11. The 27 Lines on a Smooth Cubic Surface

As an application of the theory that we have developed so far, we now want to study lines on cubic surfaces in $\mathbb{P}^3$. In Example 0.1, we have already mentioned that every smooth cubic surface has exactly 27 lines on it. Our goal is now to show this, to study the configuration of these lines, and to prove that every smooth cubic surface is birational (but not isomorphic) to $\mathbb{P}^2$. All these results are classical, dating back to the 19th century. They can be regarded historically as being among the first non-trivial statements in projective algebraic geometry.

The results of this chapter will not be needed later on. Most proofs will therefore not be given in every detail here, in particular since some of them also use methods from topology and analysis which we have not discussed in this class. The aim of this chapter is rather to give an idea of what can be done with the methods that we have developed so far.

For simplicity, we will restrict ourselves to the case of the ground field $K = \mathbb{C}$. Let us start with the discussion of a special case of a cubic surface: the Fermat cubic $V(x_0^3+x_1^3+x_2^3+x_3^3) \subset \mathbb{P}^3$ as in Example 10.21.

Lemma 11.1. *The Fermat cubic $X = V(x_0^3+x_1^3+x_2^3+x_3^3) \subset \mathbb{P}^3$ contains exactly 27 lines.*

Proof. To find the lines in X, we may assume after a permutation of the coordinates that their first Plücker coordinate in $G(2,4)$ is non-zero, i. e. as in Construction 8.18 that we have lines $L \in G(2,4)$ given as the row span of the matrix

$$\begin{pmatrix} 1 & 0 & a_2 & a_3 \\ 0 & 1 & b_2 & b_3 \end{pmatrix} \quad \text{for some } a_2,a_3,b_2,b_3 \in \mathbb{C}.$$

Then

$$L = \{(s:t:a_2s+b_2t:a_3s+b_3t) : (s:t) \in \mathbb{P}^1\},$$

and hence $L \subset X$ if and only if the polynomial $s^3+t^3+(a_2s+b_2t)^3+(a_3s+b_3t)^3$ is identically zero in s and t. This means that its coefficients of s^3, s^2t, st^2, and t^3 must be 0, i. e. that

$$a_2^3+a_3^3 = -1, \tag{1}$$

$$a_2^2b_2 = -a_3^2b_3, \tag{2}$$

$$a_2b_2^2 = -a_3b_3^2, \tag{3}$$

$$b_2^3+b_3^3 = -1. \tag{4}$$

If a_2,a_3,b_2,b_3 are all non-zero, then $(2)^2/(3)$ gives $a_2^3 = -a_3^3$, in contradiction to (1). Hence for a line in the cubic at least one of these four numbers must be zero. Again after possibly renumbering the coordinates we may assume that $a_2 = 0$. Then $a_3^3 = -1$ by (1), $b_3 = 0$ by (2), and $b_2^3 = -1$ by (4). Conversely, for such values of a_2,a_3,b_2,b_3 the above equations all hold, so that we really obtain a line in the cubic.

We thus obtain 9 lines in X by setting $a_3 = -\omega^j$ and $b_2 = -\omega^k$ for $0 \le j,k \le 2$ and $\omega = \exp(\frac{2\pi i}{3})$ a primitive third root of unity. So by finally allowing permutations of the coordinates we find that there are exactly 27 lines on X, given by the row spans of the matrices

$$\begin{pmatrix} 1 & 0 & 0 & -\omega^j \\ 0 & 1 & -\omega^k & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -\omega^j & 0 \\ 0 & 1 & 0 & -\omega^k \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & -\omega^j & 0 & 0 \\ 0 & 0 & 1 & -\omega^k \end{pmatrix}$$

for $0 \le j,k \le 2$. □

Corollary 11.2. *Let $X \subset \mathbb{P}^3$ again be the Fermat cubic as in Lemma 11.1.*

(a) *Given any line L in X, there are exactly* 10 *other lines in X that intersect L.*

(b) *Given any two disjoint lines L_1, L_2 in X, there are exactly* 5 *other lines in X meeting both L_1 and L_2.*

