8. Grassmannians

After having introduced affine and projective varieties, let us now take a break in our discussion of the general theory to construct an interesting and useful class of examples. The idea behind this construction is simple: Since the definition of projective spaces as the sets of 1-dimensional linear subspaces of K^n turned out to be a very useful concept, let us now generalize this and consider instead the sets of k-dimensional linear subspaces of K^n for an arbitrary $k = 0, \dots, n$.

Definition 8.1 (Grassmannians)**.** Let $n \in \mathbb{N}_{>0}$, and let $k \in \mathbb{N}$ with $0 \le k \le n$. We denote by $G(k,n)$ the set of all k-dimensional linear subspaces of K^n. It is called the **Grassmannian** of k-planes in K^n.

Remark 8.2. By Example 6.13 (b) and Exercise 6.31 (a), Lemma 6.18 shows that k-dimensional linear subspaces of K^n for $k > 0$ are in natural bijection with $(k-1)$-dimensional linear subspaces of $\mathbb{P}^{n-1}$. We can therefore consider $G(k,n)$ alternatively as the set of such projective linear subspaces. As the dimensions k and n are reduced by 1 in this way, our Grassmannian $G(k,n)$ of Definition 8.1 is sometimes written in the literature as $G(k-1, n-1)$ instead.

Of course, as in the case of projective spaces our goal must again be to make the Grassmannian $G(k,n)$ into a variety. In fact, we will see that it is even a projective variety in a natural way. For this we need the algebraic concept of *alternating tensor products*, a slight variant of the ordinary tensor products well-known from commutative algebra [G3, Chapter 5]. Let us briefly introduce them now.

Definition 8.3 (Alternating linear maps)**.** Let V be a vector space over K, and let $k \in \mathbb{N}$. A (k-fold) multilinear map $f \colon V^k \to W$ to another vector space W is called **alternating** if $f(v_1, \dots, v_k) = 0$ for all $v_1, \dots, v_k \in V$ such that $v_i = v_j$ for some $i \neq j$.

Remark 8.4. Let $f \colon V^k \to W$ be an alternating multilinear map, and let $v_1, \dots, v_k \in V$. Plugging in $v_i + v_j$ as the i-th and j-th argument for f we obtain by multilinearity

$$f(\dots, v_i + v_j, \dots, v_i + v_j, \dots) = f(\dots, v_i, \dots, v_i, \dots) + f(\dots, v_j, \dots, v_j, \dots) \\ + f(\dots, v_i, \dots, v_j, \dots) + f(\dots, v_j, \dots, v_i, \dots).$$

But the three terms in the first row are 0 by Definition 8.3, and hence we obtain

$$f(\dots, v_j, \dots, v_i, \dots) = -f(\dots, v_i, \dots, v_j, \dots),$$

i. e. exchanging two arguments of f multiplies the result by -1. For any permutation $\sigma \in S_k$ of the arguments (which is a composition of such exchanges) we therefore obtain

$$f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \operatorname{sign} \sigma \cdot f(v_1, \dots, v_k).$$

Example 8.5.

(a) The determinant $\det \colon \operatorname{Mat}(n \times n, K) = (K^n)^n \to K$ is an alternating n-fold multilinear map to the ground field K.

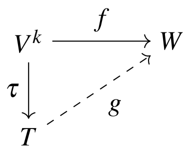
(b) The *cross product*

$$f(v,w) = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$$

of two vectors $v = (a_1, a_2, a_3)$ and $w = (b_1, b_2, b_3)$ in K^3 defines an alternating bilinear map $f \colon K^3 \times K^3 \to K^3$.

Definition 8.6 (Alternating tensor products)**.** Again let V be a vector space over K, and let $k \in \mathbb{N}$.

A **k-fold alternating tensor product** of V is a vector space T together with an alternating k-fold multilinear map $\tau \colon V^k \to T$ satisfying the following universal property: For every k-fold alternating multilinear map $f \colon V^k \to W$ to another vector space W there is a unique linear map $g \colon T \to W$ with $f = g \circ \tau$, i. e. such that the diagram on the right commutes.



