6. Projective Varieties I: Topology

In the last chapter we have studied (pre-)varieties, i. e. topological spaces that are locally isomorphic to affine varieties. In particular, the ability to glue affine varieties together allowed us to construct *compact* spaces (in the classical topology over the ground field $\mathbb{C}$) as e. g. $\mathbb{P}^1$, whereas affine varieties themselves are never compact unless they consist of only finitely many points (see Exercise 2.36 (b)). Unfortunately, the description of a variety in terms of its affine patches and gluing isomorphisms is quite inconvenient in practice, as we have seen already in some of the calculations in the last chapter. It would therefore be desirable to have a global description of these spaces that does not refer to gluing methods.

We can obtain a large class of such "compact" varieties admitting a global description by considering zero loci of polynomials in projective instead of affine spaces, generalizing projective curves as in [G2, Chapter 3] — recall that the idea of projective spaces is to add "points at infinity" to affine space similarly to how we have obtained $\mathbb{P}^1$ from $\mathbb{A}^1$ in Example 5.5 (a). It turns out that the resulting class of projective varieties is in fact very large — so large that it is actually not easy to construct a variety that is *not* an open subset of a projective variety. We will certainly not see one in these notes.

Let us quickly review the construction of projective spaces from [G2, Chapter 3], and then transfer the concept of varieties to this new setting. In this chapter we will construct these projective varieties just as topological spaces, leaving their structure as ringed spaces to Chapter 7.

Definition 6.1 (Projective spaces)**.** Let $n \in \mathbb{N}$. We define **projective n-space** over K, denoted $\mathbb{P}^n_K$ or simply $\mathbb{P}^n$, to be the set of all 1-dimensional linear subspaces of the vector space K^{n+1}.

Notation 6.2 (Homogeneous coordinates)**.** Obviously, a 1-dimensional linear subspace of K^{n+1} is uniquely determined by a non-zero vector in K^{n+1}, with two such vectors spanning the same linear subspace if and only if they are scalar multiples of each other. In other words, we have

$$\mathbb{P}^n = (K^{n+1}\backslash\{0\}) / \sim$$

with the equivalence relation

$$(x_0,\dots,x_n) \sim (y_0,\dots,y_n) \quad :\Leftrightarrow \quad x_i = \lambda y_i \text{ for some } \lambda \in K^* \text{ and all } i,$$

where $K^* = K\backslash\{0\}$ is the multiplicative group of units of K. This is usually written as $\mathbb{P}^n = (K^{n+1}\backslash\{0\})/K^*$, and the equivalence class of $(x_0,\dots,x_n)$ will be denoted by $(x_0 : \dots : x_n) \in \mathbb{P}^n$ (the notations $[x_0 : \dots : x_n]$ and $[x_0,\dots,x_n]$ are also common in the literature). So in the notation $(x_0 : \dots : x_n)$ for a point in $\mathbb{P}^n$ the numbers $x_0,\dots,x_n$ are not all zero, and they are defined only up to a common scalar multiple. They are called the **homogeneous coordinates** of the point (the reason for this name will become obvious in the course of this chapter). Note also that we will usually label the homogeneous coordinates of $\mathbb{P}^n$ by $x_0,\dots,x_n$ instead of by $x_1,\dots,x_{n+1}$. This choice is motivated by the following relation between $\mathbb{A}^n$ and $\mathbb{P}^n$.

Remark 6.3 (Affine coordinates)**.** Consider the map

$$f\colon \mathbb{A}^n \to \mathbb{P}^n,\ (x_1,\dots,x_n) \mapsto (1 : x_1 : \dots : x_n)$$

which sets $x_0 = 1$ and makes the coordinates $x_1,\dots,x_n$ of $\mathbb{A}^n$ into homogeneous coordinates of $\mathbb{P}^n$. Taking into account that the homogeneous coordinates can be rescaled, it is obviously injective with image $U_0 := \{(x_0 : \dots : x_n) : x_0 \neq 0\}$. On this image the inverse of f is given by

$$f^{-1}\colon U_0 \to \mathbb{A}^n,\ (x_0 : \dots : x_n) \mapsto \left(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right). \qquad (*)$$

With this embedding, we can thus think of $\mathbb{A}^n$ as a subset U_0 of $\mathbb{P}^n$. We call it the **affine part** of $\mathbb{P}^n$; the coordinates $\left(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right)$ of a point $(x_0 : \dots : x_n) \in U_0 \subset \mathbb{P}^n$ are called its **affine coordinates**.

The remaining points of $\mathbb{P}^n$ (where $x_0 = 0$) are of the form $(0:x_1:\cdots:x_n)$ and can be viewed as points at infinity, since by $(*)$ they would have infinite affine coordinates. By forgetting their first coordinate (which is zero anyway) they form a set that is naturally bijective to $\mathbb{P}^{n-1}$. We can thus write

$$\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1},$$

where $\mathbb{A}^n$ is the affine part and $\mathbb{P}^{n-1}$ parametrizes the points at infinity. Usually, it is more helpful to think of the points in projective space $\mathbb{P}^n$ in this way rather than as 1-dimensional linear subspaces as in Definition 6.1. After having given $\mathbb{P}^n$ the structure of a variety we will see in Proposition 7.2 and Exercise 7.3 (b) that in this decomposition $\mathbb{A}^n$ and $\mathbb{P}^{n-1}$ are open and closed subvarieties of $\mathbb{P}^n$, respectively.