Proof. As we know all the lines in X by the proof of Lemma 11.1, this is just simple checking. For example, to prove (a) we may assume by permuting coordinates and multiplying them with suitable third roots of unity that L is the row span of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix},$$

i. e. that $L = V(x_0 + x_3, x_1 + x_2)$. The other lines meeting L are then exactly the following:

$$\text{4 lines as the row span of } \begin{pmatrix} 1 & 0 & 0 & -\omega^j \\ 0 & 1 & -\omega^k & 0 \end{pmatrix} \quad \text{for } (j,k) = (1,0),(2,0),(0,1),(0,2),$$

$$\text{3 lines as the row span of } \begin{pmatrix} 1 & 0 & -\omega^j & 0 \\ 0 & 1 & 0 & -\omega^k \end{pmatrix} \quad \text{for } (j,k) = (0,0),(1,2),(2,1),$$

$$\text{3 lines as the row span of } \begin{pmatrix} 1 & -\omega^j & 0 & 0 \\ 0 & 0 & 1 & -\omega^k \end{pmatrix} \quad \text{for } (j,k) = (0,0),(1,2),(2,1).$$

The proof of part (b) is analogous. □

Let us now transfer these results to an arbitrary smooth cubic surface. This is where it gets interesting, since the equations determining the lines lying in the cubic as in the proof of Lemma 11.1 will in general be too complicated to solve them directly. Instead, we will only show that the number of lines in a smooth cubic must be the same for all cubics, so that we can then conclude by Lemma 11.1 that this number must be 27. In other words, we have to consider all smooth cubic surfaces at once.

Construction 11.3 (The incidence correspondence of lines in smooth cubic surfaces)**.** As in Exercise 10.23 (b), let $\mathbb{P}^{19} = \mathbb{P}^{\binom{3+3}{3}-1}$ be the projective space of all homogeneous degree-3 polynomials in x_0, x_1, x_2, x_3 modulo scalars, so that the space of smooth cubic surfaces is a dense open subset U of $\mathbb{P}^{19}$. More precisely, a smooth cubic surface can be given as the zero locus of an irreducible polynomial $f_c := \sum_\alpha c_\alpha x^\alpha = 0$ in multi-index notation, i. e. α runs over all quadruples of non-negative indices $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ with $\sum_i \alpha_i = 3$. The corresponding point in $U \subset \mathbb{P}^{19}$ is then the one with homogeneous coordinates $c = (c_\alpha)_\alpha$.

We can now consider the *incidence correspondence*

$$M := \{(X,L) : L \text{ is a line contained in the smooth cubic } X\} \quad \subset U \times G(2,4).$$

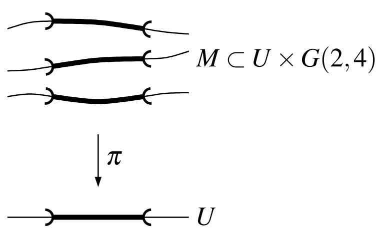
Note that it comes with a natural projection map $\pi\colon M \to U$ sending a pair (X,L) to X, and that the number of lines in a cubic surface is just its number of inverse images under π.

To show that this number of inverse images is constant on U, we will pass from the algebraic to the analytic category and prove the following statement.

Lemma 11.4. *With notations as in Construction 11.3, the incidence correspondence M is...*

(a) *closed in the Zariski topology of $U \times G(2,4)$;*

(b) *locally in the classical topology the graph of a continuously differentiable function $U \to G(2,4)$, as shown in the picture on the right.*



Proof.

