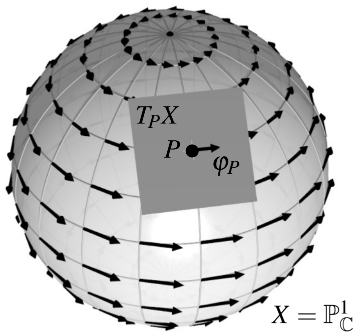
13. Sheaves of Modules

After having introduced schemes as the central objects of interest in algebraic geometry, we now have to get a more detailed understanding of sheaves to advance further. Recall from Section 3 that, intuitively speaking, we initially defined a sheaf to be a structure on a topological space X that describes function-like objects that can be glued together from local data. However, the only example of a sheaf that we have studied so far is the sheaf of regular functions on a variety or scheme.

In practice, there are many more important sheaves however that do not just describe "ordinary functions". One main feature of them occurred already for the structure sheaf on a scheme in Definition 12.16, when we saw that sections of a sheaf are not necessarily just functions to a fixed ground field with some local properties. In fact, the following example shows that even for varieties one should also be interested in "functions to a varying target":

Example 13.1 (Tangent sheaf)**.** Consider a smooth curve X, as e. g. the complex projective line $X = \mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ in the picture below on the right, which is topologically a sphere. At each point $P \in X$ we have constructed the tangent space $T_P X$ to X as a 1-dimensional vector space in Chapter 10. So we can try to construct a *tangent sheaf* T_X on X by setting

$$T_X(U) = \{\varphi = (\varphi_P)_{P \in U} : \varphi_P \in T_P X \text{ "varying nicely with } P\text{"}\}$$



for any open subset $U \subset X$ (where of course we will have to make precise what "varying nicely" means; see Chapter 15 for an exact construction of T_X). In other words, a section φ of T_X over U is just given by specifying a tangent vector to X at all points of P. This is also often called a *tangent vector field* on U. The arrows in the picture represent a global section of T_X. As the tangent spaces $T_P X$ are all 1-dimensional vector spaces, we can think of sections of T_X as "functions to a moving 1-dimensional target".

Note that there is no canonical way to multiply tangent vectors, and hence T_X will not be a sheaf of rings. There is also no reasonable notion of a "constant vector field". In fact, we will see in Example 15.12 (a) that every global section of $T_{\mathbb{P}^1}$ must have two zeros (as e. g. the north and south pole in the picture above).

But what we can do is to multiply a tangent vector field with a regular function (by multiplying the given tangent vector at every point with the value of the function), and hence T_X should be a module over the sheaf $\mathscr{O}_X$ of regular functions in a suitable sense. This leads to the central notion of a sheaf of modules that we will define now.

In the following, we will usually use the language of schemes from Chapter 12. However, in some cases (in particular for examples) we will restrict to varieties, i. e. in the sense of Convention 12.41 to separated and reduced schemes of finite type over an algebraically closed ground field, which we will then continue to denote by K.

Definition 13.2 ((Pre-)sheaves of modules)**.** Let X be a scheme.

(a) A **(pre-)sheaf of modules** on X, also called a **(pre-)sheaf of $\mathscr{O}_X$-modules**, is a (pre-)sheaf $\mathscr{F}$ on X as in Definition 3.13 such that $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module for all open subsets $U \subset X$, and such that all restriction maps are $\mathscr{O}_X$-module homomorphisms in the sense that

$$(\varphi + \psi)|_U = \varphi|_U + \psi|_U \quad \text{and} \quad (\lambda \varphi)|_U = \lambda|_U \cdot \varphi|_U$$

for all open subsets $U \subset V$ of X, $\lambda \in \mathscr{O}_X(V)$ and $\varphi, \psi \in \mathscr{F}(V)$. (Note that they are not module homomorphisms in the usual sense as $\mathscr{F}(U)$ and $\mathscr{F}(V)$ are modules over different rings $\mathscr{O}_X(U)$ resp. $\mathscr{O}_X(V)$, i. e. the scalars have to be restricted as well.)

In what follows, a (pre-)sheaf is always meant to be a (pre-)sheaf of modules unless stated otherwise. A sheaf of modules on X is also often called an $\mathscr{O}_X$**-module**.

(b) A **morphism** $f\colon \mathscr{F} \to \mathscr{G}$ of (pre-)sheaves $\mathscr{F}$ and $\mathscr{G}$ of modules on X is given by the data of $\mathscr{O}_X(U)$-module homomorphisms $f_U\colon \mathscr{F}(U) \to \mathscr{G}(U)$ for all open subsets $U \subset X$ that are compatible with restrictions, i. e. such that $f_V(\varphi)|_U = f_U(\varphi|_U)$ for all open subsets $U \subset V$ and $\varphi \in \mathscr{F}(V)$.

For simplicity of notation, we will often write the homomorphisms f_U just as f.

Example 13.3.

(a) Clearly, $\mathscr{O}_X$ is a sheaf of modules on any scheme X.

(b) If $\mathscr{F}$ and $\mathscr{G}$ are sheaves of modules on a scheme X, then so is the **direct sum** $\mathscr{F} \oplus \mathscr{G}$ defined by $(\mathscr{F} \oplus \mathscr{G})(U) := \mathscr{F}(U) \oplus \mathscr{G}(U)$.