Remark 6.4 ($\mathbb{P}^n_{\mathbb{C}}$ is compact in the classical topology)**.** In the case $K = \mathbb{C}$ one can give $\mathbb{P}^n_{\mathbb{C}}$ a standard (quotient) topology by declaring a subset $U \subset \mathbb{P}^n$ to be open if its inverse image under the quotient map $\pi\colon \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{P}^n$ is open in the standard topology. Then $\mathbb{P}^n_{\mathbb{C}}$ is compact: Let

$$S = \{(x_0,\dots,x_n) \in \mathbb{C}^{n+1} : |x_0|^2 + \cdots + |x_n|^2 = 1\}$$

be the unit sphere in $\mathbb{C}^{n+1}$. This is a compact space as it is closed and bounded. Moreover, as every point in $\mathbb{P}^n$ can be represented by a unit vector in S, the restricted map $\pi|_S\colon S \to \mathbb{P}^n$ is surjective. Hence $\mathbb{P}^n$ is compact as a continuous image of a compact set.

Remark 6.5 (Homogeneous polynomials)**.** In complete analogy to affine varieties, we now want to define projective varieties to be subsets of $\mathbb{P}^n$ that can be given as the zero locus of some polynomials in the homogeneous coordinates. Note however that if $f \in K[x_0,\dots,x_n]$ is an arbitrary polynomial, it does not make sense to write down a definition like

$$V(f) = \{(x_0:\cdots:x_n) : f(x_0,\dots,x_n) = 0\} \quad \subset \mathbb{P}^n,$$

because the homogeneous coordinates are only defined up to a common scalar. For example, if $f = x_1^2 - x_0 \in K[x_0,x_1]$ then $f(1,1) = 0$ and $f(-1,-1) \neq 0$, although $(1:1) = (-1:-1)$ in $\mathbb{P}^1$. To get rid of this problem we have to require that f is *homogeneous*, i. e. that all of its monomials have the same (total) degree d: In this case

$$f(\lambda x_0,\dots,\lambda x_n) = \lambda^d f(x_0,\dots,x_n) \text{ for all } \lambda \in K^*,$$

and so in particular we see that

$$f(\lambda x_0,\dots,\lambda x_n) = 0 \quad \Leftrightarrow \quad f(x_0,\dots,x_n) = 0,$$

so that the zero locus of f is well-defined in $\mathbb{P}^n$. So before we can start with our discussion of projective varieties we have to set up some algebraic language to be able to talk about homogeneous elements in a ring (or K-algebra).

Definition 6.6 (Graded rings and K-algebras)**.**

(a) A **graded ring** is a ring R together with Abelian subgroups $R_d \subset R$ for all $d \in \mathbb{N}$, such that:

- We have $R = \bigoplus_{d\in\mathbb{N}} R_d$, i. e. every $f \in R$ has a unique decomposition $f = \sum_{d\in\mathbb{N}} f_d$ such that $f_d \in R_d$ for all $d \in \mathbb{N}$ and only finitely many f_d are non-zero.
- For all $d,e \in \mathbb{N}$ and $f \in R_d$, $g \in R_e$ we have $fg \in R_{d+e}$.

For $f \in R\backslash\{0\}$ the biggest number $d \in \mathbb{N}$ with $f_d \neq 0$ in the decomposition $f = \sum_{d\in\mathbb{N}} f_d$ as above is called the **degree** $\deg f$ of f. The elements of $R_d\backslash\{0\}$ are said to be **homogeneous** (of degree d). We call $f = \sum_{d\in\mathbb{N}} f_d$ and $R = \bigoplus_{d\in\mathbb{N}} R_d$ as above the **homogeneous decomposition** of f and R, respectively.

(b) If R is also a K-algebra in addition to (a), we say that it is a **graded K-algebra** if $\lambda f \in R_d$ for all $\lambda \in K$, $d \in \mathbb{N}$, and $f \in R_d$.

Example 6.7. The polynomial ring $R = K[x_0,\dots,x_n]$ is obviously a graded K-algebra with

$$R_d = \left\{ \sum_{\substack{i_0,\dots,i_n\in\mathbb{N} \\ i_0+\cdots+i_n=d}} a_{i_0,\dots,i_n}\, x_0^{i_0}\cdot\,\cdots\,\cdot x_n^{i_n} : a_{i_0,\dots,i_n} \in K \text{ for all } i_0,\dots,i_n \right\}$$

for all $d \in \mathbb{N}$. In the following we will always consider it with this grading.

Exercise 6.8. Let $R \neq 0$ be a graded ring. Show that the multiplicative unit $1 \in R$ is homogeneous of degree 0.

Of course, we will also need ideals in graded rings. Naively, one might expect that we should consider ideals consisting only of homogeneous elements in this case. However, as an ideal has to be closed under multiplication with *arbitrary* ring elements, it is virtually impossible that all of its elements are homogeneous. Instead, the correct notion of homogeneous ideal is the following.

Definition 6.9 (Homogeneous ideals)**.** An ideal in a graded ring is called **homogeneous** if it can be generated by homogeneous elements.

Lemma 6.10 (Properties of homogeneous ideals)**.** *Let J, J_1, J_2 be ideals in a graded ring R.*

(a) *The ideal J is homogeneous if and only if for all $f \in J$ with homogeneous decomposition $f = \sum_{d\in\mathbb{N}} f_d$ we also have $f_d \in J$ for all d.*

(b) *If J_1 and J_2 are homogeneous then so are $J_1 + J_2$, $J_1 J_2$, $J_1 \cap J_2$, and $\sqrt{J_1}$.*

(c) *If J is homogeneous then the quotient R/J is a graded ring with homogeneous decomposition $R/J = \bigoplus_{d\in\mathbb{N}} R_d/(R_d \cap J)$.*

Proof.

(a) "$\Rightarrow$": Let $J = \langle h_i : i \in I \rangle$ for homogeneous elements $h_i \in R$ for all i, and let $f \in J$. Then $f = \sum_{i\in I} g_i h_i$ for some (not necessarily homogeneous) $g_i \in R$, of which only finitely many are non-zero. If we denote by $g_i = \sum_{e\in\mathbb{N}} g_{i,e}$ the homogeneous decompositions of these elements, the degree-d part of f for $d \in \mathbb{N}$ is

$$f_d = \sum_{\substack{i\in I, e\in\mathbb{N} \\ e+\deg h_i = d}} g_{i,e} h_i \quad \in J.$$

"$\Leftarrow$": Under the given assumption, we claim that $J = \langle h_d : h \in J, d \in \mathbb{N} \rangle$, so that J is a homogeneous ideal. In fact, the inclusion "$\subset$" follows since $h = \sum_{d\in\mathbb{N}} h_d$ for all $h \in J$, and the inclusion "$\supset$" holds by our assumption.