(a) The proof goes along the same lines as that of Lemma 11.1, just with a varying polynomial defining the cubic. More precisely, let $(X,L) \in M$. By a linear change of coordinates we may assume that $L = \operatorname{Lin}(e_1, e_2)$. Locally around this point $L \in G(2,4)$ in the Zariski topology we can use the affine coordinates on the Grassmannian as in Construction 8.18, namely $a_2, a_3, b_2, b_3 \in \mathbb{C}$ corresponding to the line in $\mathbb{P}^3$ given as the row span of the matrix

$$\begin{pmatrix} 1 & 0 & a_2 & a_3 \\ 0 & 1 & b_2 & b_3 \end{pmatrix},$$

with the point $(a_2, a_3, b_2, b_3) = (0,0,0,0)$ corresponding to L. On the space U of smooth cubic surfaces we use the coordinates $(c_\alpha)_\alpha$ as in Construction 11.3. Taking these together, in the coordinates $(c,a,b) = ((c_\alpha), a_2, a_3, b_2, b_3)$ on $U \times G(2,4)$ the incidence correspondence M is then given by

$$\begin{aligned}
(c,a,b) \in M \;&\Leftrightarrow\; f_c(s\,(1,0,a_2,a_3) + t\,(0,1,b_2,b_3)) = 0 \text{ for all } s,t \\
&\Leftrightarrow\; \sum_\alpha c_\alpha s^{\alpha_0} t^{\alpha_1} (s\,a_2 + t\,b_2)^{\alpha_2} (s\,a_3 + t\,b_3)^{\alpha_3} = 0 \text{ for all } s,t \\
&\Leftrightarrow:\; \sum_i s^i t^{3-i} F_i(c,a,b) = 0 \text{ for all } s,t \\
&\Leftrightarrow\; F_i(c,a,b) = 0 \text{ for } 0 \le i \le 3.
\end{aligned}$$

As these are polynomial equations in a,b,c, it follows that M is closed.

(b) We will show by (the complex version of) the Implicit Function Theorem [G1, Proposition 27.9] that the four equations $F_i(c,a,b) = 0$ for $i = 0,\dots,3$ determine (a_2,a_3,b_2,b_3) locally around the origin in the classical topology in terms of c. To prove this we just have to check that the Jacobian matrix $J := \frac{\partial(F_0,F_1,F_2,F_3)}{\partial(a_2,a_3,b_2,b_3)}$ is invertible if $a = b = 0$.

To compute J, note that

$$\begin{aligned}
\frac{\partial}{\partial a_2}\Big(\sum_i s^i t^{3-i} F_i\Big)\Big|_{a=b=0} &= \frac{\partial}{\partial a_2} f_c(s,t,s\,a_2 + t\,b_2, s\,a_3 + t\,b_3)\Big|_{a=b=0} \\
&= s\,\frac{\partial f_c}{\partial x_2}(s,t,0,0).
\end{aligned}$$

The (s,t)-coefficients of this polynomial are the first column in the matrix J. Similarly, the other columns are obviously $s\,\frac{\partial f_c}{\partial x_3}(s,t,0,0)$, $t\,\frac{\partial f_c}{\partial x_2}(s,t,0,0)$, and $t\,\frac{\partial f_c}{\partial x_3}(s,t,0,0)$. Hence, if the matrix J was not invertible, there would be a relation

$$(\lambda_2 s + \mu_2 t)\frac{\partial f_c}{\partial x_2}(s,t,0,0) + (\lambda_3 s + \mu_3 t)\frac{\partial f_c}{\partial x_3}(s,t,0,0) = 0$$

identically in s,t, with $(\lambda_2,\mu_2,\lambda_3,\mu_3) \in \mathbb{C}^4\backslash\{0\}$. As homogeneous polynomials in two variables always decompose into linear factors, this means that $\frac{\partial f_c}{\partial x_2}(s,t,0,0)$ and $\frac{\partial f_c}{\partial x_3}(s,t,0,0)$ must have a common linear factor, i. e. that there is a point $p = (p_0,p_1,0,0) \in L$ with $\frac{\partial f_c}{\partial x_2}(p) = \frac{\partial f_c}{\partial x_3}(p) = 0$.