Probably the most important example of non-trivial sheaves of modules is given by the following construction that generalizes the sheaves of regular functions on $\mathbb{P}^n$. As in Definition 7.1, we still consider quotients of homogeneous polynomials, but not necessarily of the same degree:

Construction 13.4 (Twisting sheaves on $\mathbb{P}^n$)**.** Let $n \in \mathbb{N}_{>0}$ and $d \in \mathbb{Z}$. For a non-empty open subset $U \subset \mathbb{P}^n$ we define

$$(\mathscr{O}_{\mathbb{P}^n}(d))(U) := \Big\{ \frac{g}{f} : f, g \in K[x_0, \dots, x_n] \text{ homogeneous with } \deg g - \deg f = d \\ \text{and } f(P) \neq 0 \text{ for all } P \in U \Big\}$$

as a subset of the quotient field of $K[x_0, \dots, x_n]$. Together with setting $(\mathscr{O}_{\mathbb{P}^n}(d))(\emptyset) := \{0\}$ we then obtain:

(a) $\mathscr{O}_{\mathbb{P}^n}(d)$ is a sheaf, since all $(\mathscr{O}_{\mathbb{P}^n}(d))(U)$ are by definition subsets of the quotient field of $K[x_0, \dots, x_n]$, with the restriction maps given by the identity on this quotient field.

(b) For $d = 0$ we have $\mathscr{O}_{\mathbb{P}^n}(0) \cong \mathscr{O}_{\mathbb{P}^n}$: The morphism $\mathscr{O}_{\mathbb{P}^n}(0) \to \mathscr{O}_{\mathbb{P}^n}$, $\frac{g}{f} \mapsto \frac{g}{f}$ is well-defined, and it is injective since $\frac{g}{f}$ is zero as a function on a non-empty open subset of $\mathbb{P}^n$ only if $g = 0$. It is also surjective, as $K[x_0, \dots, x_n]$ is a unique factorization domain, and thus the representation of the elements of its quotient field as $\frac{g}{f}$ with coprime f and g is unique (up to multiplying f and g with a constant in K^*).

In contrast, note that the sections of $\mathscr{O}_{\mathbb{P}^n}(d)$ for $d \neq 0$ are not well-defined functions as rescaling the homogeneous coordinates on $\mathbb{P}^n$ would change their value.

(c) For any $e \in \mathbb{Z}$ there are bilinear maps

$$(\mathscr{O}_{\mathbb{P}^n}(d))(U) \times (\mathscr{O}_{\mathbb{P}^n}(e))(U) \to (\mathscr{O}_{\mathbb{P}^n}(d+e))(U),\ (\varphi, \psi) \mapsto \varphi\psi.$$

In particular, for $e = 0$ this means by (b) that $\mathscr{O}_{\mathbb{P}^n}(d)$ is an $\mathscr{O}_{\mathbb{P}^n}$-module.

The sheaves $\mathscr{O}_{\mathbb{P}^n}(d)$ are called the **twisting sheaves** on $\mathbb{P}^n$. The name comes from the fact that their sections can in some sense be interpreted as functions to a twisting (i. e. varying) target, see e. g. Exercise 13.6.

Example 13.5. Let $n \in \mathbb{N}_{>0}$ and $d \in \mathbb{Z}$.

(a) For a global section $\frac{g}{f} \in (\mathscr{O}_{\mathbb{P}^n}(d))(\mathbb{P}^n)$ we need $V(f) = \emptyset$. Hence, by the Nullstellensatz f must be a constant, that we can then absorb into the numerator g. It follows that $(\mathscr{O}_{\mathbb{P}^n}(d))(\mathbb{P}^n) \cong K[x_0, \dots, x_n]_d$ is just the space of all homogeneous polynomials of degree d. In particular, we have $(\mathscr{O}_{\mathbb{P}^n}(d))(\mathbb{P}^n) = \{0\}$ for all $d < 0$.

(b) For the open subset $U_0 = \{(x_0 : x_1) \in \mathbb{P}^1 : x_0 \neq 0\}$ of $\mathbb{P}^1$ we have $\frac{1}{x_0} \in (\mathscr{O}_{\mathbb{P}^1}(-1))(U_0)$.

(c) For any homogeneous polynomial $f \in K[x_0, \dots, x_n]$ of degree e and we have a morphism of sheaves

$$\mathscr{O}_{\mathbb{P}^n}(d) \to \mathscr{O}_{\mathbb{P}^n}(d+e), \ \varphi \mapsto \varphi f$$

(i. e. it is given by the maps $(\mathscr{O}_{\mathbb{P}^n}(d))(U) \to (\mathscr{O}_{\mathbb{P}^n}(d+e))(U), \ \varphi \mapsto \varphi f$ for all open subsets $U \subset \mathbb{P}^n$).