Proposition 8.7 (Existence and uniqueness of alternating tensor products)**.** *For any vector space V and any $k \in \mathbb{N}$, there is a k-fold alternating tensor product $\tau\colon V^k \to T$ as in Definition 8.6, and it is unique up to unique isomorphism. We will write T as $\Lambda^k V$ and $\tau(v_1,\dots,v_k)$ as $v_1 \wedge \cdots \wedge v_k \in \Lambda^k V$ for all $v_1,\dots,v_k \in V$.*

Proof. The proof both of the uniqueness and the existence is entirely analogous to the case of ordinary tensor products [G3, Propositions 5.4 and 5.5]. In fact, assuming the statement for ordinary tensor products, for the existence part we could also take $T = (V \otimes \cdots \otimes V)/L$, where L is the linear subspace generated by all tensors $v_1 \otimes \cdots \otimes v_k$ with $v_1,\dots,v_k \in V$ such that $v_i = v_j$ for some $i \neq j$, which satisfies the required property by the universal property of the ordinary tensor product together with Definition 8.3. □

Example 8.8. Assume that V is a finite-dimensional vector space with $n := \dim V$, and let $e_1,\dots,e_n$ be a basis of V.

(a) In the same way as the tensors $e_{i_1} \otimes \cdots \otimes e_{i_k}$ for all $i_1,\dots,i_k \in \{1,\dots,n\}$ form a basis of the ordinary k-fold tensor product $V \otimes \cdots \otimes V$ [G3, Example 5.10 (a)], the alternating tensors $e_{i_1} \wedge \cdots \wedge e_{i_k}$ for all *strictly increasing* indices $i_1 < \cdots < i_k$ in $\{1,\dots,n\}$ form a basis of $\Lambda^k V$. In particular, we have $\dim \Lambda^k V = \binom{n}{k}$.

(b) Clearly, we have $\Lambda^0 V \cong K$ and $\Lambda^1 V \cong V$. Moreover, by (a) we have $\dim \Lambda^n K^n = 1$; an isomorphism $\Lambda^n K^n \cong K$ is given by the determinant as in Example 8.5 (a).

(c) For $V = K^3$ and two vectors $v = a_1e_1 + a_2e_2 + a_3e_3$ and $w = b_1e_1 + b_2e_2 + b_3e_3$ in K^3 we have

$$v \wedge w = (a_1b_2 - a_2b_1)\, e_1 \wedge e_2 + (a_1b_3 - a_3b_1)\, e_1 \wedge e_3 + (a_2b_3 - a_3b_2)\, e_2 \wedge e_3 \quad \in \Lambda^2 K^3.$$

As $e_1 \wedge e_2$, $e_1 \wedge e_3$, and $e_2 \wedge e_3$ form a basis of $\Lambda^2 K^3 \cong K^3$ by (a), we therefore see that (up to a simple change of basis) $v \wedge w$ is just the cross product of v and w as in Example 8.5 (b), i. e. the cross product gives a concrete isomorphism $\Lambda^2 K^3 \cong K^3$.

In this example, note that the coordinates of $v \wedge w$ are just the three 2×2 minors (i. e. the determinants of all 2×2 submatrices) of the 2×3 matrix with rows v and w. This is in fact a general phenomenon:

Remark 8.9 (Alternating tensor products and determinants)**.** Let $0 \le k \le n$, and let $v_1,\dots,v_k \in K^n$ with basis expansions $v_i = \sum_j a_{i,j} e_j$ for $i = 1,\dots,k$ with respect to the standard basis. For strictly increasing indices $i_1 < \cdots < i_k$ let us determine the coefficient of the basis vector $e_{i_1} \wedge \cdots \wedge e_{i_k}$ of $\Lambda^k K^n$ as in Example 8.8 (a) in the tensor product $v_1 \wedge \cdots \wedge v_k$. First of all, by multilinearity we have

$$v_1 \wedge \cdots \wedge v_k = \sum_{j_1,\dots,j_k} a_{1,j_1} \cdots a_{k,j_k} \cdot e_{j_1} \wedge \cdots \wedge e_{j_k}.$$