But as the line L lies in the cubic $V(f_c)$, we also have $f_c(s,t,0,0) = 0$ for all s,t. Differentiating this with respect to s and t gives $\frac{\partial f_c}{\partial x_0}(p) = 0$ and $\frac{\partial f_c}{\partial x_1}(p) = 0$, respectively. Hence all partial derivatives of f_c vanish at $p \in L \subset X$. By the Jacobi criterion of Exercise 10.13 (b) this means that p would be a singular point of X, in contradiction to our assumption. Hence
18 J must be invertible, which proves (b). □

Corollary 11.5. *Every smooth cubic surface contains exactly* 27 *lines.*

Proof. In this proof we will work with the classical topology throughout. Let $X \in U$ be a fixed smooth cubic, and let $L \subset \mathbb{P}^3$ be an arbitrary line. We distinguish two cases:

Case 1: If L lies in X, Lemma 11.4 (b) shows that there is an open neighborhood $V_L \times W_L$ of (X,L) in $U \times G(2,4)$ in which the incidence correspondence M is the graph of a continuously differentiable function. In particular, every cubic in V_L contains exactly one line in W_L.

Case 2: If L does not lie in X there is an open neighborhood $V_L \times W_L$ of (X,L) such that no cubic in V_L contains any line (since the incidence correspondence is closed by Lemma 11.4 (a)).

Now let L vary. As the Grassmannian $G(2,4)$ is projective, and hence compact, there are finitely many W_L that cover $G(2,4)$. Let V be the intersection of the corresponding V_L, which is then again an open neighborhood of X. By construction, in this neighborhood V all cubic surfaces have the same number of lines (namely the number of W_L coming from case 1). As this argument holds for any cubic, we conclude that the number of lines contained in a cubic surface is a locally constant function on U.

To see that this number is also globally constant, it therefore suffices to show that U is connected. But this follows from Exercise 10.23 (b): We know that U is the complement of a proper Zariski-closed subset in $\mathbb{P}^{19}$. But as such a closed subset has complex codimension at least 1 and hence real codimension at least 2, taking this subset away from the smooth and connected space $\mathbb{P}^{19}$ leaves us again with a connected space. □

Remark 11.6.

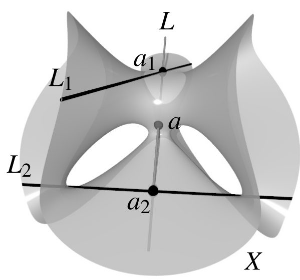
(a) In topological terms, the argument of the proof of Corollary 11.5 says that the projection map $\pi\colon M \to U$ of Construction 11.3 is a 27-sheeted covering map.

(b) Applying the methods of Lemma 11.4 and Corollary 11.5 to suitable incidence correspondences involving two resp. three lines in cubic surfaces, one can show similarly that the statements of Corollary 11.2 hold for an arbitrary smooth cubic surface X as well: There are exactly 10 lines in X meeting any given one, and exactly 5 lines in X meeting any two disjoint given ones.

Note that a cubic surface X is clearly not isomorphic to $\mathbb{P}^2$: By Remark 11.6 (b) there are two disjoint lines on X, whereas in $\mathbb{P}^2$ any two curves intersect by Exercise 6.30 (b). However, we will now see that X is birational to $\mathbb{P}^2$, and that it is in fact isomorphic to a blow-up of $\mathbb{P}^2$ at six points.

Proposition 11.7. *Any smooth cubic surface is birational to $\mathbb{P}^2$.*

Proof. By Remark 11.6 (b) there are two disjoint lines $L_1, L_2 \subset X$. The following mutually inverse rational maps $X \dashrightarrow L_1 \times L_2$ and $L_1 \times L_2 \dashrightarrow X$ show that X is birational to $L_1 \times L_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$, and hence to $\mathbb{P}^2$:



"$X \dashrightarrow L_1 \times L_2$": By Exercise 6.28, for every point a not on L_1 or L_2 there is a unique line L in $\mathbb{P}^3$ through L_1, L_2, and a. Take the rational map from X to $L_1 \times L_2$ sending a to $(a_1, a_2) := (L_1 \cap L, L_2 \cap L)$, which is obviously well-defined away from $L_1 \cup L_2$.