(b) If J_1 and J_2 are generated by homogeneous elements, then clearly so are $J_1 + J_2$ (which is generated by $J_1 \cup J_2$) and $J_1 J_2$. Moreover, J_1 and J_2 then satisfy the equivalent condition of (a), and thus so does $J_1 \cap J_2$.

It remains to be shown that $\sqrt{J_1}$ is homogeneous. We will check the condition of (a) for any $f \in \sqrt{J_1}$ by induction over the degree d of f. Writing $f = f_0 + \cdots + f_d$ in its homogeneous decomposition, we get

$$f^n = (f_0 + \cdots + f_d)^n = f_d^n + (\text{terms of lower degree}) \quad \in J_1$$

for some $n \in \mathbb{N}$, hence $f_d^n \in J_1$ by (a), and thus $f_d \in \sqrt{J_1}$. But then $f - f_d = f_0 + \cdots + f_{d-1}$ lies in $\sqrt{J_1}$ as well, and so by the induction hypothesis we also see that $f_0, \ldots, f_{d-1} \in \sqrt{J_1}$.

(c) It is clear that $R_d/(R_d \cap J) \to R/J$, $\overline{f} \mapsto \overline{f}$ is an injective group homomorphism, so that we can consider $R_d/(R_d \cap J)$ as a subgroup of R/J for all d.

Now let $f \in R$ be arbitrary, with homogeneous decomposition $f = \sum_{d\in\mathbb{N}} f_d$. Then we have $\overline{f} = \sum_{d\in\mathbb{N}} \overline{f_d}$ with $\overline{f_d} \in R_d/(R_d \cap J)$, so $\overline{f}$ also has a homogeneous decomposition. Moreover, this decomposition is unique: If $\sum_{d\in\mathbb{N}} \overline{f_d} = \sum_{d\in\mathbb{N}} \overline{g_d}$ are two such decompositions of the same element in R/J then $\sum_{d\in\mathbb{N}} (f_d - g_d)$ lies in J. Hence, by (a) we have $f_d - g_d \in J$ for all d as well, which means that $\overline{f_d} = \overline{g_d} \in R_d/(R_d \cap J)$. □

With this preparation we can now define projective varieties in the same way as affine ones. For simplicity, for a homogeneous polynomial $f \in K[x_0, \ldots, x_n]$ and a point $x = (x_0 : \cdots : x_n) \in \mathbb{P}^n$ we will write the condition $f(x_0, \ldots, x_n) = 0$ (which is well-defined by Remark 6.5) also as $f(x) = 0$.

Definition 6.11 (Projective varieties and their ideals)**.** Let $n \in \mathbb{N}$.

(a) Let $S \subset K[x_0,\dots,x_n]$ be a set of homogeneous polynomials. Then the (projective) **zero locus** of S is defined as

$$V(S) := \{x \in \mathbb{P}^n : f(x) = 0 \text{ for all } f \in S\} \quad \subset \mathbb{P}^n.$$

Subsets of $\mathbb{P}^n$ that are of this form are called **projective varieties**. For $S = (f_1,\dots,f_k)$ we will write $V(S)$ also as $V(f_1,\dots,f_k)$.

(b) For a homogeneous ideal $J \trianglelefteq K[x_0,\dots,x_n]$ we set

$$V(J) := \{x \in \mathbb{P}^n : f(x) = 0 \text{ for all homogeneous } f \in J\} \quad \subset \mathbb{P}^n.$$

Clearly, if J is the ideal generated by a set S of homogeneous polynomials then $V(J) = V(S)$.

(c) If $X \subset \mathbb{P}^n$ is any subset we define its **ideal** to be

$$I(X) := \langle f \in K[x_0,\dots,x_n] \text{ homogeneous} : f(x) = 0 \text{ for all } x \in X \rangle \quad \trianglelefteq K[x_0,\dots,x_n].$$

(Note that the homogeneous polynomials vanishing on X do not form an ideal yet, so that we have to take the ideal generated by them.)

If we want to distinguish these projective constructions from the affine ones in Definitions 1.2 (b) and 1.8 we will denote them by $V_p(S)$ and $I_p(X)$, and the affine ones by $V_a(S)$ and $I_a(X)$, respectively.

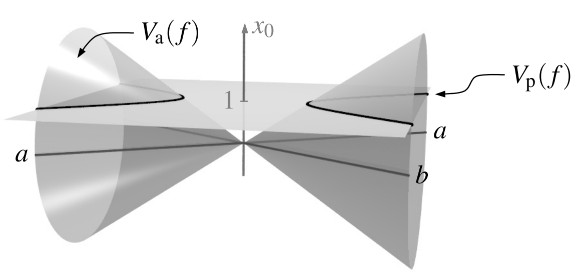
Example 6.12.

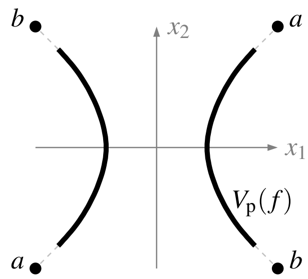
(a) As in the affine case, the empty set $\emptyset = V_p(1)$ and the whole space $\mathbb{P}^n = V_p(0)$ are projective varieties.

(b) If $f_1,\dots,f_r \in K[x_0,\dots,x_n]$ are homogeneous linear polynomials then $V_p(f_1,\dots,f_r) \subset \mathbb{P}^n$ is a projective variety. Projective varieties that are of this form are called **linear subspaces** of $\mathbb{P}^n$.

