14. Quasi-coherent Sheaves

In the last chapter we have introduced sheaves of modules on schemes as an important tool in algebraic geometry. It turns out however that for many applications these arbitrary sheaves of modules are too general to be useful, and that one wants to restrict instead to a slightly smaller class of sheaves that has a closer relation to commutative algebra.

More precisely, if $X = \operatorname{Spec} R$ is an affine scheme we would expect that a module over R determines a sheaf of modules on X. This is indeed the case, and almost any sheaf of modules on X appearing in practice comes from an R-module in this way. For computations, this means that statements about such a sheaf can finally be reduced to statements about the R-module, and thus to pure commutative algebra. But it does not follow from the definitions that a sheaf on X has to come from an R-module, so we will say that it is quasi-coherent if it does, and in most cases restrict our attention to these quasi-coherent sheaves. For a general scheme, we accordingly require that this property holds on an affine open cover.

Let us start by showing exactly how an R-module M determines a sheaf of modules $\tilde{M}$ on the affine scheme $X = \operatorname{Spec} R$. Note that the following construction literally reproduces Definition 12.16 of the structure sheaf in the case of the module $M = R$ (so that we obtain $\tilde{R} = \mathscr{O}_X$).

Definition 14.1 (Sheaf associated to a module)**.** Let $X = \operatorname{Spec} R$ be an affine scheme, and let M be an R-module. For an open subset $U \subset X$ we set

$$\begin{aligned}\tilde{M}(U) := \{\varphi = (\varphi_P)_{P\in U} : \ & \varphi_P \in M_P \text{ for all } P \in U, \text{ and} \\ & \text{for all } P \text{ there is an open neighborhood } U_P \text{ of } P \text{ in } U \\ & \text{and } g \in M, f \in R \text{ with } \varphi_Q = \frac{g}{f} \text{ for all } Q \in U_P\}.\end{aligned}$$

It is clear from the local nature of the definition that $\tilde{M}$ is a sheaf. Moreover, $\tilde{M}(U)$ is by construction a module over $\tilde{R}(U) = \mathscr{O}_X(U)$, and hence $\tilde{M}$ is a sheaf of modules on X. We call it the **sheaf associated to** M.

As sheaves associated to modules are defined in the same way as the structure sheaf, they also share the same important basic properties:

Proposition 14.2. *Let $X = \operatorname{Spec} R$ be an affine scheme, and let M be an R-module.*

(a) *For every $P \in X$ the stalk $\tilde{M}_P$ of $\tilde{M}$ at P is isomorphic to the localization M_P.*

(b) *For every $f \in R$ we have $\tilde{M}(D(f)) \cong M_f$. In particular, setting $f = 1$ we obtain $\tilde{M}(X) = M$.*

Proof. In the case $M = R$, these are exactly the statements of Lemma 12.18 and Proposition 12.20. For an arbitrary module M the proofs can be copied literally since we never had to multiply numerators in the fractions that define the sections of the sheaf. □

The following example shows that, unfortunately, not every sheaf of modules on an affine scheme $\operatorname{Spec} R$ is associated to an R-module, so that we have to define this as an additional property. It is a rather strange construction however that will usually not occur in applications.

Example 14.3. For the ring $R = K[x]_{\langle x\rangle}$, the affine scheme $X = \operatorname{Spec} R$ describes $\mathbb{A}^1$ locally around the origin. It has only two points, namely $\langle x\rangle$ (corresponding to the origin) and $\langle 0\rangle$ (the generic point of $\mathbb{A}^1$). Topologically, the only non-trivial open subset of X is $U := D(x) = \{\langle 0\rangle\}$, with $\mathscr{O}_X(U) = R_x = K[x]_{\langle 0\rangle}$ by Lemma 12.18. Hence we can completely specify a presheaf of modules $\mathscr{F}$ on X by setting

$$\mathscr{F}(X) = \mathscr{F}(\emptyset) = \{0\} \quad \text{and} \quad \mathscr{F}(U) = K[x]_{\langle 0\rangle}$$

with the zero maps as restriction homomorphisms. Note that this is trivially a sheaf — simply because no open subset of X has any non-trivial open cover at all.

But $\mathscr{F}$ cannot be of the form $\tilde{M}$ for an R-module M, since otherwise it would follow from Proposition 14.2 (b) that $M = \mathscr{F}(X) = \{0\}$, and thus that $\mathscr{F} = \tilde{M}$ is the zero sheaf, which is clearly not the case.

