4. Morphisms

So far we have defined and studied regular functions on an affine variety X. They can be thought of as the morphisms (i. e. the "nice" maps) from open subsets of X to the ground field $K = \mathbb{A}^1$. We now want to extend this notion of morphisms to maps to other affine varieties than just $\mathbb{A}^1$ (and in fact also to maps between more general varieties in Chapter 5). It turns out that there is a very natural way to define these morphisms once you know what the regular functions are on the source and target variety. So let us start by attaching the data of the regular functions to the structure of an affine variety, or rather more generally of a topological space.

Definition 4.1 (Ringed spaces)**.**

(a) A **ringed space** is a topological space X together with a sheaf of rings on X. In this situation the given sheaf will always be denoted $\mathscr{O}_X$ and called the **structure sheaf** of the ringed space. Usually we will write this ringed space simply as X, with the structure sheaf $\mathscr{O}_X$ being understood.

(b) An affine variety will always be considered as a ringed space together with its sheaf of regular functions as the structure sheaf.

(c) An open subset U of a ringed space X (e. g. of an affine variety) will always be considered as a ringed space with the structure sheaf being the restriction $\mathscr{O}_X|_U$ as in Definition 3.16.

With this idea that the regular functions make up the structure of an affine variety the obvious idea to define a morphism $f\colon X \to Y$ between affine varieties (or more generally ringed spaces) is now that they should preserve this structure in the sense that for any regular function $\varphi\colon U \to K$ on an open subset U of Y the composition $\varphi \circ f\colon f^{-1}(U) \to K$ is again a regular function.

However, there is a slight technical problem with this approach. Whereas there is no doubt about what the composition $\varphi \circ f$ above should mean for a regular function φ on an affine variety, this notion is a priori undefined for general ringed spaces: Recall that in this case by Definition 3.13 the structure sheaves $\mathscr{O}_X$ and $\mathscr{O}_Y$ are given by the data of arbitrary rings $\mathscr{O}_X(U)$ and $\mathscr{O}_Y(V)$ for open subsets $U \subset X$ and $V \subset Y$. So although we usually think of the elements of $\mathscr{O}_X(U)$ and $\mathscr{O}_Y(V)$ as functions on U resp. V there is nothing in the definition that guarantees us such an interpretation, and consequently there is no well-defined notion of composing these sections of the structure sheaves with the map $f\colon X \to Y$. So in order to be able to proceed without too many technicalities let us assume for the moment that all our sheaves are in fact sheaves of functions with some properties:

Convention 4.2 (Sheaves of rings = sheaves of K-valued functions)**.** From now on until we discuss schemes in Chapter 12, for every sheaf of rings $\mathscr{F}$ on a topological space X we will assume that $\mathscr{F}(U)$ for any open subset $U \subset X$ is a subring of the ring of all functions from U to K (with the usual pointwise addition and multiplication) containing all constant functions, and that the restriction maps are the ordinary restrictions of such functions. In particular, this makes every such sheaf also into a sheaf of K-algebras, with the scalar multiplication by elements of K again given by pointwise multiplication. So in short we can say:

Every sheaf of rings is assumed to be a sheaf of K-valued functions.

With this convention we can now go ahead and define morphisms between ringed spaces as motivated above.

Definition 4.3 (Morphisms of ringed spaces)**.** Let $f\colon X \to Y$ be a map of ringed spaces.

(a) For any map $\varphi\colon U \to K$ from an open subset U of Y to the ground field K we denote the composition $\varphi \circ f\colon f^{-1}(U) \to K$ (which is well-defined by Convention 4.2) by $f^*\varphi$. It is called the **pull-back** of φ by f.

(b) The map f is called a **morphism** (of ringed spaces) if it is continuous, and if for all open subsets $U \subset Y$ and $\varphi \in \mathscr{O}_Y(U)$ we have $f^*\varphi \in \mathscr{O}_X(f^{-1}(U))$. So in this case pulling back by f yields K-algebra homomorphisms

$$f^*\colon \mathscr{O}_Y(U) \to \mathscr{O}_X(f^{-1}(U)),\ \ \varphi \mapsto f^*\varphi.$$

(c) We say that f is an **isomorphism** (of ringed spaces) if it has a two-sided inverse, i. e. if it is bijective, and both $f\colon X \to Y$ and $f^{-1}\colon Y \to X$ are morphisms.