Exercise 6.13. Let $a \in \mathbb{P}^n$ be a point. Show that the one-point set $\{a\}$ is a projective variety, and compute explicit generators for the ideal $I_p(\{a\}) \trianglelefteq K[x_0,\dots,x_n]$.

Example 6.14. Let $f = x_1^2 - x_2^2 - x_0^2 \in \mathbb{C}[x_0,x_1,x_2]$. The real part of the affine zero locus $V_a(f) \subset \mathbb{A}^3$ of this homogeneous polynomial is the 2-dimensional cone shown in the picture below on the left. According to Definition 6.11, its projective zero locus $V_p(f) \subset \mathbb{P}^2$ is the set of all 1-dimensional linear subspaces contained in this cone — but we have seen in Remark 6.3 already that we should rather think of $\mathbb{P}^2$ as the affine plane $\mathbb{A}^2$ (embedded in $\mathbb{A}^3$ at $x_0 = 1$) together with some points at infinity. With this interpretation the real part of $V_p(f)$ consists of the hyperbola shown below on the right (whose equation $x_1^2 - x_2^2 - 1 = 0$ can be obtained by setting $x_0 = 1$ in f), together with two points a and b at infinity. In the 3-dimensional picture on the left, these two points correspond to the two 1-dimensional linear subspaces parallel to the plane at $x_0 = 1$, in the 2-dimensional picture of the affine part in $\mathbb{A}^2$ on the right they can be thought of as points at infinity in the corresponding directions. Note that, in the latter interpretation, "opposite" points at infinity are actually the same, since they correspond to the same 1-dimensional linear subspace in $\mathbb{C}^3$.





We see in this example that the affine and projective zero locus of f carry essentially the same geometric information — the difference is just whether we consider the cone as a set of individual points, or as a union of 1-dimensional linear subspaces in $\mathbb{A}^3$. Let us now formalize and generalize this correspondence.

Definition 6.15 (Cones)**.** Let $\pi\colon \mathbb{A}^{n+1}\backslash\{0\} \to \mathbb{P}^n$, $(x_0,\dots,x_n) \mapsto (x_0:\dots:x_n)$.

(a) An affine variety $X \subset \mathbb{A}^{n+1}$ is called a **cone** if $0 \in X$, and $\lambda x \in X$ for all $\lambda \in K$ and $x \in X$. In other words, it consists of the origin together with a union of lines through 0.

(b) For a cone $X \subset \mathbb{A}^{n+1}$ we call

$$\mathbb{P}(X) := \pi(X\backslash\{0\}) = \{(x_0:\dots:x_n) \in \mathbb{P}^n : (x_0,\dots,x_n) \in X\} \quad \subset \mathbb{P}^n$$

the **projectivization** of X.

(c) For a projective variety $X \subset \mathbb{P}^n$ we call

$$C(X) := \{0\} \cup \pi^{-1}(X) = \{0\} \cup \{(x_0,\dots,x_n) : (x_0:\dots:x_n) \in X\} \quad \subset \mathbb{A}^{n+1}$$

the **cone** over X (note that this is obviously a cone in the sense of (a)).

Remark 6.16 (Cones and homogeneous ideals)**.**

(a) If $S \subset K[x_0,\dots,x_n]$ is a set of non-constant homogeneous polynomials then $V_\mathrm{a}(S)$ is a cone: Clearly, we then have $0 \in V_\mathrm{a}(S)$. Moreover, let $\lambda \in K$ and $x \in V_\mathrm{a}(S)$. Then $f(x) = 0$ for all $f \in S$, hence $f(\lambda x) = \lambda^{\deg f} f(x) = 0$, and so $\lambda x \in V_\mathrm{a}(S)$ as well.

(b) Conversely, the ideal $I(X)$ of a cone $X \subset \mathbb{A}^{n+1}$ is homogeneous: Let $f \in I(X)$ with homogeneous decomposition $f = \sum_{d\in\mathbb{N}} f_d$. Then for all $x \in X$ we have $f(x) = 0$, and therefore also

$$0 = f(\lambda x) = \sum_{d\in\mathbb{N}} \lambda^d f_d(x)$$

for all $\lambda \in K$ since X is a cone. This means that we have the zero polynomial in λ, i. e. that $f_d(x) = 0$ for all d, and thus $f_d \in I(X)$. Hence $I(X)$ is homogeneous by Lemma 6.10 (a).

Lemma 6.17 (Cones $\leftrightarrow$ projective varieties)**.** *There is a bijection*

$$\begin{array}{rcl} \{\textit{cones in } \mathbb{A}^{n+1}\} & \stackrel{1:1}{\longleftrightarrow} & \{\textit{projective varieties in } \mathbb{P}^n\} \\ X & \longmapsto & \mathbb{P}(X) \\ C(X) & \longleftarrow & X. \end{array}$$

Proof. For a set $S \subset K[x_0,\dots,x_n]$ of non-constant homogeneous polynomials we have by construction

$$\mathbb{P}(V_\mathrm{a}(S)) = V_\mathrm{p}(S) \qquad \text{and} \qquad C(V_\mathrm{p}(S)) = V_\mathrm{a}(S).$$

But $V_\mathrm{a}(S)$ is really a cone by Remark 6.16 (a), every cone is of this form by Remark 6.16 (b) (namely for a set S of homogeneous generators of its homogeneous ideal), and every projective variety is of the form $V_\mathrm{p}(S)$. Hence we obtain the bijection as desired. □

In other words, the correspondence between cones and projective varieties works by passing from the affine to the projective zero locus (and vice versa) of the same set of homogeneous polynomials, as in Example 6.14. Note that in this way linear subspaces of $\mathbb{A}^{n+1}$ correspond exactly to linear
09 subspaces of $\mathbb{P}^n$ in the sense of Example 6.12 (b).