Note that the indices $j_1,\dots,j_k$ in the products $e_{j_1} \wedge \cdots \wedge e_{j_k}$ in the terms of this sum are not necessarily in strictly ascending order. So to figure out the coefficient of $e_{i_1} \wedge \cdots \wedge e_{i_k}$ in $v_1 \wedge \cdots \wedge v_k$ we have to sort the indices in each sum first; the resulting coefficient is then by Remark 8.4

$$\sum \operatorname{sign} \sigma \cdot a_{1,i_{\sigma(1)}} \cdots a_{k,i_{\sigma(k)}},$$

where the sum is taken over all permutations σ. By definition, this is exactly the determinant of the maximal quadratic submatrix of the coefficient matrix $(a_{j,i})_{j,i}$ obtained by taking only the columns $i_1,\dots,i_k$. In other words, the coordinates of $v_1 \wedge \cdots \wedge v_k$ in the basis of Example 8.8 (a) are just all the maximal minors of the matrix whose rows are $v_1,\dots,v_k$. So the alternating tensor product can be viewed as a convenient way to encode all these minors in a single object.

As a consequence, we will see now that alternating tensor products can be used to encode the linear dependence and linear spans of vectors in a very elegant way.

Lemma 8.10. *Let $v_1,\dots,v_k \in K^n$ for some $k \le n$. Then $v_1 \wedge \cdots \wedge v_k = 0$ if and only if $v_1,\dots,v_k$ are linearly dependent.*

Proof. By Remark 8.9, we have $v_1 \wedge \cdots \wedge v_k = 0$ if and only if all maximal minors of the matrix with rows $v_1, \dots, v_k$ are zero. But this is the case if and only if this matrix does not have full rank, i. e. if and only if $v_1, \dots, v_k$ are linearly dependent. □

Lemma 8.11. *Let $v_1, \dots, v_k \in K^n$ and $w_1, \dots, w_k \in K^n$ both be linearly independent. Then the alternating tensor products $v_1 \wedge \cdots \wedge v_k$ and $w_1 \wedge \cdots \wedge w_k$ are linearly dependent in $\Lambda^k K^n$ if and only if* $\operatorname{Lin}(v_1, \dots, v_k) = \operatorname{Lin}(w_1, \dots, w_k)$.

Proof. As we have assumed both $v_1, \dots, v_k$ and $w_1, \dots, w_k$ to be linearly independent, we know by Lemma 8.10 that $v_1 \wedge \cdots \wedge v_k$ and $w_1 \wedge \cdots \wedge w_k$ are both non-zero.

"⇒" Assume that $v_1 \wedge \cdots \wedge v_k = \lambda\, w_1 \wedge \cdots \wedge w_k$ for some $\lambda \in K$. Then we have

$$w_i \wedge v_1 \wedge \cdots \wedge v_k = \lambda\, w_i \wedge w_1 \wedge \cdots \wedge w_k = 0$$

for all i since the vector w_i appears twice in this alternating product. Hence the vectors $w_i, v_1, \dots, v_k$ are linearly dependent by Lemma 8.10, which means that $w_i \in \operatorname{Lin}(v_1, \dots, v_k)$, and thus $\operatorname{Lin}(w_1, \dots, w_k) \subset \operatorname{Lin}(v_1, \dots, v_k)$. The other inclusion then follows by symmetry.

"⇐" If $v_1, \dots, v_k$ and $w_1, \dots, w_k$ span the same subspace of K^n then the basis $w_1, \dots, w_k$ can be obtained from $v_1, \dots, v_k$ by a finite sequence of basis transformations $v_i \to v_i + \lambda\, v_j$ and $v_i \to \lambda\, v_i$ for $\lambda \in K$ and $i \neq j$. But as

$$v_1 \wedge \cdots \wedge v_{i-1} \wedge (v_i + \lambda\, v_j) \wedge v_{i+1} \wedge \cdots \wedge v_n = v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_n$$

and

$$v_1 \wedge \cdots \wedge (\lambda\, v_i) \wedge \cdots \wedge v_n = \lambda\, v_1 \wedge \cdots \wedge v_n,$$

these transformations change the alternating product at most by a multiplicative scalar. □

We can now use our results to realize the Grassmannian $G(k,n)$ as a subset of a projective space.