Morphisms and isomorphisms of (open subsets of) affine varieties are morphisms (resp. isomorphisms) as ringed spaces.

Remark 4.4.

(a) The requirement of f being continuous is necessary in Definition 4.3 (b) to formulate the second condition: It ensures that $f^{-1}(U)$ is open in X if U is open in Y, i. e. that $\mathscr{O}_X(f^{-1}(U))$ is well-defined.

(b) Without our Convention 4.2, i. e. for ringed spaces without a natural notion of a pull-back of elements of $\mathscr{O}_Y(U)$, one would actually have to include suitable ring homomorphisms $\mathscr{O}_Y(U) \to \mathscr{O}_X(f^{-1}(U))$ in the data needed to specify a morphism. In other words, in this case a morphism would no longer be just a set-theoretic map satisfying certain properties. This happens for schemes that we will discuss in Chapter 12 (see Definition 12.25), and it would indeed be the "correct" general notion of morphisms of arbitrary ringed spaces. For the moment however, we will not do this here as it would clearly make our current discussion of morphisms more complicated than necessary.

Remark 4.5 (Properties of morphisms)**.** The following two properties of morphisms are obvious from the definition:

(a) Compositions of morphisms are morphisms: If $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms of ringed spaces then so is $g \circ f\colon X \to Z$.

(b) Restrictions of morphisms are morphisms: If $f\colon X \to Y$ is a morphism of ringed spaces and $U \subset X$ and $V \subset Y$ are open subsets such that $f(U) \subset V$ then the restricted map $f|_U\colon U \to V$ is again a morphism of ringed spaces.

Conversely, morphisms satisfy a "gluing property" similar to that of a sheaf in Definition 3.13:

Lemma 4.6 (Gluing property for morphisms)**.** *Let $f\colon X \to Y$ be a map of ringed spaces. Assume that there is an open cover $\{U_i : i \in I\}$ of X such that all restrictions $f|_{U_i}\colon U_i \to Y$ are morphisms. Then f is a morphism.*

Proof. By Definition 4.3 (b) we have to check two things:

(a) The map f is continuous: Let $V \subset Y$ be an open subset. Then

$$f^{-1}(V) = \bigcup_{i\in I}(U_i \cap f^{-1}(V)) = \bigcup_{i\in I}(f|_{U_i})^{-1}(V).$$

But as all restrictions $f|_{U_i}$ are continuous the sets $(f|_{U_i})^{-1}(V)$ are open in U_i, and hence open in X. So $f^{-1}(V)$ is open in X, which means that f is continuous.

Of course, this is just the well-known topological statement that continuity is a local property.

(b) The map f pulls back sections of $\mathscr{O}_Y$ to sections of $\mathscr{O}_X$: Let $V \subset Y$ be an open subset and $\varphi \in \mathscr{O}_Y(V)$. Then $(f^*\varphi)|_{U_i\cap f^{-1}(V)} = (f|_{U_i\cap f^{-1}(V)})^*\varphi \in \mathscr{O}_X(U_i \cap f^{-1}(V))$ since $f|_{U_i}$ (and thus also $f|_{U_i\cap f^{-1}(V)}$ by Remark 4.5 (b)) is a morphism. By the gluing property for sheaves in Definition 3.13 this means that $f^*\varphi \in \mathscr{O}_X(f^{-1}(V))$. □

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Let us now apply our definition of morphisms to (open subsets of) affine varieties. The following proposition can be viewed as a confirmation that our constructions above were reasonable: As one would certainly expect, a morphism to an affine variety $Y \subset \mathbb{A}^n$ is simply given by an n-tuple of regular functions whose image lies in Y.