Definition 14.4 (Quasi-coherent sheaves)**.** A sheaf of modules $\mathscr{F}$ on a scheme X is called **quasi-coherent** if there is an affine open cover $\{U_i : i \in I\}$ of X such that on every $U_i = \operatorname{Spec} R_i$ the restricted sheaf $\mathscr{F}|_{U_i}$ is isomorphic to the sheaf $\tilde{M}_i$ associated to an R_i-module M_i.

Remark 14.5.

(a) It can be shown that for a quasi-coherent sheaf $\mathscr{F}$ on a scheme X the restriction to *any* affine open subset $U = \operatorname{Spec} R \subset X$ is in fact of the form $\mathscr{F}|_U \cong \tilde{M}$ for an R-module M [H, Proposition II.5.4].

(b) There is also the notion of a *coherent* sheaf, which is given just as in Definition 14.4 with the additional requirement that M_i is a *finitely generated* R_i-module for all i. We will not need this stronger condition in these notes, although it is clearly satisfied for almost all sheaves occurring in practice.

Example 14.6.

(a) Of course, the structure sheaf $\mathscr{O}_X$ is quasi-coherent on any scheme X, since for any open subset $U = \operatorname{Spec} R \subset X$ we have $\mathscr{O}_X|_U \cong \tilde{R}$.

(b) Consequently, for any $n \in \mathbb{N}_{>0}$ and $d \in \mathbb{Z}$ the twisting sheaf $\mathscr{O}_{\mathbb{P}^n}(d)$ is quasi-coherent on $\mathbb{P}^n$ as it is locally isomorphic to the structure sheaf by Example 13.5 (d).

As mentioned already, the following lemma now shows that most statements about quasi-coherent sheaves can be translated immediately into commutative algebra statements about modules, and that almost all constructions that can be performed with quasi-coherent sheaves will again yield quasi-coherent sheaves.

Lemma 14.7. *Let $X = \operatorname{Spec} R$ be an affine scheme.*

(a) *For any two R-modules M and N there is a bijection*

$$\{\text{morphisms of sheaves } \tilde{M} \to \tilde{N}\} \quad \stackrel{1:1}{\longleftrightarrow} \quad \{R\text{-module homomorphisms } M \to N\}.$$

(b) *A sequence of R-modules $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact if and only if the corresponding sequence of associated sheaves $0 \to \tilde{M}_1 \to \tilde{M}_2 \to \tilde{M}_3 \to 0$ is exact on X.*

(c) *For any R-modules M, N we have*

$$\tilde{M} \oplus \tilde{N} = (M \oplus N)^{\sim}, \quad \tilde{M} \otimes \tilde{N} = (M \otimes N)^{\sim}, \quad \text{and} \quad \tilde{M}^{\vee} = (M^{\vee})^{\sim}.$$

In particular, kernels, images, quotients, direct sums, tensor products, and duals of quasi-coherent sheaves are again quasi-coherent on any scheme X.

Proof.

(a) Given a morphism $\tilde{M} \to \tilde{N}$, taking global sections gives us an R-module homomorphism $M \to N$ by Proposition 14.2 (b). Conversely, an R-module homomorphism $M \to N$ gives rise by localization to morphisms $M_P \to N_P$ for all $P \in X$, and therefore by definition determines a morphism $\tilde{M} \to \tilde{N}$ of the associated sheaves. It is clear from the construction that these two operations are inverse to each other.

(b) It is known from commutative algebra that a sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of R-modules is exact if and only if for all $P \in \operatorname{Spec} R$ the sequence of localized modules $0 \to (M_1)_P \to (M_2)_P \to (M_3)_P \to 0$ is exact [G3, Proposition 6.27]. But by Proposition 14.2 (a) this is just the sequence of stalks $0 \to (\tilde{M}_1)_P \to (\tilde{M}_2)_P \to (\tilde{M}_3)_P \to 0$. So the statement follows immediately since by Lemma 13.21 exactness of a sequence of sheaves can be checked on the stalks.