(d) For $d \neq 0$, the sheaf $\mathscr{O}_{\mathbb{P}^n}(d)$ and the structure sheaf $\mathscr{O}_{\mathbb{P}^n}$ are not isomorphic on $\mathbb{P}^n$ since their spaces of global sections $(\mathscr{O}_{\mathbb{P}^n}(d))(\mathbb{P}^n) = K[x_0, \dots, x_n]_d$ and $\mathscr{O}_{\mathbb{P}^n}(\mathbb{P}^n) = K$ differ by (a). But they become isomorphic when restricted to any open subset $U_i = \{(x_0 : \cdots : x_n) : x_i \neq 0\}$ for $i = 0, \dots, n$ by

$$f\colon \mathscr{O}_{\mathbb{P}^n}|_{U_i} \to \mathscr{O}_{\mathbb{P}^n}(d)|_{U_i}, \ \varphi \mapsto \varphi x_i^d \quad \text{with inverse} \quad f^{-1}\colon \mathscr{O}_{\mathbb{P}^n}(d)|_{U_i} \to \mathscr{O}_{\mathbb{P}^n}|_{U_i}, \ \varphi \mapsto \frac{\varphi}{x_i^d}.$$

Exercise 13.6 (The tautological sheaf on $\mathbb{P}^n$)**.** Recall that $\mathbb{P}^n$ is by definition the set of 1-dimensional linear subspaces of K^{n+1}. Hence we can define the *tautological sheaf* $\mathscr{F}$ on $\mathbb{P}^n$ by

$$\mathscr{F}(U) = \{\varphi\colon U \to K^{n+1} \text{ a morphism with } \varphi(P) \in P \text{ for all (closed) points } P \in U\}$$

for all open subsets $U \subset \mathbb{P}^n$. (In other words, the value of φ at every point $P \in U \subset \mathbb{P}^n$ has to lie in the corresponding 1-dimensional linear subspace of K^{n+1}, so that in particular $\mathscr{F}$ is visibly a sheaf of functions to a 1-dimensional target that varies with the point in the source.)

Prove that $\mathscr{F}$ is isomorphic to the twisting sheaf $\mathscr{O}_{\mathbb{P}^n}(-1)$.

Remark 13.7 (Morphisms on stalks)**.** For a sheaf of modules $\mathscr{F}$ on a scheme X, the $\mathscr{O}_X$-module structure makes each stalk $\mathscr{F}_P$ for $P \in X$ into an $\mathscr{O}_{X,P}$-module. As in Definition 12.26, a morphism $f\colon \mathscr{F} \to \mathscr{G}$ of sheaves on X then determines well-defined $\mathscr{O}_{X,P}$-module homomorphisms

$$f_P\colon \mathscr{F}_P \to \mathscr{G}_P, \ \overline{(U,\varphi)} \to \overline{(U, f_U(\varphi))}$$

between the stalks.

Exercise 13.8 (Sheaf isomorphisms are local)**.** Show that a morphism $f\colon \mathscr{F} \to \mathscr{G}$ of sheaves on a scheme X is an isomorphism if and only if the induced map $f_P\colon \mathscr{F}_P \to \mathscr{G}_P$ on the stalk as in Remark 13.7 is an isomorphism for all $P \in X$.

The goal of this chapter is to introduce and study the basic constructions that one can make with sheaves of modules. The simplest one is probably the push-forward of a sheaf along a morphism of schemes.

Definition 13.9 (Push-forward of sheaves)**.** Let $f\colon X \to Y$ be a morphism of schemes, and let $\mathscr{F}$ be a sheaf on X. For all open subsets $U \subset Y$ we set

$$(f_*\mathscr{F})(U) := \mathscr{F}(f^{-1}(U)).$$

It is checked immediately that this is a sheaf on Y; in fact it is a sheaf of $\mathscr{O}_Y$-modules by setting $\lambda \cdot \varphi := f^*\lambda \cdot \varphi$ for all $\lambda \in \mathscr{O}_Y(U)$ and $\varphi \in \mathscr{F}(f^{-1}(U))$. 22

Example 13.10. Let $f\colon X \to Y$ be a morphism of schemes.

(a) The data of the pull-back maps $f_U^*\colon \mathscr{O}_Y(U) \to \mathscr{O}_X(f^{-1}(U))$ that come with f as in Definition 12.26 define exactly a morphism of sheaves

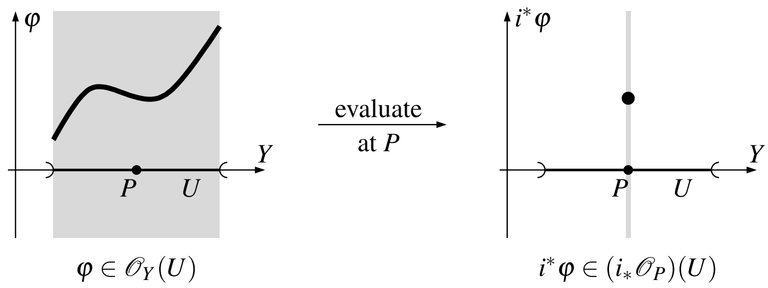
$$f^*\colon \mathscr{O}_Y \to f_*\mathscr{O}_X, \ \varphi \mapsto f^*\varphi.$$

In other words, one could simplify Definition 12.26 by saying that the data to define a morphism between locally ringed spaces X and Y is just a continuous map $f\colon X \to Y$ together with a morphism of sheaves $f^*\colon \mathscr{O}_Y \to f_*\mathscr{O}_X$ (such that $(f_P^*)^{-1}(I_P) = I_{f(P)}$ with f_P^* as in Remark 13.7).

(b) As a more concrete example, let P be a point on a variety Y, and let $i\colon P \to Y$ be the inclusion map. Then we have $\mathscr{O}_P(P) = K$ and $\mathscr{O}_P(\emptyset) = \{0\}$. Hence, the sheaf $i_*\mathscr{O}_P$ is given by

$$(i_*\mathscr{O}_P)(U) = \mathscr{O}_P(i^{-1}(U)) = \begin{cases} K & \text{if } P \in U, \\ \{0\} & \text{if } P \notin U \end{cases} \quad \text{for all open subsets } U \subset Y,$$

and the morphism $\mathscr{O}_Y \to i_*\mathscr{O}_P$, $\varphi \mapsto i^*\varphi$ is given by evaluating a regular function φ on $U \subset Y$ at the point P (if it lies in U), as shown in the following picture.