Proposition 4.7 (Morphisms between affine varieties)**.** *Let U be an open subset of an affine variety X, and let $Y \subset \mathbb{A}^n$ be another affine variety. Then the morphisms $f\colon U \to Y$ are exactly the maps of the form*

$$f = (\varphi_1, \dots, \varphi_n)\colon U \to Y,\ x \mapsto (\varphi_1(x), \dots, \varphi_n(x))$$

with $\varphi_i \in \mathscr{O}_X(U)$ for all $i = 1, \dots, n$.

In particular, the morphisms from U to $\mathbb{A}^1$ are exactly the regular functions in $\mathscr{O}_X(U)$.

Proof. First assume that $f\colon U \to Y$ is a morphism. For $i = 1, \dots, n$ the i-th coordinate function y_i on $Y \subset \mathbb{A}^n$ is clearly regular on Y, and so $\varphi_i := f^* y_i \in \mathscr{O}_X(f^{-1}(Y)) = \mathscr{O}_X(U)$ by Definition 4.3 (b). But this is just the i-th component function of f, and so we have $f = (\varphi_1, \dots, \varphi_n)$.

Conversely, let now $f = (\varphi_1, \dots, \varphi_n)$ with $\varphi_1, \dots, \varphi_n \in \mathscr{O}_X(U)$ and $f(U) \subset Y$. First of all f is continuous: Let Z be any closed subset of Y. Then Z is of the form $V(g_1, \dots, g_m)$ for some $g_1, \dots, g_m \in A(Y)$, and

$$f^{-1}(Z) = \{x \in U : g_i(\varphi_1(x), \dots, \varphi_n(x)) = 0 \text{ for all } i = 1, \dots, m\}.$$

But the functions $x \mapsto g_i(\varphi_1(x), \dots, \varphi_n(x))$ are regular on U since locally plugging in quotients of polynomial functions for the variables of a polynomial gives again locally a quotient of polynomial functions. Hence $f^{-1}(Z)$ is closed in U by Lemma 3.4, and thus f is continuous. Similarly, if $\varphi \in \mathscr{O}_Y(W)$ is a regular function on some open subset $W \subset Y$ then

$$f^*\varphi = \varphi \circ f\colon f^{-1}(W) \to K,\ x \mapsto \varphi(\varphi_1(x), \dots, \varphi_n(x))$$

is regular again, since if we replace the variables in a quotient of polynomial functions by other quotients of polynomial functions we obtain again a quotient of polynomial functions. Hence f is a morphism. □

For affine varieties themselves (rather than their open subsets) we obtain as a consequence the following useful corollary that translates our geometric notion of morphisms entirely into the language of commutative algebra.

Corollary 4.8. *For any two affine varieties X and Y there is a bijection*

$$\begin{aligned} \{\text{morphisms } X \to Y\} \quad &\stackrel{1:1}{\longleftrightarrow} \quad \{K\text{-algebra homomorphisms } A(Y) \to A(X)\} \\ f \quad &\longmapsto \quad f^*. \end{aligned}$$

In particular, isomorphisms of affine varieties correspond exactly to K-algebra isomorphisms in this way.

Proof. By Definition 4.3 it is clear that any morphism $f\colon X \to Y$ determines a K-algebra homomorphism $f^*\colon \mathscr{O}_Y(Y) \to \mathscr{O}_X(X)$, i. e. $f^*\colon A(Y) \to A(X)$ by Proposition 3.8.

Conversely, let $g\colon A(Y) \to A(X)$ be a K-algebra homomorphism. Assume that $Y \subset \mathbb{A}^n$ and denote by $y_1, \dots, y_n$ the coordinate functions of $\mathbb{A}^n$. Then $\varphi_i := g(y_i) \in A(X) = \mathscr{O}_X(X)$ for all $i = 1, \dots, n$. If we set $f = (\varphi_1, \dots, \varphi_n)\colon X \to \mathbb{A}^n$ then we obtain for any $h \in K[y_1, \dots, y_n]$

$$(f^*h)(x) = h(f(x)) = h(\varphi_1(x), \dots, \varphi_n(x)) \stackrel{(*)}{=} g(h)(x) \quad \text{for all } x \in X,$$

where $(*)$ holds since both sides of the equation are K-algebra homomorphisms in h and equal to $\varphi_i(x)$ on the generators y_i for $i = 1, \dots, n$ of $K[y_1, \dots, y_n]$.