(c) Similarly to (b), this follows from the commutative algebra fact that direct sums, tensor products, and taking duals commute with localization. More precisely, e. g. in the case of direct sums the natural isomorphisms $M_P \oplus N_P \cong (M \oplus N)_P$ [G3, Corollary 6.22] for all $P \in X$ give rise to a morphism of sheaves $\tilde{M} \oplus \tilde{N} \to (M \oplus N)^{\sim}$, and this is in fact an isomorphism by Exercise 13.8 as it is an isomorphism on the stalks (which are precisely $M_P \oplus N_P$ and $(M \oplus N)_P$, respectively). □

There is only one construction with sheaves that we have considered so far and that is not covered by Lemma 14.7: the push-forward of sheaves as in Definition 13.9. In fact, in full generality the push-forward of a quasi-coherent sheaf need not be quasi-coherent, but counterexamples are hard to construct and certainly not of any relevance to us. For simplicity, let us restrict here to the case of the push-forward along an inclusion of a closed subscheme, which is the only case that we will need and for which it is easily shown that it preserves quasi-coherence.

Lemma 14.8 (Ideal sequence)**.** *Let $i\colon Y \to X$ be the inclusion of a closed subscheme.*

(a) *If $\mathscr{F}$ is a quasi-coherent sheaf on Y then $i_*\mathscr{F}$ is quasi-coherent on X.*

(b) *There is an exact sequence*

$$0 \to \mathscr{I}_{Y/X} \to \mathscr{O}_X \to i_*\mathscr{O}_Y \to 0$$

*of quasi-coherent sheaves on X, where the second non-trivial map is the pull-back of regular functions as in Example 13.10 (a). Its kernel $\mathscr{I}_{Y/X}$ is called the **ideal sheaf** of Y in X.*

Proof.

(a) First assume that $X = \operatorname{Spec} R$ is affine. Then Y is an affine subscheme of X by Construction 12.33 (b). It is thus of the form $Y = \operatorname{Spec} R/J$ for an ideal $J \trianglelefteq R$, with the inclusion $i\colon Y \to X$ corresponding to the quotient ring homomorphism $R \to R/J$.

As $\mathscr{F}$ is quasi-coherent on Y, it is of the form $\tilde{M}$ for an (R/J)-module M. Now by Definition 13.9 the sections of $i_*\mathscr{F}$ are just the same as those of $\mathscr{F}$ (on the inverse image open subset). Hence, $i_*\mathscr{F}$ is again just the sheaf associated to M, just considered as an R-module using the map $R \to R/J$.

In the general case, we apply this argument to every subset in an affine open cover of X to see that $i_*\mathscr{F}$ is quasi-coherent on X.

(b) We only have to show that the pull-back morphism $\mathscr{O}_X \to i_*\mathscr{O}_Y$ is surjective, as we then obtain an exact sequence as stated with $\mathscr{I}_{Y/X}$ its kernel sheaf. By Lemma 13.21 this can be checked on an affine open cover, so let us assume that $X = \operatorname{Spec} R$ is affine. But then as in (a) the sheaves $\mathscr{O}_X$ and $i_*\mathscr{O}_Y$ are just the sheaves associated to the R-modules R and R/J for an ideal $J \trianglelefteq R$, respectively. As the map $R \to R/J$ is clearly surjective, the statement now follows from Lemma 14.7 (b), and we see that the sequence of the lemma corresponds to the exact sequence of R-modules $0 \to J \to R \to R/J \to 0$. □

Example 14.9 (The skyscraper sequence revisited)**.** Let $P = (1:0) \in X = \mathbb{P}^1$, with inclusion map $i\colon P \to \mathbb{P}^1$. Moreover, denote the standard open cover of $\mathbb{P}^1$ by $\{U_0, U_1\}$ with $U_i = \{(x_0:x_1) : x_i \neq 0\}$ for $i = 0, 1$.

As $i_*\mathscr{O}_P = K_P$ is the skyscraper sheaf of Example 13.10 (b), the ideal sequence of Lemma 14.8 (b) for the map i is just the skyscraper sequence of Example 13.22. In particular, we see:

(a) The ideal sheaf $\mathscr{I}_{P/\mathbb{P}^1}$ is isomorphic to $\mathscr{O}_{\mathbb{P}^1}(-1)$. We will generalize this statement in Exercise 14.12.

(b) The skyscraper sheaf K_P is quasi-coherent: On $U_0 = \operatorname{Spec} K[x_1]$, the proof of Lemma 14.8 shows that it is the sheaf associated to the $K[x_1]$-module $K[x_1]/\langle x_1 \rangle \cong K$, and on U_1 it is clearly the zero sheaf.