The sections of $i_*\mathscr{O}_P$ can thus be interpreted as "functions on Y that only have a value at P". The sheaf $i_*\mathscr{O}_P$ is therefore usually denoted K_P ("the field K concentrated at the point P") and called the **skyscraper sheaf** on Y at P (because of the shape of the shaded region above in which the graph of the function lies).

Next, we would expect that a morphism of sheaves of modules has a kernel and an image sheaf. Whereas the definition of the kernel is very straightforward, it turns out however that the construction of the image is more complicated. So in order to see the difference, let us prove in detail that the kernel sheaf can be obtained as expected.

Construction 13.11 (Kernel sheaf)**.** Let $f\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on a scheme X. For any open subset $U \subset X$ we set

$$(\operatorname{Ker} f)(U) := \operatorname{Ker}(f_U\colon \mathscr{F}(U) \to \mathscr{G}(U)).$$

It is clear that this defines a presheaf $\operatorname{Ker} f$ on X with the obvious restriction maps.

We claim that $\operatorname{Ker} f$ is in fact a sheaf on X, i. e. that it satisfies the gluing property. To show this, let $\{U_i : i \in I\}$ be an open cover of an open subset $U \subset X$, and assume that we are given sections $\varphi_i \in \operatorname{Ker} f_{U_i} \subset \mathscr{F}(U_i)$ that agree on the overlaps. As $\mathscr{F}$ is a sheaf, they glue to a unique section $\varphi \in \mathscr{F}(U)$, and we just have to check that $\varphi \in (\operatorname{Ker} f)(U)$, i. e. that $f_U(\varphi) = 0 \in \mathscr{G}(U)$. But $\mathscr{G}$ is a sheaf as well, so we can check this locally on the given open cover: For all $i \in I$ we have $f_U(\varphi)|_{U_i} = f_{U_i}(\varphi|_{U_i}) = f_{U_i}(\varphi_i) = 0$.

Hence $\operatorname{Ker} f$ is a sheaf on X. We call it the **kernel sheaf** of f.

What the above argument boils down to is simply that the property of being in the kernel, i. e. of being mapped to zero under a morphism, is a *local* property — a function-like object is zero if it is zero on every subset of an open cover. So the kernel is again a sheaf. In contrast, if we defined the image in the same way, then for a section $\varphi \in \mathscr{G}(U)$ of a sheaf $\mathscr{G}$ on an open set U to be in the image of a morphism $f\colon \mathscr{F} \to \mathscr{G}$ we would need *one* section of $\mathscr{F}$ on all of U that maps to φ under f. This is not a local property, and hence replacing the kernel with the image in Construction 13.11 will only give us a presheaf in general. Let us see this explicitly, and denote this presheaf by $\operatorname{Im}' f$ as it is just a preliminary construction that will be replaced by the "correct" one in Definition 13.19 (a).

Construction 13.12 (Image presheaf)**.** For a morphism $f\colon \mathscr{F} \to \mathscr{G}$ of sheaves on a scheme X and any open subset $U \subset X$ we set

$$(\operatorname{Im}' f)(U) := \operatorname{Im}(f_U\colon \mathscr{F}(U) \to \mathscr{G}(U)).$$

As in Construction 13.11, it is immediate that this defines a presheaf $\operatorname{Im}' f$ on X with the obvious restriction maps.

To see that $\operatorname{Im}' f$ is in general not a sheaf, consider the inclusion map $i\colon \mathbb{A}^1 \to \mathbb{A}^2$, $x_1 \mapsto (x_1,0)$ and its corresponding morphism of sheaves $f = i^*\colon \mathscr{O}_{\mathbb{A}^2} \to i_*\mathscr{O}_{\mathbb{A}^1}$ on $\mathbb{A}^2$ as in Example 13.10 (a). For the open cover of $U := \mathbb{A}^2 \backslash \{(0,0)\}$ by $U_k := \{(x_1,x_2) : x_k \neq 0\}$ for $k \in \{1,2\}$ we then have by Corollary 3.10 and Example 3.11:

$$\begin{aligned}
\mathscr{O}_{\mathbb{A}^2}(U_1) &= K[x_1,x_2]_{x_1} &&\text{and} & i_*\mathscr{O}_{\mathbb{A}^1}(U_1) &= K[x_1]_{x_1} &&\Rightarrow & (\operatorname{Im}' f)(U_1) &= K[x_1]_{x_1},\\
\mathscr{O}_{\mathbb{A}^2}(U_2) &= K[x_1,x_2]_{x_2} &&\text{and} & i_*\mathscr{O}_{\mathbb{A}^1}(U_2) &= \{0\} &&\Rightarrow & (\operatorname{Im}' f)(U_2) &= \{0\},\\
\mathscr{O}_{\mathbb{A}^2}(U_1\cap U_2) &= K[x_1,x_2]_{x_1x_2} &&\text{and} & i_*\mathscr{O}_{\mathbb{A}^1}(U_1\cap U_2) &= \{0\} &&\Rightarrow & (\operatorname{Im}' f)(U_1\cap U_2) &= \{0\},\\
\mathscr{O}_{\mathbb{A}^2}(U) &= K[x_1,x_2] &&\text{and} & i_*\mathscr{O}_{\mathbb{A}^1}(U) &= K[x_1]_{x_1} &&\Rightarrow & (\operatorname{Im}' f)(U) &= K[x_1].
\end{aligned}$$