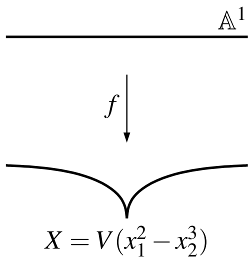
First of all this shows that $h(f(x)) = 0$ for all $h \in I(Y)$, since these polynomials are zero in $A(Y)$, so that g vanishes on them. Hence the image of f lies in $V(I(Y)) = Y$, i. e. we have constructed a map $f\colon X \to Y$. As its coordinate functions are regular, it is indeed a morphism by Proposition 4.7, and moreover the above relation shows that $f^* = g$ so that we get the bijection as stated in the corollary.

The additional statement about isomorphisms now follows immediately since $(f\circ g)^* = g^*\circ f^*$ and $(g\circ f)^* = f^*\circ g^*$ for all $f\colon X\to Y$ and $g\colon Y\to X$. □

Example 4.9 (Isomorphisms $\neq$ bijective morphisms)**.** Let $X = V(x_1^2 - x_2^3) \subset \mathbb{A}^2$ be the cubic curve as in the picture below on the right. It has a singular point at the origin in the sense of [G2, Definition 2.22] (or Definition 10.7 (a)).

Now consider the map

$$f\colon \mathbb{A}^1 \to X,\ t\mapsto (t^3,t^2)$$



which is a morphism by Proposition 4.7. Its corresponding K-algebra homomorphism $f^*\colon A(X)\to A(\mathbb{A}^1)$ as in Corollary 4.8 is given by

$$\begin{aligned} K[x_1,x_2]/(x_1^2-x_2^3) &\to K[t] \\ \overline{x_1} &\mapsto t^3 \\ \overline{x_2} &\mapsto t^2 \end{aligned}$$

which can be seen by composing f with the two coordinate functions of $\mathbb{A}^2$.

Note that f is bijective with inverse map

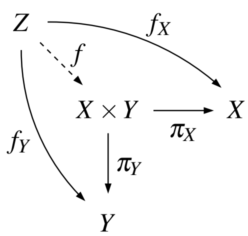
$$f^{-1}\colon X\to\mathbb{A}^1,\ (x_1,x_2)\mapsto \begin{cases} \frac{x_1}{x_2} & \text{if } x_2\neq 0 \\ 0 & \text{if } x_2 = 0. \end{cases}$$

But f is not an isomorphism (i. e. f^{-1} is not a morphism), since otherwise by Corollary 4.8 the map f^* above would have to be an isomorphism as well — which is false since the linear polynomial t is clearly not in its image. So we have to be careful not to confuse isomorphisms with bijective morphisms.

Another consequence of Proposition 4.7 concerns our definition of the product $X\times Y$ of two affine varieties X and Y in Example 1.5 (d). Recall from Example 2.4 (c) that $X\times Y$ does not carry the product topology — which might seem strange at first. The following proposition however justifies this choice, since it shows that our definition of the product satisfies the so-called *universal property* that giving a morphism to $X\times Y$ is the same as giving a morphism each to X and Y. In fact, when we introduce more general varieties in the next chapter we will *define* their products using this certainly desirable universal property.

Proposition 4.10 (Universal property of products)**.** *Let X and Y be affine varieties, and let $\pi_X\colon X\times Y\to X$ and $\pi_Y\colon X\times Y\to Y$ be the projection morphisms from the product onto the two factors. Then for every affine variety Z and two morphisms $f_X\colon Z\to X$ and $f_Y\colon Z\to Y$ there is a unique morphism $f\colon Z\to X\times Y$ such that $f_X = \pi_X\circ f$ and $f_Y = \pi_Y\circ f$.*

In other words, giving a morphism from an affine variety to the product $X\times Y$ is the same as giving a morphism to each of the factors X and Y.