"$L_1 \times L_2 \dashrightarrow X$": Map any pair of points $(a_1, a_2) \in L_1 \times L_2$ to the third intersection point of X with the line L through a_1 and a_2. This is well-defined whenever L is not contained in X. □

Proposition 11.8. *Any smooth cubic surface is isomorphic to $\mathbb{P}^2$ blown up in 6 (suitably chosen) points.*

Proof. We will only sketch the proof. Let X be a smooth cubic surface, and consider the rational map $f\colon X \dashrightarrow L_1 \times L_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ as in the proof of Proposition 11.7.

First of all we claim that f is actually a morphism. To see this, note that there is a different description for f: If $a \in X \backslash L_1$, let H be the unique plane in $\mathbb{P}^3$ that contains L_1 and a, and set $f_2(a) = H \cap L_2$. If one defines $f_1(a)$ analogously, then $f(a) = (f_1(a), f_2(a))$. Now if the point a lies on L_1, let H be the tangent plane to X at a, and again set $f_2(a) = H \cap L_2$. Extending f_1 similarly, one can show that this extends $f = (f_1, f_2)$ to a well-defined morphism $X \to \mathbb{P}^1 \times \mathbb{P}^1$ on all of X.

Now let us investigate where the inverse map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow X$ is not well-defined. As already mentioned in the proof of Proposition 11.7, this is the case if the point $(a_1, a_2) \in L_1 \times L_2$ is chosen so that $\overline{a_1 a_2} \subset X$. In this case, the whole line $\overline{a_1 a_2}$ will be mapped to (a_1, a_2) by f, and it can be checked that f is actually locally the blow-up of this point. By Remark 11.6 (b) there are exactly 5 such lines $\overline{a_1 a_2}$ on X. Hence X is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in 5 points, i. e. by Lemma 9.26 the blow-up of $\mathbb{P}^2$ in 6 suitably chosen points. □

Remark 11.9. It is interesting to see the lines on a cubic surface X in the picture of Proposition 11.8 in which we think of X as a blow-up of $\mathbb{P}^2$ in 6 points. It turns out that the 27 lines correspond to the following curves that we know already (and that are all isomorphic to $\mathbb{P}^1$):

- the 6 exceptional hypersurfaces,
- the strict transforms of the $\binom{6}{2} = 15$ lines through two of the blown-up points,
- the strict transforms of the $\binom{6}{5} = 6$ conics through five of the blown-up points (see Exercise 7.29 (c)).

In fact, it is easy to check by the above explicit description of the isomorphism of X with the blow-up of $\mathbb{P}^2$ that these curves on the blow-up actually correspond to lines on the cubic surface.

It is also interesting to see again in this picture that every such "line" meets 10 of the other "lines", as mentioned in Remark 11.6 (b):

- Every exceptional hypersurface intersects the 5 lines and the 5 conics that pass through this blown-up point.
- Every line through two of the blown-up points meets
  - the 2 exceptional hypersurfaces of the blown-up points,
  - the $\binom{4}{2} = 6$ lines through two of the four remaining points,
  - the 2 conics through the four remaining points and one of the blown-up points.
- Every conic through five of the blown-up points meets the 5 exceptional hypersurfaces at these points, as well as the 5 lines through one of these five points and the remaining point.

Exercise 11.10. As in Exercise 10.23 (b) let $U \subset \mathbb{P}^{\binom{4+5}{4}-1} = \mathbb{P}^{125}$ be the set of all smooth (3-dimensional) hypersurfaces of degree 5 in $\mathbb{P}^4$.

(a) Using the Jacobi criterion, show that the incidence correspondence

$$\{(X,L) \in U \times G(2,5) : L \text{ is a line contained in } X\}$$

is smooth of dimension 125, i. e. of the same dimension as U.

(b) Although (a) suggests that a smooth hypersurface of degree 5 in $\mathbb{P}^4$ contains only finitely many lines, show that the Fermat hypersurface $V(x_0^5 + \cdots + x_4^5) \subset \mathbb{P}^4$ contains infinitely many lines.

(Hint: Consider lines of the form $L = \{(a_0 s : a_1 s : a_2 t : a_3 t : a_4 t) : (s:t) \in \mathbb{P}^1\}$ for suitable $a_0, \dots, a_4 \in \mathbb{C}$.)