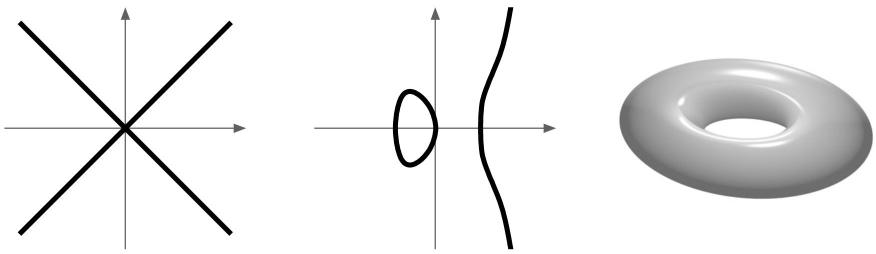
0. Introduction

The main goal of algebraic geometry is to study solution sets of polynomial equations in several variables. So, in its easiest form, if $f_1,\dots,f_k \in K[x_1,\dots,x_n]$ are polynomials in n variables over a given ground field K we want to consider the set

$$X = \{x \in K^n : f_1(x) = \cdots = f_k(x) = 0\}$$

which is called an *(affine) variety*. Usually, we will ask *geometric* questions about such a variety X and try to answer them by an *algebraic* study of the polynomials $f_1,\dots,f_k$ in the polynomial ring. Hence the following two previously taught classes are of particular importance and will be prerequisites for our work:

- In the "Plane Algebraic Curves" class [G2] we have considered the case $n = 2$ and $k = 1$ in detail, i. e. zero loci of a single polynomial in two variables, which we can then think of as a curve in the plane. As illustrated by the examples in the picture below, we have studied many geometric properties of such curves: The real curve in (a) has a singular point at the origin [G2, Definition 2.22] and consists of two parts that are algebraic curves themselves [G2, Definition 1.5 (c)], the real curve in (b) has two connected components in the usual topology of the plane [G2, Proposition 5.10], and the complex curve in (c) (which is then a real surface) is topologically a torus [G2, Proposition 5.16]. We have also determined how many intersection points two curves can have [G2, Chapter 4], and how many rational functions with prescribed zeros and poles we can find on a curve [G2, Chapter 8]. It is one of the main goals of this class to consider questions of similar type in the higher-dimensional case.



(a) (b) (c)

- In the "Commutative Algebra" class [G3] we would study the given polynomials algebraically, i. e. consider the ideal $I = \langle f_1,\dots,f_k \rangle$ in the polynomial ring $K[x_1,\dots,x_n]$, or its quotient ring $R = K[x_1,\dots,x_n]/I$, which is called the coordinate ring of the variety X and can be thought of as the ring of polynomial functions on X. We can then ask algebraic questions about this: For example, is R an integral domain, or a unique factorization domain [G3, Definition 8.1]? What is the Krull dimension of R [G3, Definition 11.1], or the primary decomposition of I [G3, Definition 8.15]?

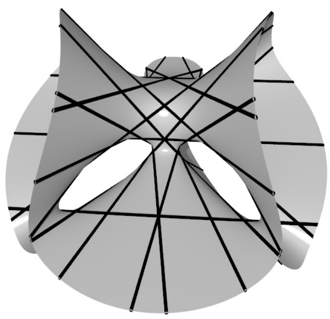
In these notes, we will set up a category that can be interpreted geometrically as well as algebraically, so that we can combine geometric and algebraic methods in our study. For example, in the above setting we can either think geometrically of the variety X or algebraically of the coordinate ring R — they both carry exactly the same information, and (geometric) morphisms of varieties will precisely correspond to (algebraic) K-algebra homomorphisms between their coordinate rings.