Hence the sections $\frac{1}{x_1} \in (\operatorname{Im}' f)(U_1)$ and $0 \in (\operatorname{Im}' f)(U_2)$ are (trivially) compatible on the overlap in $(\operatorname{Im}' f)(U_1 \cap U_2) = \{0\}$, but they do not glue to a section in $(\operatorname{Im}' f)(U) = K[x_1]$. This means that $\operatorname{Im}' f$ is not a sheaf. We call $\operatorname{Im}' f$ the **image presheaf** of f.

In fact, there are many more natural constructions with sheaves of modules that only yield a presheaf in general. Let us give one more example of this that will be important later.

Construction 13.13 (Tensor presheaf)**.** Let $\mathscr{F}$ and $\mathscr{G}$ be two sheaves on a scheme X. We define the **tensor presheaf** $\mathscr{F} \otimes' \mathscr{G}$ on X by

$$(\mathscr{F} \otimes' \mathscr{G})(U) := \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U)$$

for any open subset $U \subset X$ (as in the previous constructions it is clear that this is a presheaf).

The following example shows that $\mathscr{F} \otimes' \mathscr{G}$ is in general not a sheaf: On $X = \mathbb{P}^1$ consider the twisting sheaves $\mathscr{F} = \mathscr{O}_X(1)$ and $\mathscr{G} = \mathscr{O}_X(-1)$, and the open subsets $U_i = \{(x_0 : x_1) : x_i \neq 0\}$ for $i \in \{0,1\}$ that cover X. Then

$$x_0 \otimes \frac{1}{x_0} \in (\mathscr{F} \otimes' \mathscr{G})(U_0) \quad \text{and} \quad x_1 \otimes \frac{1}{x_1} \in (\mathscr{F} \otimes' \mathscr{G})(U_1),$$

and these two sections are compatible on $U_0 \cap U_1$ since

$$x_0 \otimes \frac{1}{x_0} = \frac{x_1}{x_0} \cdot x_0 \otimes \frac{x_0}{x_1} \cdot \frac{1}{x_0} = x_1 \otimes \frac{1}{x_1} \quad \in (\mathscr{F} \otimes' \mathscr{G})(U_0 \cap U_1)$$

as $\frac{x_1}{x_0}$ and $\frac{x_0}{x_1}$ are regular on $U_0 \cap U_1$. So if $\mathscr{F} \otimes' \mathscr{G}$ was a sheaf, these sections would have to glue to a global section in

$$(\mathscr{F} \otimes' \mathscr{G})(X) = \mathscr{F}(X) \otimes_{\mathscr{O}_X(X)} \mathscr{G}(X) \overset{13.5\,(a)}{=} \mathscr{F}(X) \otimes_{\mathscr{O}_X(X)} \{0\} = \{0\},$$

which is of course impossible. Hence $\mathscr{F} \otimes' \mathscr{G}$ is not a sheaf.

The way out of this trouble is called sheafification: a process that naturally associates to any presheaf $\mathscr{F}'$ a sheaf $\mathscr{F}$ that describes essentially the same function-like objects, but with the conditions on them made local. We have used this construction implicitly already in the construction of the structure sheaf of an affine variety X or scheme $\operatorname{Spec} R$, when we initially wanted a regular function to be a quotient of elements of $A(X)$ resp. R, but then defined it to only be locally of this form to obtain a sheaf.

Definition 13.14 (Sheafification)**.** Let $\mathscr{F}'$ be a presheaf on a scheme X. For an open subset $U \subset X$ we set

$$\begin{aligned}
\mathscr{F}(U) := \{\varphi = (\varphi_P)_{P\in U} : {}& \varphi_P \in \mathscr{F}'_P \text{ for all } P \in U, \text{ and}\\
& \text{for all } P \text{ there is an open neighborhood } U_P \text{ of } P \text{ in } U\\
& \text{and a section } s \in \mathscr{F}'(U_P) \text{ with } \varphi_Q = s_Q \text{ for all } Q \in U_P\},
\end{aligned}$$

where $s_Q \in \mathscr{F}'_Q$ denotes the germ of s in Q. (As usual, we can write the condition in the last line as "$\varphi = s$ on U_P".)

It is obvious from the local nature of the definition that $\mathscr{F}$ is a sheaf. It is called the **sheafification** of $\mathscr{F}'$ or **sheaf associated to** $\mathscr{F}'$.

Example 13.15.

(a) Let X be an affine variety. If we had defined the presheaf $\mathscr{O}'_X$ of functions on X that are (globally) quotients of polynomials, i. e.

$$\mathscr{O}'_X(U) = \left\{\varphi : U \to K \text{ such that there are } f,g \in A(X) \text{ with } \varphi = \frac{g}{f} \text{ on } U\right\}$$

for all open subsets $U \subset X$, then the structure sheaf $\mathscr{O}_X$ would just be the sheafification of $\mathscr{O}'_X$. We have seen in Example 3.3 that $\mathscr{O}'_X$ is in general not a sheaf already and thus differs from $\mathscr{O}_X$.