Of course, we would also expect a projective version of the Nullstellensatz as in Proposition 1.10, i. e. that $I_\mathrm{p}(V_\mathrm{p}(J)) = \sqrt{J}$ for any homogeneous ideal J in $K[x_0,\dots,x_n]$. This is *almost* true and can in fact be proved by reduction to the affine case — there is one exception however: as the origin in $\mathbb{A}^{n+1}$ does not correspond to a point in projective space $\mathbb{P}^n$, its ideal $\langle x_0,\dots,x_n\rangle$ has to be excluded from the correspondence between varieties and ideals.

Definition 6.18 (Irrelevant ideal)**.** The (radical homogeneous) ideal

$$I_0 := (x_0,\dots,x_n) \quad \trianglelefteq K[x_0,\dots,x_n]$$

is called the **irrelevant ideal**.

Proposition 6.19 (Projective Nullstellensatz).

(a) *For any projective variety $X \subset \mathbb{P}^n$ we have $V_p(I_p(X)) = X$.*

(b) *For any homogeneous ideal $J \trianglelefteq K[x_0,\dots,x_n]$ with $\sqrt{J} \neq I_0$ we have $I_p(V_p(J)) = \sqrt{J}$.*

In particular, there is an inclusion-reversing bijection

$$\begin{array}{rcl} \{\textit{projective varieties in } \mathbb{P}^n\} & \stackrel{1:1}{\longleftrightarrow} & \left\{ \begin{array}{c} \textit{homogeneous radical ideals in } K[x_0,\dots,x_n] \\ \textit{not equal to the irrelevant ideal} \end{array} \right\} \\ X & \longmapsto & I_p(X) \\ V_p(J) & \longleftarrow & J. \end{array}$$

Proof. The equality in (a), the inclusion "$\supset$" of (b), and the fact that the operations $V_p(\cdot)$ and $I_p(\cdot)$ reverse inclusions are easy and follow in exactly the same way as in the affine case in Proposition 1.10.

For the remaining inclusion "$\subset$" of (b) let J be a homogeneous ideal in $K[x_0,\dots,x_n]$ with $\sqrt{J} \neq I_0$. Then

$$\begin{aligned} I_p(V_p(J)) &= \langle f \in K[x_0,\dots,x_n] \text{ homogeneous} : f(x) = 0 \text{ for all } x \in V_p(J) \rangle \\ &= \langle f \in K[x_0,\dots,x_n] \text{ homogeneous} : f(x) = 0 \text{ for all } x \in V_a(J)\backslash\{0\} \rangle. \end{aligned}$$

As the affine zero locus of polynomials is closed, we can rewrite this as

$$I_p(V_p(J)) = \langle f \in K[x_0,\dots,x_n] \text{ homogeneous} : f(x) = 0 \text{ for all } x \in \overline{V_a(J)\backslash\{0\}} \rangle.$$

But now $V_a(J) \neq \{0\}$ as otherwise $\sqrt{J} = I_a(V_a(J)) = I_0$, which we excluded. So $V_a(J)$ is either empty or (by Remark 6.16 (a)) a cone containing at least one line through the origin. In both cases we obviously get $\overline{V_a(J)\backslash\{0\}} = V_a(J)$, so that

$$I_p(V_p(J)) = \langle f \in K[x_0,\dots,x_n] \text{ homogeneous} : f(x) = 0 \text{ for all } x \in V_a(J) \rangle.$$

As the ideal of the cone $V_a(J)$ is homogeneous by Remark 6.16 (b) this can be rewritten as $I_p(V_p(J)) = I_a(V_a(J))$, which is equal to $\sqrt{J}$ by the affine Nullstellensatz.

The additional bijection statement now follows from (a) and (b), together with the observation that $I_p(X)$ is always radical by (b), and never equal to I_0 as otherwise we would obtain the contradiction $I_0 = I_p(V_p(I_0)) = I_p(\emptyset) = K[x_0,\dots,x_n]$. □

Remark 6.20 (Properties of $V_p(\cdot)$ and $I_p(\cdot)$)**.** The operations $V_p(\cdot)$ and $I_p(\cdot)$ satisfy the same properties as their affine counterparts in Lemmas 1.4, 1.7, and 1.12. More precisely, in the same way as in the affine case we obtain:

(a) For any two subsets $S_1, S_2 \subset K[x_0,\dots,x_n]$ consisting of homogeneous polynomials we have $V_p(S_1) \cup V_p(S_2) = V_p(S_1 S_2)$; for any family (S_i) of subsets of $K[x_0,\dots,x_n]$ of homogeneous polynomials we have $\bigcap_i V_p(S_i) = V_p(\bigcup_i S_i)$.

(b) If $J_1, J_2 \trianglelefteq K[x_0,\dots,x_n]$ are homogeneous ideals then

$$V_p(J_1) \cup V_p(J_2) = V_p(J_1 J_2) = V_p(J_1 \cap J_2) \quad \text{and} \quad V_p(J_1) \cap V_p(J_2) = V_p(J_1 + J_2).$$

(c) For any two projective varieties X_1, X_2 in $\mathbb{P}^n$ we have $I_p(X_1 \cap X_2) = \sqrt{I_p(X_1) + I_p(X_2)}$ unless the latter is the irrelevant ideal (which is only possible if X_1 and X_2 are disjoint). Moreover, we have $I_p(X_1 \cup X_2) = I_p(X_1) \cap I_p(X_2)$.

Next, and also as in the affine case, let us associate a coordinate ring to a projective variety, and consider zero loci and ideals in a relative setting.