(c) The theory of quasi-coherent sheaves easily reproves the exactness of the original skyscraper sequence

$$0 \to \mathscr{O}_{\mathbb{P}^1}(-1) \to \mathscr{O}_{\mathbb{P}^1} \to K_P \to 0$$

on $\mathbb{P}^1$: By Lemma 13.21, it suffices to check this after restricting to U_0 and U_1. On U_0, the twisting sheaf $\mathscr{O}_{\mathbb{P}^1}(-1)$ is isomorphic to $\mathscr{O}_{\mathbb{P}^1}$ by Example 13.5 (d), so the sequence corresponds by (b) to the sequence of $K[x_1]$-modules

$$0 \longrightarrow K[x_1] \xrightarrow{\cdot x_1} K[x_1] \longrightarrow K[x_1]/\langle x_1 \rangle \longrightarrow 0,$$

which is clearly exact. On U_1 (which does not contain P) the skyscraper sheaf is trivial, and hence the exactness of the sequence is just the statement of Example 13.5 (d) that $\mathscr{O}_X(-1)|_{U_1} \cong \mathscr{O}_X|_{U_1}$ by multiplication with x_1. 24

Remark 14.10 (Ideal sheaves)**.** In general, an *ideal sheaf* on a scheme X is a "subsheaf of $\mathscr{O}_X$", i. e. a sheaf of modules $\mathscr{I}$ on X together with an injective morphism $\mathscr{I} \to \mathscr{O}_X$.

(a) By Lemma 14.8, every closed subscheme of X determines a quasi-coherent ideal sheaf on X. Conversely, one can check that every quasi-coherent ideal sheaf on X determines a closed subscheme of X: We can cover X by affine open subsets $\operatorname{Spec} R_i \subset X$, on which the sheaf is then given by ideals $J_i \trianglelefteq R_i$, and glue the affine schemes $\operatorname{Spec} R_i/J_i$.

(b) Similarly, the blow-up construction of Chapter 9 can be generalized to blow up an arbitrary variety X at a quasi-coherent sheaf of ideals (and hence by (a) also at a closed subscheme), as already mentioned in Construction 9.16 (c): On an affine open subvariety $U \subset X$ the sheaf is given by an ideal in $A(U)$, so that we can blow up this ideal and glue the resulting varieties when U ranges over an open cover of X.

As we have now seen that almost all our constructions will yield quasi-coherent sheaves, and that quasi-coherent sheaves are very useful for applying commutative algebra techniques, let us agree:

From now on, all sheaves on a scheme will be assumed to be quasi-coherent.

According to Lemma 14.8, push-forwards of sheaves will therefore from now on only be considered along inclusions of closed subschemes.

Exercise 14.11. Let $P = (1:0) \in \mathbb{P}^1$, and denote as usual by K_P the skyscraper sheaf at P. Determine all morphisms of sheaves of modules

(a) from $\mathscr{O}_{\mathbb{P}^1}$ to $\mathscr{O}_{\mathbb{P}^1}(-1)$;

(b) from $\mathscr{O}_{\mathbb{P}^1}(-1)$ to K_P;

(c) from K_P to $\mathscr{O}_{\mathbb{P}^1}$.

Exercise 14.12. Let $X \subset \mathbb{P}^n$ be an irreducible hypersurface of degree d. Show that the ideal sheaf of X in $\mathbb{P}^n$ is given by $\mathscr{I}_{X/\mathbb{P}^n} \cong \mathscr{O}_{\mathbb{P}^n}(-d)$.

Exercise 14.13. Find a number $d \in \mathbb{Z}$ and morphisms of sheaves f and g such that the sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1} \xrightarrow{f} \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1) \xrightarrow{g} \mathscr{O}_{\mathbb{P}^1}(d) \longrightarrow 0$$

is exact on $\mathbb{P}^1$.

There is one last fundamental construction of sheaves that we will need in these notes and that is in some sense dual to the push-forward: If $f\colon X \to Y$ is a morphism of schemes and $\mathscr{F}$ a sheaf on Y, there is a notion of a pull-back sheaf $f^*\mathscr{F}$ on X. Note that a priori this clearly looks more natural than the push-forward since sheaves describe function-like objects, and the natural operation for functions is the pull-back. In fact, the pull-back of sheaves that we are going to define now has by far much nicer properties than the push-forward, but surprisingly its construction is more difficult. We will proceed in two steps, first describing the affine situation and then gluing it to a global object on an arbitrary scheme.