(b) As in Construction 13.13, consider again on $X = \mathbb{P}^1$ the two sections $s_i := x_i \otimes \frac{1}{x_i}$ of $\mathscr{O}_X(1) \otimes' \mathscr{O}_X(-1)$ on the open subsets $U_i = \{(x_0 : x_1) : x_i \neq 0\}$ for $i \in \{0,1\}$. Although they are compatible on $U_0 \cap U_1$, we had seen that they do not glue to a global section of $\mathscr{O}_X(1) \otimes' \mathscr{O}_X(-1)$. As expected, they now do glue however to a global section of the sheafification of $\mathscr{O}_X(1) \otimes' \mathscr{O}_X(-1)$, namely to the family $\varphi = (\varphi_P)_{P \in X}$ with $\varphi_P = (s_0)_P$ for all $P \in U_0$ and $\varphi_P = (s_1)_P$ for all $P \in U_1$.

Remark 13.16. Any presheaf $\mathscr{F}'$ on a scheme X admits a natural morphism to its sheafification $\mathscr{F}$ defined by

$$\mathscr{F}'(U) \to \mathscr{F}(U),\ s \mapsto (s_P)_{P \in U}$$

for all open subsets $U \subset X$. In fact, there are some more immediate and important properties of sheafification that we should mention. For example, as it does not affect any local properties we would expect that it does not alter the stalks. Moreover, if we start with a sheaf (for which the conditions on the sections are local already) then the sheafification should not change anything. Let us quickly prove these two properties.

Lemma 13.17 (Properties of sheafification)**.** *Let $\mathscr{F}'$ be a presheaf on a scheme X, and let $\mathscr{F}$ be its sheafification.*

(a) *For all $P \in X$ we have $\mathscr{F}_P \cong \mathscr{F}'_P$.*

(b) *If $\mathscr{F}'$ is already a sheaf then $\mathscr{F} \cong \mathscr{F}'$.*

Proof.

(a) For all $P \in X$ there is a natural $\mathscr{O}_{X,P}$-module homomorphism $\mathscr{F}_P \to \mathscr{F}'_P$, $\overline{(U,\varphi)} \mapsto \varphi_P$ that sends the class of a family $\varphi = (\varphi_Q)_{Q \in U} \in \mathscr{F}(U)$ to its element at P. Conversely, the natural morphism $\mathscr{F}' \to \mathscr{F}$ of Remark 13.16 gives rise to a homomorphism $\mathscr{F}'_P \to \mathscr{F}_P$ by Remark 13.7. It is obvious that these two maps are inverse to each other, so we have $\mathscr{F}_P \cong \mathscr{F}'_P$.

(b) If $\mathscr{F}'$ is already a sheaf then the natural morphism $\mathscr{F}' \to \mathscr{F}$ of Remark 13.16 is a morphism of sheaves. As it induces isomorphisms on the stalks by (a), it is itself an isomorphism by Exercise 13.8. $\square$

Exercise 13.18 (Universal property of sheafification)**.** Let $\mathscr{F}'$ be a presheaf on a scheme X, and denote by $h\colon \mathscr{F}' \to \mathscr{F}$ the natural morphism of Remark 13.16.

Prove that any morphism $f'\colon \mathscr{F}' \to \mathscr{G}$ to a sheaf $\mathscr{G}$ factors uniquely through h, i. e. there is a unique morphism $f\colon \mathscr{F} \to \mathscr{G}$ with $f' = f \circ h$.

Of course, the idea is now to append the process of sheafification to any construction involving sheaves that might just give us a presheaf. Let us list some important examples.

Definition 13.19 (Constructions with sheaves)**.** Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on a scheme X.

(a) For a morphism $f\colon \mathscr{F} \to \mathscr{G}$ we define the **image sheaf** $\operatorname{Im} f$ to be the sheafification of the image presheaf $\operatorname{Im}' f$ of Construction 13.12. Note that it admits a natural morphism to $\mathscr{G}$ by the universal property of Exercise 13.18.

We say that f is **injective** if $\operatorname{Ker} f = \{0\}$, we say that f is **surjective** if $\operatorname{Im} f \cong \mathscr{G}$ (more precisely, if the natural map $\operatorname{Im} f \to \mathscr{G}$ is an isomorphism).

If f is injective then all maps $\mathscr{F}(U) \to \mathscr{G}(U)$ for open subsets $U \subset X$ are injective by Construction 13.11, so we can define the **quotient sheaf** $\mathscr{G}/\mathscr{F}$ to be the sheafification of the presheaf $U \mapsto \mathscr{G}(U)/\mathscr{F}(U)$.

A sequence

$$\cdots \longrightarrow \mathscr{F}_i \xrightarrow{f_i} \mathscr{F}_{i+1} \xrightarrow{f_{i+1}} \mathscr{F}_{i+2} \longrightarrow \cdots$$

of sheaves and morphisms on X is called **exact** if $\operatorname{Im} f_i \cong \operatorname{Ker} f_{i+1}$ for all i. (More precisely, we require that $f_{i+1} \circ f_i$ is the zero map, which induces morphisms $\operatorname{Im}' f_i \to \operatorname{Ker} f_{i+1}$ and hence $\operatorname{Im} f_i \to \operatorname{Ker} f_{i+1}$ by Exercise 13.18; we require this to be an isomorphism.)