Many of the geometric questions studied for plane curves in [G2] were related to their *topology*, asking in some sense for the "shape" of the varieties. Such topological questions are still very interesting when we now extend our objects of study to arbitrary dimensions, but there are also entirely new types of questions that we could ask. Here are two examples:

Example 0.1 (Lines on cubic surfaces)**.** Consider the real cubic surface

$$X = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : 1+x_1^3+x_2^3+x_3^3-(1+x_1+x_2+x_3)^3 = 0\} \quad \subset \mathbb{R}^3.$$

It is shown in the picture on the right, where we have used a linear projection to map the real 3-dimensional space onto the drawing plane (and not just a topologically correct picture). We can therefore see from the picture that there are some straight lines contained in X. In fact, we will show in Chapter 11 that (after a suitable compactification) every smooth cubic surface has exactly 27 lines on it. This is another sort of question that one can ask about varieties: Do they contain curves with some prescribed properties (in this case lines), and if so, how many? This branch of algebraic geometry is usually called *enumerative geometry*.



Example 0.2 (Curves in 3-space)**.** Even curves can give rise to new phenomena if they are not contained in the plane. Consider e. g. the space curve

$$X = \{(x_1,x_2,x_3) = (t^3,t^4,t^5) : t \in \mathbb{C}\} \quad \subset \mathbb{C}^3.$$

We have given it parametrically here, but it is in fact easy to see that we can describe it equally well in terms of polynomial equations as

$$X = \{(x_1,x_2,x_3) \in \mathbb{C}^3 : x_1^3 = x_2x_3,\ x_2^2 = x_1x_3,\ x_3^2 = x_1^2x_2\}.$$

What is striking here is that we have three equations, although we would expect that a 1-dimensional object in 3-dimensional space should be given by only two equations. But if we leave out any of the three equations above, we would change the set that it describes: Leaving out e. g. the last equation $x_3^2 = x_1^2x_2$ would yield the whole x_3-axis $\{(x_1,x_2,x_3) : x_1 = x_2 = 0\}$ as additional points that do satisfy the first two equations but not the last one. Similarly, leaving out the first or second equation would again add a coordinate line to X.

So, in contrast to the linear algebra situation, it is not clear whether a given object of codimension d can be given by d equations. Even worse, for a given set of equations it is in general a difficult task to figure out what dimension their solution has. There do exist algorithms to find this out for any given set of polynomials — similarly to the Gaussian algorithm for linear equations — but they are so complicated that one would usually want to use a computer program to perform the calculations. This is a simple example of an application of *computer algebra* to algebraic geometry.

However, if we replace the three equations above (over the complex numbers) by

$$x_1^3 = x_2x_3,\ x_2^2 = x_1x_3,\ x_3^2 = x_1^2x_2 + \varepsilon$$

for a (small) non-zero $\varepsilon \in \mathbb{C}$, the resulting set of solutions would actually become 0-dimensional, as expected from three equations in 3-dimensional space. So we see that very small changes in the equations can make a very big difference in the resulting solution set. This means that we usually cannot apply numerical methods to our problems.

Remark 0.3 (Algebraic geometry over different ground fields)**.** Depending on the chosen ground field, algebraic geometry also has relations to the following fields of mathematics:

(a) Over the ground field $\mathbb{R}$ or $\mathbb{C}$ we can use *real* resp. *complex analysis* to study varieties, as we occasionally did already for plane curves e. g. in [G2, Chapter 7 or Remark 8.5]. In fact, many results in algebraic geometry can also be proven using analytic methods.

(b) When using (extensions of) finite fields or the rational numbers as the ground field one enters the area of *number theory*. For example, the famous Fermat's Last Theorem is just asking a seemingly simple question about a variety $\{(x_1,x_2,x_3) \in \mathbb{Q}^3 : x_1^n + x_2^n = x_3^n\}$ over the rationals, namely for which $n \in \mathbb{N}$ it contains any non-trivial points at all. As you probably know, the proof of the fact that there are no such non-trivial points for $n \geq 3$ is very complicated — but even if this type of question is very different from the questions that we would usually ask for real or complex varieties, the proof uses many concepts and techniques from the theory of algebraic geometry.

With this many relations to other fields of mathematics, it is obvious that we have to restrict our attention in this class to a rather small subset of the possible questions and applications. As already mentioned, our focus will mainly be on algebraic methods and geometric questions. In this way, we can keep the prerequisites for this class to a minimum, requiring except for the "Plane Algebraic Curves" and "Commutative Algebra" classes only a very basic topological knowledge up to the definitions and first properties of topological spaces, open and closed sets, neighborhoods, and continuous maps.