Construction 14.14 (Pull-back of sheaves)**.** Let $f\colon X\to Y$ be a morphism of schemes, and let $\mathscr{F}$ be a sheaf on Y.

(a) First assume that $X=\operatorname{Spec}R$ and $Y=\operatorname{Spec}S$ are affine, so that f corresponds to a ring homomorphism $S\to R$. Moreover, as $\mathscr{F}$ is quasi-coherent by our general assumption we have $\mathscr{F}=\tilde{M}$ for an S-module M.

Then both M and R are S-modules, and hence we can form the tensor product $M\otimes_S R$ as an R-module. Its associated sheaf $(M\otimes_S R)\tilde{}$ on X is called the pull-back $f^*\mathscr{F}$ of $\mathscr{F}$ along f.

(b) Now let X and Y be arbitrary. To construct a sheaf $f^*\mathscr{F}$ on X, consider first an open subset U of X. Unfortunately, we cannot consider $\mathscr{F}(f(U))$ as $f(U)$ is in general not an open subset of Y. The best we can do is to adapt the definition of a stalk (see Construction 3.17) and define a sheaf $f^{-1}\mathscr{F}$ as the sheafification of the presheaf

$$U\mapsto\{(V,\varphi): V\subset Y \text{ open with } f(U)\subset V, \text{ and } \varphi\in\mathscr{F}(V)\}/\sim,$$

where $(V,\varphi)\sim(V',\varphi')$ if there is an open subset W with $f(U)\subset W\subset V\cap V'$ such that $\varphi|_W=\varphi'|_W$. Note that the stalk of this presheaf, and hence by Lemma 13.17 (a) also of $f^{-1}\mathscr{F}$, at a point $P\in X$ is just $\mathscr{F}_{f(P)}$ as expected.

This is not yet the desired sheaf however, as $f^{-1}\mathscr{F}$ is still made up from sections of $\mathscr{F}$ on open subsets of Y, and thus there is no way to multiply them with regular functions on (open subsets of) X. In other words, $f^{-1}\mathscr{F}$ is not an $\mathscr{O}_X$-module. It is an $f^{-1}\mathscr{O}_Y$-module however, and so is $\mathscr{O}_X$ by pull-back of functions. Hence we can finally define

$$f^*\mathscr{F}:=f^{-1}\mathscr{F}\otimes_{f^{-1}\mathscr{O}_Y}\mathscr{O}_X,$$

where the tensor product again involves sheafification.

Note that in the affine setting this reduces to (a): For $X=\operatorname{Spec}R$, $Y=\operatorname{Spec}S$, and $\mathscr{F}=\tilde{M}$ for an S-module M we have an isomorphism

$$\begin{aligned}(f^*\mathscr{F})_P &\cong (f^{-1}\mathscr{F})_P\otimes_{(f^{-1}\mathscr{O}_Y)_P}\mathscr{O}_{X,P}\cong\mathscr{F}_{f(P)}\otimes_{\mathscr{O}_{Y,f(P)}}\mathscr{O}_{X,P}\overset{14.2\,(a)}{\cong} M_{f(P)}\otimes_{S_{f(P)}}R_P\\ &\cong (M\otimes_S R)_P\end{aligned}$$

for all $P\in X$. This can be used to construct a morphism of sheaves $f^*\mathscr{F}\to(M\otimes_S R)\tilde{}$ by mapping a section $\varphi\in f^*\mathscr{F}(U)$ to the collection of stalks $(\varphi_P)_{P\in U}\in(M\otimes_S R)\tilde{}(U)$ for any open subset $U\subset X$. But this morphism is now even an isomorphism of sheaves as we can check that property on the stalks by Exercise 13.8 and we have just seen that

$$(f^*\mathscr{F})_P\cong(M\otimes_S R)_P\overset{14.2\,(a)}{\cong}(M\otimes_S R)\tilde{}_P.$$

In fact, for concrete (local) computations involving the pull-back sheaf $f^*\mathscr{F}$ we will almost always use the much simpler description of (a); part (b) is mainly needed to show that there is a globally well-defined sheaf that restricts to (a) in the affine case.

Remark 14.15.

(a) For any morphism $f\colon X\to Y$ of schemes we have $f^*\mathscr{O}_Y=\mathscr{O}_X$ by definition. Moreover, the pull-back construction commutes with direct sums, i. e. we have $f^*(\mathscr{F}\oplus\mathscr{G})\cong f^*\mathscr{F}\oplus f^*\mathscr{G}$ for all sheaves $\mathscr{F}$ and $\mathscr{G}$ on Y.