Proof. Obviously, the only way to obtain the relations $f_X = \pi_X\circ f$ and $f_Y = \pi_Y\circ f$ is to take the map $f\colon Z\to X\times Y,\ z\mapsto (f_X(z), f_Y(z))$. But this is clearly a morphism by Proposition 4.7: As f_X and f_Y must be given by regular functions in each coordinate, the same is then true for f. □

Remark 4.11 (Coordinate ring of a product)**.** The universal property of the product in Proposition 4.10 corresponds exactly to the universal property of tensor products using the translation between morphisms of affine varieties and K-algebra homomorphisms of Corollary 4.8. Hence the coordinate ring $A(X\times Y)$ of the product is just the tensor product $A(X)\otimes_K A(Y)$.

Exercise 4.12 (Affine conics)**.** An irreducible quadric curve in $\mathbb{A}^2$ is also called an *affine conic*. Show that every affine conic over a field of characteristic not equal to 2 is isomorphic to exactly one of the varieties $X_1 = V(x_2 - x_1^2)$ and $X_2 = V(x_1x_2 - 1)$, with an isomorphism given by a linear coordinate transformation followed by a translation.

Exercise 4.13. Let $f\colon X \to Y$ be a morphism of affine varieties and $f^*\colon A(Y) \to A(X)$ the corresponding homomorphism of the coordinate rings. Are the following statements true or false?

(a) f is surjective if and only if f^* is injective.

(b) f is injective if and only if f^* is surjective.

(c) If $f\colon \mathbb{A}^1 \to \mathbb{A}^1$ is an isomorphism then f is affine linear, i. e. of the form $f(x) = ax+b$ for some $a,b \in K$.

(d) If $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ is an isomorphism then f is affine linear, i. e. it is of the form $f(x) = Ax+b$ for some $A \in \mathrm{Mat}(2\times 2, K)$ and $b \in K^2$.

Construction 4.14 (Affine varieties from finitely generated reduced K-algebras)**.** Corollary 4.8 allows us to construct affine varieties in a different way: Let R be a finitely generated K-algebra, and assume that it is *reduced*, i. e. that it has no nilpotent elements. We can then pick generators $a_1,\dots,a_n$ for R and obtain a surjective K-algebra homomorphism

$$g\colon K[x_1,\dots,x_n] \to R,\ f \mapsto f(a_1,\dots,a_n).$$

By the homomorphism theorem we therefore see that $R \cong K[x_1,\dots,x_n]/J$, where J is the kernel of g. Moreover, J is a radical ideal as we have assumed R to be reduced. Hence $X = V(J)$ is an affine variety in $\mathbb{A}^n$ with coordinate ring $A(X) \cong R$.

Note that this construction of X from R depends on the choice of generators of R, and so we can get different affine varieties that way. However, Corollary 4.8 implies that all these affine varieties will be isomorphic since they have isomorphic coordinate rings — they just differ in their embeddings in affine spaces.

This motivates us to make a (very minor) redefinition of the term "affine variety" to allow for objects that are isomorphic to an affine variety in the old sense, but that do not come with an intrinsic description as the zero locus of some polynomials in a fixed affine space.

Definition 4.15 (Slight redefinition of affine varieties)**.** From now on, an **affine variety** will be a ringed space that is isomorphic to an affine variety in the old sense of Definition 1.2 (b).

Remark 4.16. With this new definition, the result of Construction 4.14 can be reformulated by saying that there is a natural bijection

$$\{\text{affine varieties}\}/\text{isomorphisms} \quad \stackrel{1:1}{\longleftrightarrow} \quad \{\text{finitely generated reduced } K\text{-algebras}\}/\text{isomorphisms}$$

that also extends to morphisms, i. e. morphisms of affine varieties correspond exactly to homomorphisms of K-algebras in this picture.