(b) The **tensor sheaf** $\mathscr{F} \otimes \mathscr{G}$ is the sheafification of the presheaf $U \mapsto \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U)$ of Construction 13.13.

The **dual sheaf** $\mathscr{F}^\vee$ is the sheafification of the presheaf $U \mapsto \operatorname{Hom}_{\mathscr{O}_X(U)}(\mathscr{F}(U), \mathscr{O}_X(U))$.

Exercise 13.20. Let $f\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on a scheme X. Moreover, for a point $P \in X$ let $f_P\colon \mathscr{F}_P \to \mathscr{G}_P$ be the induced map on the stalks as in Remark 13.7. Prove:

(a) $(\operatorname{Ker} f)_P = \operatorname{Ker}(f_P)$.

(b) $(\operatorname{Im} f)_P = \operatorname{Im}(f_P)$.

In particular, conclude the following generalization of Exercise 13.8: The morphism f is injective (resp. surjective) if and only if f_P is injective (resp. surjective) for all $P \in X$.

Note that, with the definition of surjectivity and hence exactness involving sheafification, it is in general no longer true that a morphism of sheaves on a scheme X is surjective (resp. a sequence of morphisms of sheaves on X is exact) if and only if it is on sections for all open subsets $U \subset X$ — we will see this explicitly in Example 13.22 (b). However, we will see partial results in this direction in Exercise 13.26, and it is in any case possible to check exactness of a sequence locally in the following sense: 23

Lemma 13.21. *Let*

$$\cdots \to \mathscr{F}_i \to \mathscr{F}_{i+1} \to \mathscr{F}_{i+2} \to \cdots$$

be a sequence of sheaves and morphisms on a scheme X. Then the following statements are equivalent:

(a) *The sequence $\cdots \to \mathscr{F}_i \to \mathscr{F}_{i+1} \to \mathscr{F}_{i+2} \to \cdots$ is exact.*

(b) *The restricted sequence $\cdots \to \mathscr{F}_i|_U \to \mathscr{F}_{i+1}|_U \to \mathscr{F}_{i+2}|_U \to \cdots$ is exact for any open subset $U \subset X$.*

(c) *There is an open cover $\{U_k : k \in I\}$ of X such that for all $k \in I$ the restricted sequence $\cdots \to \mathscr{F}_i|_{U_k} \to \mathscr{F}_{i+1}|_{U_k} \to \mathscr{F}_{i+2}|_{U_k} \to \cdots$ is exact.*

(d) *The induced sequence $\cdots \to (\mathscr{F}_i)_P \to (\mathscr{F}_{i+1})_P \to (\mathscr{F}_{i+2})_P \to \cdots$ on the stalks is exact for all $P \in X$.*

Proof. (a) $\Rightarrow$ (d): If $\operatorname{Im} f_i \cong \operatorname{Ker} f_{i+1}$ then $(\operatorname{Im} f_i)_P \cong (\operatorname{Ker} f_{i+1})_P$ and hence $\operatorname{Im}((f_i)_P) = \operatorname{Ker}((f_{i+1})_P)$ for all $P \in X$ by Exercise 13.20.

(d) $\Rightarrow$ (a): If $\operatorname{Im}((f_i)_P) = \operatorname{Ker}((f_{i+1})_P)$ for all $P \in X$ then $(\operatorname{Im} f_i)_P \cong (\operatorname{Ker} f_{i+1})_P$ again by Exercise 13.20. By Lemma 13.17 (b), we can identify $\mathscr{F}_i$, $\mathscr{F}_{i+1}$, and $\mathscr{F}_{i+2}$ with their sheafifications, in which the maps between them are given by the maps on the stalks. Hence $f_{i+1} \circ f_i$ is the zero map. As in Definition 13.19 (a), this means that we have a morphism $\operatorname{Im} f_i \to \operatorname{Ker} f_{i+1}$. But this has to be an isomorphism by Exercise 13.8 since the induced map on the stalks is an isomorphism by assumption.

The equivalences (b) $\Leftrightarrow$ (d) and (c) $\Leftrightarrow$ (d) follow in the same way since for any open subset U containing a point P the stalk $\mathscr{F}_P$ depends only on $\mathscr{F}|_U$. $\square$

Example 13.22 (Skyscraper sequence)**.**

(a) Let $P = (1:0) \in X = \mathbb{P}^1$, and denote by $i\colon P \to X$ the inclusion morphism. Consider the **skyscraper sequence**

$$0 \longrightarrow \mathscr{O}_X(-1) \stackrel{f}{\longrightarrow} \mathscr{O}_X \stackrel{g}{\longrightarrow} K_P \longrightarrow 0,$$

where f is given by multiplication with x_1 as in Example 13.5 (c), and $g\colon \mathscr{O}_X \to i_*\mathscr{O}_P$ is the evaluation at P as in Example 13.10 (b). We claim that this sequence is exact. In fact, in the next chapter we will establish simple ways to check exactness of a sequence like this (see Example 14.9), but for now let us verify this directly:

- The kernel $\operatorname{Ker} f$ consists of all sections φ of $\mathscr{O}_X(-1)$ such that $\varphi x_1 = 0$ (in the quotient field of $K[x_0,x_1]$). This obviously requires $\varphi = 0$, and hence f is injective.
- The image presheaf $\operatorname{Im}' f$ contains all regular functions of the form φx_1 for a section φ of $\mathscr{O}_X(-1)$. These are exactly the regular functions with value 0 at P. As this is a local condition we see that $\operatorname{Im}' f$ is already a sheaf, and thus for any open subset $U \subset X$ that
$$(\operatorname{Im} f)(U) = (\operatorname{Im}' f)(U) = \{\psi \in \mathscr{O}_X(U) : \psi(P) = 0 \text{ if } P \in U\} = (\operatorname{Ker} g)(U).$$
- We clearly have $\operatorname{Im}' g = K_P$, as constant functions already map surjectively to K_P. So again, $\operatorname{Im}' g$ is already a sheaf, and we conclude that $\operatorname{Im} g = K_P$ as well, i. e. that g is surjective.