(b) Let $\mathscr{F}$ be a sheaf on a scheme X. For a closed point $P\in X$ and the inclusion map $i\colon P\to X$, the sheaf $i^*\mathscr{F}$ on P has only one non-trivial space of sections $i^*\mathscr{F}(P)$, which by abuse of notation we will simply write as $i^*\mathscr{F}$. The geometric meaning of this space is easiest to see in the affine picture of Construction 14.14 (a): If $X=\operatorname{Spec}R$ is affine, so that $P\trianglelefteq R$ is a maximal ideal and $\mathscr{F}=\tilde{M}$ for an R-module M, we have

$$i^*\mathscr{F}=M\otimes_R R/P=M/PM$$

as a vector space over the residue field $K(P)=R/P$ (see Definition 12.4 (a)).

This vector space $i^*\mathscr{F}$ over $K(P)$ is called the **fiber** of $\mathscr{F}$ at P. Every section φ of $\mathscr{F}$ over an open subset containing P determines by Definition 14.1 an element φ_P in $\mathscr{F}_P = M_P$, and hence by taking its quotient modulo PM a **value** $\varphi(P)$ in $i^*\mathscr{F}$. For the structure sheaf $\mathscr{F} = \mathscr{O}_X$, this just coincides with Definition 12.4 (b) of the value of a regular function. So for an arbitrary sheaf we can still define values of sections at points — they just do not lie in the residue fields any more, but in vector spaces over them. Note however that, just as in the case of regular functions in Example 12.21 (a), these values do not necessarily determine the section.

As we now have defined a push-forward as well as a pull-back of sheaves, let us quickly study to what extent these two operations are inverse to each other. The proof of the following lemma is a good example how statements from commutative algebra can be transferred directly into the language of quasi-coherent sheaves.

Lemma 14.16 (Push-forward and pull-back of sheaves)**.** *Let $i\colon Y \to X$ be the inclusion of a closed subscheme. Then for all sheaves $\mathscr{F}$ on Y and $\mathscr{G}$ on X we have:*

(a) $i^* i_* \mathscr{F} \cong \mathscr{F}$;

(b) ***(Projection formula)*** $i_*(\mathscr{F} \otimes i^*\mathscr{G}) \cong (i_*\mathscr{F}) \otimes \mathscr{G}$.

In particular, for $\mathscr{F} = \mathscr{O}_Y$ we obtain $i_* i^* \mathscr{G} \cong (i_*\mathscr{O}_Y) \otimes \mathscr{G}$.

Proof. According to Lemma 13.21 we can check these isomorphisms locally, so we may assume that $X = \operatorname{Spec} R$ is affine, and consequently $Y = \operatorname{Spec} R/J$ for an ideal $J \trianglelefteq R$. Moreover, we have $\mathscr{F} = \tilde{M}$ for an R/J-module M and $\mathscr{G} = \tilde{N}$ for an R-module N. To prove the stated isomorphisms of quasi-coherent sheaves it then suffices by Lemma 14.7 to show that the corresponding modules are isomorphic. So recall by (the proof of) Lemma 14.8 (a) that the push-forward is given by considering an R/J-module as an R-module, and by Construction 14.14 (a) that the pull-back is given by the tensor product with R over R/J. This means:

(a) The sheaf $i^* i_* \mathscr{F}$ is associated to the R/J-module $M \otimes_R R/J$, which by commutative algebra is canonically isomorphic to M.

(b) We have that

$$i_*(\mathscr{F} \otimes i^*\mathscr{G}) \text{ is the sheaf associated to the } R\text{-module } M \otimes_{R/J} (N \otimes_R R/J), \text{ and}$$
$$(i_*\mathscr{F}) \otimes \mathscr{G} \text{ is the sheaf associated to the } R\text{-module } M \otimes_R N,$$

and again it is known from commutative algebra that these two modules are naturally isomorphic. □

To conclude this chapter, let us finally introduce a class of sheaves that are even nicer than the quasi-coherent ones: those that on an affine open cover are associated to *free* modules.