Note that all our concepts and results immediately carry over to an affine variety X in this new sense: For example, all topological concepts are defined as X is still a topological space, regular functions are just sections of the structure sheaf $\mathscr{O}_X$, the coordinate ring $A(X)$ can be considered to be $\mathscr{O}_X(X)$ by Proposition 3.8, and products involving X can be defined using any embedding of X in affine space (yielding a product that is unique up to isomorphisms).

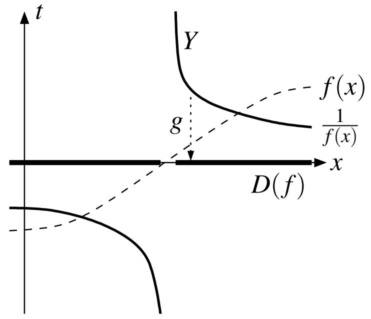
Probably the most important examples of affine varieties in this new sense that do not look like affine varieties a priori are the distinguished open subsets of Definition 3.6:

Proposition 4.17 (Distinguished open subsets are affine varieties)**.** *Let X be an affine variety, and let $f \in A(X)$. Then the distinguished open subset $D(f)$ is an affine variety with coordinate ring $A(D(f)) \cong A(X)_f$.*

Proof. Clearly,

$$Y := \{(x,t) \in X \times \mathbb{A}^1 : t\,f(x) = 1\} \quad \subset X \times \mathbb{A}^1$$

is an affine variety as it is the zero locus of the polynomial $t\,f(x) - 1$ in the affine variety $X \times \mathbb{A}^1$. In the picture on the right, the graph of f is shown by a dashed line, and the graph of $\frac{1}{f}$ (and hence Y) as a thin solid line. The affine variety Y is isomorphic to $D(f)$ (i. e. the thick line in the picture) by the projection morphism



$$g\colon Y \to D(f),\ (x,t) \mapsto x$$

$$\text{with inverse} \quad g^{-1}\colon D(f) \to Y, x \mapsto \left(x, \frac{1}{f(x)}\right).$$

So $D(f)$ is an affine variety, and by Proposition 3.8 and Corollary 3.10 we see that its coordinate ring is $A(D(f)) \cong \mathscr{O}_X(D(f)) \cong A(X)_f$. □

Example 4.18 ($\mathbb{A}^2 \backslash \{0\}$ is not an affine variety)**.** As in Example 3.11 let $X = \mathbb{A}^2$ and consider the open subset $U = \mathbb{A}^2 \backslash \{0\}$ of X. Then even in the new sense of Definition 4.15 the ringed space U is not an affine variety: Otherwise its coordinate ring would be $\mathscr{O}_X(U)$ by Proposition 3.8, and thus just the polynomial ring $K[x,y]$ by Example 3.11. But this is the same as the coordinate ring of $X = \mathbb{A}^2$, and hence Corollary 4.8 would imply that U and X are isomorphic, with the isomorphism given by the identity map. This is obviously not true, and hence we conclude that U is not an affine variety.

However, we can cover U by the two (distinguished) open subsets

$$D(x_1) = \{(x_1,x_2) : x_1 \neq 0\} \quad \text{and} \quad D(x_2) = \{(x_1,x_2) : x_2 \neq 0\}$$

which are affine by Proposition 4.17. This leads us to the idea that we should also consider ringed spaces that can be patched together from affine varieties. We will do this in the next chapter.

Exercise 4.19. Which of the following ringed spaces are isomorphic over $\mathbb{C}$?

(a) $\mathbb{A}^1 \backslash \{1\}$
(b) $V(x_1^2 + x_2^2) \subset \mathbb{A}^2$
(c) $V(x_2 - x_1^2, x_3 - x_1^3) \backslash \{0\} \subset \mathbb{A}^3$
(d) $V(x_1 x_2) \subset \mathbb{A}^2$
(e) $V(x_2^2 - x_1^3 - x_1^2) \subset \mathbb{A}^2$
(f) $V(x_1^2 - x_2^2 - 1) \subset \mathbb{A}^2$ 07