(b) Now take the additional point $Q = (0:1) \in X$ and consider the *double skyscraper sequence*

$$0 \longrightarrow \mathscr{O}_X(-2) \stackrel{\cdot x_0 x_1}{\longrightarrow} \mathscr{O}_X \stackrel{h}{\longrightarrow} K_P \oplus K_Q \longrightarrow 0,$$

where the second non-trivial map h is now the evaluation at P and Q. As this sequence just restricts to the ordinary skyscraper sequence of (a) on $U_i = \{(x_0:x_1) : x_i \neq 0\}$ for $i \in \{0,1\}$, we see by Lemma 13.21 that this sequence is still exact.

In particular, the map h is surjective. But note that h is not surjective on global sections as $\mathscr{O}_X(X) \cong K$ only contains constant functions by Corollary 7.23, and hence $(\operatorname{Im}' h)(X) \cong K$ only contains constant functions on $\{P,Q\}$ — whereas $(K_P \oplus K_Q)(X) = K^2$. Sheafification will make this condition local, so that $(\operatorname{Im} h)(X)$ contains *locally* constant functions on $\{P,Q\}$ and we have in fact $\operatorname{Im} h \cong K_P \oplus K_Q$.

Example 13.23 (Tensor products of twisting sheaves)**.** Let $n \in \mathbb{N}_{>0}$ and $d,e \in \mathbb{Z}$. By Construction 13.4 (c) there are $\mathscr{O}_{\mathbb{P}^n}(U)$-module homomorphisms

$$(\mathscr{O}_{\mathbb{P}^n}(d))(U) \otimes_{\mathscr{O}_{\mathbb{P}^n}(U)} (\mathscr{O}_{\mathbb{P}^n}(e))(U) \to (\mathscr{O}_{\mathbb{P}^n}(d+e))(U),\ (\varphi,\psi) \mapsto \varphi\psi$$

for all open subsets $U \subset \mathbb{P}^n$. This defines presheaf homomorphisms $\mathscr{O}_{\mathbb{P}^n}(d) \otimes' \mathscr{O}_{\mathbb{P}^n}(e) \to \mathscr{O}_{\mathbb{P}^n}(d+e)$, and hence morphisms $\mathscr{O}_{\mathbb{P}^n}(d) \otimes \mathscr{O}_{\mathbb{P}^n}(e) \to \mathscr{O}_{\mathbb{P}^n}(d+e)$ by the universal property of sheafification of Exercise 13.18.

But on the open subsets $U_i = \{(x_0:\cdots:x_n) : x_i \neq 0\}$ this morphism just restricts by Example 13.5 (d) to $\mathscr{O}_{\mathbb{P}^n}|_U \otimes \mathscr{O}_{\mathbb{P}^n}|_U \to \mathscr{O}_{\mathbb{P}^n}|_U$, which is clearly an isomorphism. Hence we conclude by Lemma 13.21 that

$$\mathscr{O}_{\mathbb{P}^n}(d) \otimes \mathscr{O}_{\mathbb{P}^n}(e) \cong \mathscr{O}_{\mathbb{P}^n}(d+e) \quad \text{by} \quad \varphi \otimes \psi \mapsto \varphi\psi.$$

Exercise 13.24. Find $d \in \mathbb{Z}$ and morphisms f and g such that the sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1} \stackrel{f}{\longrightarrow} \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1) \stackrel{g}{\longrightarrow} \mathscr{O}_{\mathbb{P}^1}(d) \longrightarrow 0$$

is exact on $\mathbb{P}^1$.

Exercise 13.25. Prove that $\mathscr{O}_{\mathbb{P}^n}(d)^\vee \cong \mathscr{O}_{\mathbb{P}^n}(-d)$ for all $n \in \mathbb{N}_{>0}$ and $d \in \mathbb{Z}$.

Exercise 13.26. Let X be a scheme.

(a) Prove that a sequence of sheaves $\cdots \to \mathscr{F}_i \to \mathscr{F}_{i+1} \to \mathscr{F}_{i+2} \to \cdots$ on X is exact if the sequence of sections $\cdots \to \mathscr{F}_i(U) \to \mathscr{F}_{i+1}(U) \to \mathscr{F}_{i+2}(U) \to \cdots$ is exact for all open subsets $U \subset X$.

(b) Let $f\colon \mathscr{F} \to \mathscr{G}$ be an injective morphism of sheaves on X. Prove that then the image presheaf $\operatorname{Im}' f$ is already a sheaf.

Conclude from this the following partial converse to (a): If $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3$ is an exact sequence of sheaves on X then the sequence of sections $0 \to \mathscr{F}_1(U) \to \mathscr{F}_2(U) \to \mathscr{F}_3(U)$ is also exact for all open subsets $U \subset X$.