Definition 14.17 (Locally free sheaves)**.** A sheaf of modules $\mathscr{F}$ on a scheme X is called **locally free** if there is an affine open cover $\{U_i : i \in I\}$ of X such that on every $U_i = \operatorname{Spec} R_i$ the restricted sheaf $\mathscr{F}|_{U_i}$ is isomorphic to the sheaf $\tilde{M}_i$ associated to a free R_i-module M_i of finite rank (i. e. to a finite direct sum of copies of $\mathscr{O}_{U_i}$). If this rank is the same for all $i \in I$ this number is also called the **rank** of $\mathscr{F}$, denoted $\operatorname{rk}\mathscr{F}$.

Remark 14.18.

(a) For a locally free sheaf $\mathscr{F}$ of rank r on a scheme X the fiber at any point $P \in X$ as in Remark 14.15 (b) is an r-dimensional vector space over the residue field $K(P)$. Hence, sections of such a sheaf can be thought of as "functions that take values in a varying r-dimesional vector space", as in our original motivation in Example 13.1 where we informally introduced the tangent sheaf in this way. For this reason, a locally free sheaf is also often called a **vector bundle**, resp. a **line bundle** if it is of rank 1.

(b) In contrast to the case of quasi-coherent sheaves in Remark 14.5 (a), for a locally free sheaf $\mathscr{F}$ on a scheme X it is in general not true on *every* affine open subset $U = \operatorname{Spec} R$ that it is of the form $\mathscr{F}|_U \cong \tilde{M}$ for a free R-module M. A simple example can be obtained if X is the (reducible) 0-dimensional variety consisting of two points P and Q, with coordinate ring $R = K \times K$ by Proposition 2.7. Then the R-module $M = K \times \{0\}$ is clearly not free, but the sheaf $\tilde{M}$ on X is locally free as it is isomorphic to the structure sheaf (resp. the zero sheaf) on the affine open subset $\{P\}$ (resp. $\{Q\}$) of X.

Example 14.19.

(a) The structure sheaf is a line bundle, i. e. locally free of rank 1, on any scheme X. The twisting sheaves $\mathscr{O}_{\mathbb{P}^n}(d)$ of Construction 13.4 are locally isomorphic to the structure sheaf by Example 13.5 (d), and hence line bundles as well. In contrast, the skyscraper sheaf K_P for a point P on $\mathbb{P}^1$ as in Example 13.10 (b) is not locally free, as its fiber at P is 1-dimensional, whereas all other fibers are 0-dimensional.

(b) For any construction in commutative algebra that preserves free modules, the corresponding operation on locally free sheaves will again yield locally free sheaves of the corresponding ranks. Hence, for example, if $\mathscr{F}$ and $\mathscr{G}$ are locally free on a scheme X with $\operatorname{rk}\mathscr{F} = n$ and $\operatorname{rk}\mathscr{G} = m$ then

- the direct sum $\mathscr{F} \oplus \mathscr{G}$ is locally free of rank $n+m$;
- the tensor product $\mathscr{F} \otimes \mathscr{G}$ is locally free of rank nm;
- the dual $\mathscr{F}^\vee$ is locally free of rank n; and
- the pull-back $f^*\mathscr{F}$ for a morphism $f\colon Y \to X$ is locally free of rank n.

(c) In Definition 8.6 and Proposition 8.7, we have constructed the alternating tensor product $\Lambda^k V$ for a vector space V and a natural number $k \in \mathbb{N}$. This construction works in the very same way for modules over a ring, so just as in Construction 13.13 and Definition 13.19 (b) we can define a k-fold alternating tensor product $\Lambda^k\mathscr{F}$ of a sheaf $\mathscr{F}$ on a scheme X as the sheaf associated to the presheaf $U \mapsto \Lambda^k(\mathscr{F}(U))$.

As in Example 8.8 (a), the k-fold alternating tensor product of a free module of rank n is again free of rank $\binom{n}{k}$. Hence, in the same way as in (b) the alternating tensor product $\Lambda^k\mathscr{F}$ of a locally free sheaf $\mathscr{F}$ of rank n is locally free of rank $\binom{n}{k}$. In particular, the highest non-trivial alternating tensor product $\Lambda^n\mathscr{F}$ is a line bundle.

Clearly, the twisting sheaves on projective spaces are the most important line bundles. Using the pull-back construction we can now extend their definition to arbitrary projective schemes.

