mathbb{P}^n$, i. e. the union in $\mathbb{P}^n$ of all lines intersecting both X and Y.