Construction 6.21 (Relative version of zero $V_p(\cdot)$ and $I_p(\cdot)$)**.** Let $Y \subset \mathbb{P}^n$ be a projective variety. In analogy to Definition 1.15 we call

$$S(Y) := K[x_0,\dots,x_n]/I(Y)$$

the **homogeneous coordinate ring** of Y. By Lemma 6.10 (c) it is a graded ring, so that it makes sense to talk about homogeneous elements of $S(Y)$. Moreover, the condition $f(x) = 0$ is still well-defined for a homogeneous element $f \in S(Y)$ and a point $x \in Y$, and thus we can define as in Definition 6.11

$$V(J) := \{x \in Y : f(x) = 0 \text{ for all homogeneous } f \in J\} \qquad \text{for a homogeneous ideal } J \trianglelefteq S(Y)$$

(and similarly for a set of homogeneous polynomials in $S(Y)$), and

$$I(X) := \langle f \in S(Y) \text{ homogeneous} : f(x) = 0 \text{ for all } x \in X \rangle \qquad \text{for a subset } X \subset Y.$$

As before, in case of possible confusion we will decorate V and I with the subscript Y and/or p to denote the relative and projective situation, respectively. Subsets of Y that are of the form $V_Y(J)$ for a homogeneous ideal $J \trianglelefteq S(Y)$ will be called **projective subvarieties** of Y; these are obviously exactly the projective varieties contained in Y.

As in the affine case, the Nullstellensatz and the properties of $V(\cdot)$ and $I(\cdot)$ can again be transferred to this relative setting in the obvious way.

Remark 6.22. A remark that is sometimes useful is that every projective subvariety X of a projective variety $Y \subset \mathbb{P}^n$ can be written as the zero locus of finitely many homogeneous polynomials in $S(Y)$ *of the same degree*. This follows easily from the fact that $V_p(f) = V_p(x_0^d f, \dots, x_n^d f)$ for all homogeneous $f \in S(Y)$ and every $d \in \mathbb{N}$. However, it is not true that every homogeneous ideal in $S(Y)$ can be generated by homogeneous elements of the same degree.

We can now proceed to define a topology on projective varieties. As in the affine setting, it follows by (the relative version of) Remark 6.20 (a) that arbitrary intersections and finite unions of subvarieties of a projective variety X are again subvarieties, and hence we can define the Zariski topology on X in the same way as in the affine case:

Definition 6.23 (Zariski topology)**.** The **Zariski topology** on a projective variety X is the topology whose closed sets are exactly the projective subvarieties of X, i. e. the subsets of the form $V_p(S)$ for some set $S \subset S(X)$ of homogeneous elements.

Of course, from now on we will always use this topology for projective varieties and their subsets. Note that, in the same way as in Remark 2.3, this is well-defined in the sense that the Zariski topology on a projective variety $X \subset \mathbb{P}^n$ agrees with the subspace topology of X in $\mathbb{P}^n$. Moreover, since we want to consider $\mathbb{A}^n$ as a subset of $\mathbb{P}^n$ as in Remark 6.3 we should also check that the Zariski topology on $\mathbb{A}^n$ is the same as the subspace topology of $\mathbb{A}^n$ in $\mathbb{P}^n$. To do this, we need the following definition.

Construction 6.24 (Homogenization and dehomogenization)**.**

(a) For a homogeneous polynomial $f \in K[x_0,\dots,x_n]$, the **dehomogenization** of f is defined to be the polynomial $f^{\mathrm{i}} := f(x_0 = 1) \in K[x_1,\dots,x_n]$ obtained from f by setting $x_0 = 1$. In general, it will be an inhomogeneous polynomial (hence the notation f^{i}). Note that evaluation at $x_0 = 1$ is a ring homomorphism, i. e. we have

$$(fg)^{\mathrm{i}} = f^{\mathrm{i}} g^{\mathrm{i}} \quad \text{and} \quad (f+g)^{\mathrm{i}} = f^{\mathrm{i}} + g^{\mathrm{i}}$$

for all $f, g \in K[x_0,\dots,x_n]$. As it is surjective, we can also apply this construction directly to ideals: For a homogeneous ideal $J \trianglelefteq K[x_0,\dots,x_n]$, the *dehomogenization* $J^{\mathrm{i}} := \{f^{\mathrm{i}} : f \in J\}$ is again an ideal in $K[x_1,\dots,x_n]$.

(b) For the opposite direction, let

$$f = \sum_{i_1,\dots,i_n \in \mathbb{N}} a_{i_1,\dots,i_n} x_1^{i_1} \cdot \cdots \cdot x_n^{i_n} \quad \in K[x_1,\dots,x_n]$$

be a (non-zero) polynomial of degree d. We define its **homogenization** to be

$$\begin{aligned} f^{\mathrm{h}} &:= x_0^d \, f\left(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right) \\ &= \sum_{i_1,\dots,i_n\in\mathbb{N}} a_{i_1,\dots,i_n}\, x_0^{d-i_1-\cdots-i_n}\, x_1^{i_1}\cdot\,\cdots\,\cdot x_n^{i_n} \quad \in K[x_0,\dots,x_n]; \end{aligned}$$

obviously this is a homogeneous polynomial of degree d. For all $f,g \in K[x_1,\dots,x_n]$ of degrees d and e, respectively, we have

$$(fg)^{\mathrm{h}} = x_0^{d+e}\, f\left(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right)\cdot g\left(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right) = f^{\mathrm{h}}\cdot g^{\mathrm{h}},$$

but in contrast to (a) the polynomial $(f+g)^{\mathrm{h}}$ is clearly not equal to $f^{\mathrm{h}}+g^{\mathrm{h}}$ in general — in fact, $f^{\mathrm{h}}+g^{\mathrm{h}}$ is usually not even homogeneous. So in order to apply this construction to an ideal $J \trianglelefteq K[x_1,\dots,x_n]$, we have to define the ideal $J^{\mathrm{h}} \trianglelefteq K[x_0,\dots,x_n]$ to be the ideal generated by the homogenizations f^{h} of all non-zero $f\in J$.

Example 6.25. For $f = x_1^2 - x_2^2 - 1 \in K[x_1,x_2]$ we have $f^{\mathrm{h}} = x_1^2 - x_2^2 - x_0^2 \in K[x_0,x_1,x_2]$, and then back $(f^{\mathrm{h}})^{\mathrm{i}} = x_1^2 - x_2^2 - 1 = f$.