Notation 14.20 (Twisting sheaves on projective schemes)**.** Let X be a closed subscheme of a projective space $\mathbb{P}^n$ (e. g. a projective variety). Then for any $d \in \mathbb{Z}$ we define the **twisting sheaf** $\mathscr{O}_X(d)$ on X by

$$\mathscr{O}_X(d) := i^*\mathscr{O}_{\mathbb{P}^n}(d),$$

where $i\colon X \to \mathbb{P}^n$ is the inclusion. By Example 14.19 (b), it is a line bundle on X. Similarly to Construction 13.4, its sections are locally given by quotients $\frac{g}{f}$ with $f, g \in S(X)$ homogeneous and $\deg g - \deg f = d$.

Finally in this chapter, let us transfer two important results from commutative algebra about free modules to the language of locally free sheaves.

Lemma 14.21 (Exact sequences and locally free sheaves)**.** *Let* $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ *be an exact sequence of quasi-coherent sheaves on a scheme* X.

(a) *For another quasi-coherent sheaf* $\mathscr{F}$ *on* X *the sequence*

$$0 \to \mathscr{F}_1 \otimes \mathscr{F} \to \mathscr{F}_2 \otimes \mathscr{F} \to \mathscr{F}_3 \otimes \mathscr{F} \to 0$$

is also exact on X *if* $\mathscr{F}$ *is locally free or all* $\mathscr{F}_i$ *are locally free.*

(b) *For any morphism $f\colon Y \to X$ of schemes the sequence*
$$0 \to f^*\mathscr{F}_1 \to f^*\mathscr{F}_2 \to f^*\mathscr{F}_3 \to 0$$
is exact on Y if all $\mathscr{F}_i$ are locally free.

Proof.

(a) Restricting to an open subset of X we may assume that $X = \operatorname{Spec} R$ is affine, $\mathscr{F}_i = \tilde{M}_i$ and $\mathscr{F} = \tilde{M}$ for R-modules M_i and M with $i = 1,2,3$, and that M or all M_i are free of finite rank. But then the statement follows from Lemma 14.7 (b) since in this case it is known from commutative algebra that the original exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ remains exact after taking the tensor product with M [G3, Exercise 5.24].

(b) This time we may assume (by restricting to an open subset $U \subset X$ and then to an open subset of $f^{-1}(U)$ in Y) that $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$ are affine, and that $\mathscr{F}_i = \tilde{M}_i$ for free R-modules M_i of finite rank with $i = 1,2,3$. As in (a), the tensor product sequence
$$0 \to M_1 \otimes_R S \to M_2 \otimes_R S \to M_3 \otimes_R S \to 0$$
is then again exact, and by Construction 14.14 (a) this is precisely the sequence of modules corresponding to the sequence of pull-back sheaves. Hence, the statement follows again from Lemma 14.7 (b). □

Lemma 14.22. *Let $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ be an exact sequence of locally free sheaves of ranks n, $n+m$, and m, respectively. Then there is an isomorphism of line bundles*
$$\Lambda^n \mathscr{F}_1 \otimes \Lambda^m \mathscr{F}_3 \cong \Lambda^{n+m} \mathscr{F}_2.$$

Proof. As in the proof of the previous lemma, it suffices to show the statement for free modules. So let
$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$
be an exact sequence of free modules of ranks n, $n+m$, and m, respectively, over a ring R. We claim that there is then a natural isomorphism
$$\Lambda^n M_1 \otimes \Lambda^m M_3 \to \Lambda^{n+m} M_2$$
$$(x_1 \wedge \cdots \wedge x_n) \otimes (y_1 \wedge \cdots \wedge y_m) \mapsto f(x_1) \wedge \cdots \wedge f(x_n) \wedge g^{-1}(y_1) \wedge \cdots \wedge g^{-1}(y_m), \qquad (*)$$
where $g^{-1}(y_i)$ means to pick any inverse image of y_i under g. In fact, this map is well-defined, as any two inverse images will differ by an element of the form $f(x)$ for some $x \in M_1$, and the alternating tensor product $f(x_1) \wedge \cdots \wedge f(x_n) \wedge f(x)$ is zero since $\operatorname{rk} M_1 = n$. Moreover, the map $(*)$ is an isomorphism: Both sides are isomorphic to R, and if $x_1, \dots, x_n$ and $y_1, \dots, y_m$ are bases of M_1 and M_3, respectively, then the element on the left and right hand side of $(*)$ is a basis of $\Lambda^n M_1 \otimes \Lambda^m M_3$ and $\Lambda^{n+m} M_2$, respectively. □ 25