Remark 6.26 ($\mathbb{A}^n$ as an open subset of $\mathbb{P}^n$)**.** Recall from Remark 6.3 that we want to identify the subset $U_0 = \{(x_0:\cdots:x_n)\in\mathbb{P}^n : x_0\neq 0\}$ of $\mathbb{P}^n$ with $\mathbb{A}^n$ by the bijective map

$$F\colon\ \mathbb{A}^n \to U_0,\ (x_1,\dots,x_n)\mapsto (1:x_1:\cdots:x_n).$$

Obviously, U_0 is an open subset of $\mathbb{P}^n$. Moreover, with the above identification the subspace topology of $U_0 = \mathbb{A}^n \subset \mathbb{P}^n$ is the affine Zariski topology:

(a) If $X = V_{\mathrm{p}}(J)\cap\mathbb{A}^n$ is a closed set in the subspace topology (with $J \trianglelefteq K[x_0,\dots,x_n]$ a homogeneous ideal) then $X = V_{\mathrm{a}}(J^{\mathrm{i}})$ is also Zariski-closed.

(b) If $X = V_{\mathrm{a}}(J)\subset\mathbb{A}^n$ is Zariski-closed (with $J\trianglelefteq K[x_1,\dots,x_n]$) then $X = V_{\mathrm{p}}(J^{\mathrm{h}})\cap\mathbb{A}^n$ is closed in the subspace topology as well.

In other words we can say that the map $F\colon\ \mathbb{A}^n\to U_0$ above is a homeomorphism. In fact, after having given $\mathbb{P}^n$ the structure of a variety we will see in Proposition 7.2 that it is even an isomorphism of varieties.

Having defined the Zariski topology on projective varieties (or more generally on subsets of $\mathbb{P}^n$) we can now immediately apply all topological concepts of Chapter 2 to this new situation. In particular, the notions of connectedness, irreducibility, and dimension are well-defined for projective varieties (and have the same geometric interpretation as in the affine case). Let us study some examples using these concepts.

Remark 6.27 ($\mathbb{P}^n$ is irreducible of dimension n)**.** Of course, by symmetry of the coordinates, it follows from Remark 6.26 that all subsets $U_i = \{(x_0:\cdots:x_n) : x_i\neq 0\}$ of $\mathbb{P}^n$ for $i=0,\dots,n$ are homeomorphic to $\mathbb{A}^n$ as well. As these subsets cover $\mathbb{P}^n$ and have non-empty intersections, we conclude by Exercise 2.21 (b) that $\mathbb{P}^n$ is irreducible, and by Exercise 2.34 (a) that $\dim\mathbb{P}^n = n$.

Exercise 6.28. Let $L_1,L_2\subset\mathbb{P}^3$ be two disjoint lines (i. e. 1-dimensional linear subspaces in the sense of Example 6.12 (b)), and let $a\in\mathbb{P}^3\backslash(L_1\cup L_2)$. Show that there is a unique line $L\subset\mathbb{P}^3$ through a that intersects both L_1 and L_2.

Is the corresponding statement for lines and points in $\mathbb{A}^3$ true as well?

Exercise 6.29.

(a) Prove that a graded ring R is an integral domain if and only if for all *homogeneous* elements $f,g\in R$ with $fg=0$ we have $f=0$ or $g=0$.

(b) Show that a projective variety X is irreducible if and only if its homogeneous coordinate ring $S(X)$ is an integral domain.

Exercise 6.30. In this exercise we want to show that an intersection of projective varieties is never empty unless one would expect it to be empty for dimensional reasons — so e. g. the phenomenon of parallel non-intersecting lines in the plane does not occur in projective space (which we have seen already in Remark 6.3).

So let $X,Y \subset \mathbb{P}^n$ be non-empty projective varieties. Show:

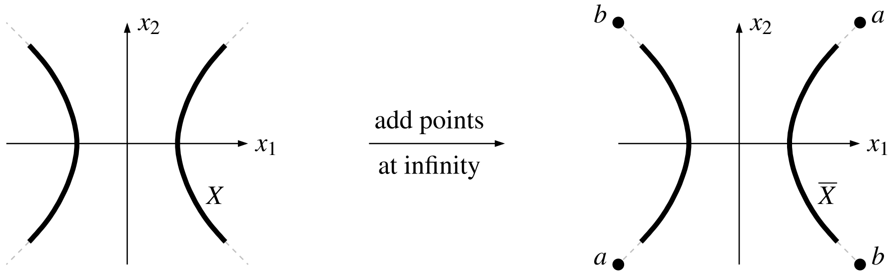
(a) The dimension of the cone $C(X) \subset \mathbb{A}^{n+1}$ is $\dim X + 1$.

(b) If $\dim X + \dim Y \geq n$ then $X \cap Y \neq \emptyset$.

We have just seen in Remark 6.26 (b) that for an affine variety $X = V(J) \subset \mathbb{A}^n$ the homogenization J^{h} gives an ideal such that the closed set $V_{\mathrm{p}}(J^{\mathrm{h}}) \subset \mathbb{P}^n$ restricts to X on $\mathbb{A}^n \subset \mathbb{P}^n$. In fact, we will now show that $V_{\mathrm{p}}(J^{\mathrm{h}})$ is even the *smallest* closed set in $\mathbb{P}^n$ containing X, i. e. the closure $\overline{X}$ of X in $\mathbb{P}^n$. As this will be a "compact" space in the sense of Remarks 6.3 and 6.4 we can think of this closure $\overline{X}$ as being obtained by compactifying X by some "points at infinity". For example, if we start with the affine hyperbola $X = V_{\mathrm{a}}(x_1^2 - x_2^2 - 1) \subset \mathbb{A}^2$ in the picture below on the left, its closure

$$\overline{X} = V_{\mathrm{p}}\left((x_1^2 - x_2^2 - 1)^{\mathrm{h}}\right) = V_{\mathrm{p}}(x_1^2 - x_2^2 - x_0^2) \subset \mathbb{P}^2$$

adds the two points a and b at infinity as in Example 6.14.



Proposition 6.31 (Computation of the projective closure)**.** *Let $J \trianglelefteq K[x_1,\dots,x_n]$ be an ideal. Consider its affine zero locus $X = V_{\mathrm{a}}(J) \subset \mathbb{A}^n$, and its closure $\overline{X}$ in $\mathbb{P}^n$.*

(a) *We have $\overline{X} = V_{\mathrm{p}}(J^{\mathrm{h}})$.*

(b) *If $J = \langle f \rangle$ is a non-zero principal ideal then $\overline{X} = V_{\mathrm{p}}(f^{\mathrm{h}})$.*

Proof.

(a) Clearly, the set $V_{\mathrm{p}}(J^{\mathrm{h}})$ is closed and contains X. In order to show that $V_{\mathrm{p}}(J^{\mathrm{h}})$ is the smallest closed set containing X let $Y \supset X$ be any closed set; we have to prove that $Y \supset V_{\mathrm{p}}(J^{\mathrm{h}})$. As Y is closed we have $Y = V_{\mathrm{p}}(J')$ for some homogeneous ideal J'. Now any homogeneous element of J' can be written as $x_0^d f^{\mathrm{h}}$ for some $d \in \mathbb{N}$ and $f \in K[x_1,\dots,x_n]$, and for this element we have

$$
\begin{aligned}
& x_0^d f^{\mathrm{h}} \text{ is zero on } X \subset \mathbb{P}^n && (X \text{ is a subset of } Y)\\
\Rightarrow\ & f \text{ is zero on } X \subset \mathbb{A}^n && (x_0 \neq 0 \text{ on } X \subset \mathbb{A}^n)\\
\Rightarrow\ & f \in I_{\mathrm{a}}(X) = I_{\mathrm{a}}(V_{\mathrm{a}}(J)) = \sqrt{J} && (\text{Proposition } 1.10)\\
\Rightarrow\ & f^m \in J \text{ for some } m \in \mathbb{N} &&\\
\Rightarrow\ & (f^{\mathrm{h}})^m = (f^m)^{\mathrm{h}} \in J^{\mathrm{h}} \text{ for some } m \in \mathbb{N} && (\text{Construction } 6.24 \text{ (b)})\\
\Rightarrow\ & f^{\mathrm{h}} \in \sqrt{J^{\mathrm{h}}} &&\\
\Rightarrow\ & x_0^d f^{\mathrm{h}} \in \sqrt{J^{\mathrm{h}}}. &&
\end{aligned}
$$

We therefore conclude that $J' \subset \sqrt{J^{\mathrm{h}}}$, and so $Y = V_{\mathrm{p}}(J') \supset V_{\mathrm{p}}(\sqrt{J^{\mathrm{h}}}) = V_{\mathrm{p}}(J^{\mathrm{h}})$ as desired.

(b) As $\langle f \rangle = \{fg : g \in K[x_1,\dots,x_n]\}$, we have

$$\overline{X} = V_p((fg)^h : g \in K[x_1,\dots,x_n]) = V_p(f^h g^h : g \in K[x_1,\dots,x_n]) = V_p(f^h)$$

by (a) and Construction 6.24 (b). □

Remark 6.32 (Ideal of hypersurfaces in $\mathbb{P}^n$)**.** Let X be a hypersurface in $\mathbb{P}^n$, and assume without loss of generality that it does not contain the set of points at infinity $V_p(x_0)$ as a component. Then $Y := X \cap \mathbb{A}^n$ is an affine hypersurface whose closure is again X. By Remark 2.38 we know that its ideal $I(Y)$ is principal, generated by a polynomial $g \in K[x_1,\dots,x_n]$.

If we now set $f = g^h \in K[x_0,\dots,x_n]$ then $V_p(f) = \overline{Y} = X$ by Proposition 6.31 (b). Moreover, as g has no repeated factors the same is true for f, and hence we even have $I(X) = \langle f \rangle$. In other words, just as in the affine case the ideal of any projective hypersurface is principal, and thus we can transfer our definition of degree to the projective case:

Definition 6.33 (Degree of a projective hypersurface)**.** Let X be a hypersurface in $\mathbb{P}^n$, with ideal $I(X) = \langle f \rangle$ as in Remark 6.32. As in the affine case in Definition 2.39, the degree of f is then also called the **degree** of X, again denoted $\deg X$. We also use the terms **linear**, **quadric**, or **cubic** for projective hypersurfaces of degrees 1, 2, or 3, respectively.

Example 6.34. In contrast to Proposition 6.31 (b), for general ideals it usually does not suffice to only homogenize a set of generators. As an example, consider the ideal $J = \langle x_1, x_2 - x_1^2 \rangle \trianglelefteq K[x_1,x_2]$ with affine zero locus $X = V_a(J) = \{0\} \subset \mathbb{A}^2$. This one-point set is also closed in $\mathbb{P}^2$, and thus $\overline{X} = \{(1:0:0)\}$ is just the corresponding point in homogeneous coordinates. But if we homogenize the two given generators of J we obtain the homogeneous ideal $\langle x_1, x_0x_2 - x_1^2 \rangle$ with projective zero locus $\{(1:0:0),(0:0:1)\} \supsetneq \overline{X}$.

For those of you who know some computer algebra: One can show however that it suffices to homogenize a *Gröbner basis* of J. This makes the problem of finding $\overline{X}$ computationally feasible since in contrast to Proposition 6.31 (a) we only have to homogenize finitely many polynomials.

Exercise 6.35. Sketch the set of real points of the complex affine curve $X = V(x_1^3 - x_1x_2^2 + 1) \subset \mathbb{A}^2_{\mathbb{C}}$ and compute the points at infinity of its projective closure $\overline{X} \subset \mathbb{P}^2_{\mathbb{C}}$. 10