Construction 8.12 (Plücker embedding)**.** Let $0 \le k \le n$, and consider the map $f\colon G(k,n) \to \mathbb{P}^{\binom{n}{k}-1}$ given by sending a linear subspace $\operatorname{Lin}(v_1, \dots, v_k) \in G(k,n)$ to the class of the alternating tensor $v_1 \wedge \cdots \wedge v_k \in \Lambda^k K^n \cong K^{\binom{n}{k}}$ in $\mathbb{P}^{\binom{n}{k}-1}$.

Note that this is well-defined: $v_1 \wedge \cdots \wedge v_k$ is non-zero by Lemma 8.10, and representing the same subspace by a different basis does not change the resulting point in $\mathbb{P}^{\binom{n}{k}-1}$ by the part "⇐" of Lemma 8.11. Moreover, the map f is injective by the part "⇒" of Lemma 8.11. We call it the **Plücker embedding** of $G(k,n)$; for a k-dimensional linear subspace $L \in G(k,n)$ the (homogeneous) coordinates of $f(L)$ in $\mathbb{P}^{\binom{n}{k}-1}$ are called the **Plücker coordinates** of L. By Remark 8.9, they are just all the maximal minors of the matrix whose rows are $v_1, \dots, v_k$.

In the following, we will always consider $G(k,n)$ as a subset of $\mathbb{P}^{\binom{n}{k}-1}$ using this Plücker embedding.

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Example 8.13.

(a) The Plücker embedding of $G(1,n)$ simply maps a linear subspace $\operatorname{Lin}(a_1 e_1 + \cdots + a_n e_n)$ to the point $(a_1 : \cdots : a_n) \in \mathbb{P}^{\binom{n}{1}-1} = \mathbb{P}^{n-1}$. Hence $G(1,n) = \mathbb{P}^{n-1}$ as expected.

(b) Consider the 2-dimensional subspace $L = \operatorname{Lin}(e_1 + e_2, e_1 + e_3) \in G(2,3)$ of K^3. As

$$(e_1 + e_2) \wedge (e_1 + e_3) = -e_1 \wedge e_2 + e_1 \wedge e_3 + e_2 \wedge e_3,$$

the coefficients $(-1:1:1)$ of this vector are the Plücker coordinates of L in $\mathbb{P}^{\binom{3}{2}-1} = \mathbb{P}^2$. Alternatively, these are the three maximal minors of the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

whose rows are the given spanning vectors $e_1 + e_2$ and $e_1 + e_3$ of L. Note that a change of these spanning vectors will just perform row operations on this matrix, which changes the maximal minors at most by a common constant factor. This shows again in this example that the homogeneous Plücker coordinates of L are well-defined.

So far we have embedded the Grassmannian $G(k,n)$ into a projective space, but we still have to see that it is a closed subset, i. e. a projective variety. By Construction 8.12, $G(k,n)$ consists exactly of the classes in $\mathbb{P}^{\binom{n}{k}-1}$ of all non-zero alternating tensors in $\Lambda^k K^n$ that can be written as so-called *pure tensors*, i. e. as $v_1 \wedge \cdots \wedge v_k$ for some $v_1,\dots,v_k \in K^n$ — and not just as a linear combination of such expressions. Hence we have to find suitable equations describing these pure tensors in $\Lambda^k K^n$. The key lemma to achieve this is the following.

Lemma 8.14. *For a fixed non-zero $\omega \in \Lambda^k K^n$ with $k<n$ consider the K-linear map*

$$f\colon K^n \to \Lambda^{k+1} K^n,\ \ v \mapsto v \wedge \omega.$$

Then $\operatorname{rk} f \geq n-k$, with equality holding if and only if $\omega = v_1 \wedge \cdots \wedge v_k$ for some $v_1,\dots,v_k \in K^n$.

Example 8.15. Let $k=2$ and $n=4$.

(a) For $\omega = e_1 \wedge e_2$ the map f of Lemma 8.14 is given by

$$\begin{aligned} f(a_1e_1+a_2e_2+a_3e_3+a_4e_4) &= (a_1e_1+a_2e_2+a_3e_3+a_4e_4) \wedge e_1 \wedge e_2 \\ &= a_3 e_1 \wedge e_2 \wedge e_3 + a_4 e_1 \wedge e_2 \wedge e_4, \end{aligned}$$

for $a_1,a_2,a_3,a_4 \in K$, and thus has rank $\operatorname{rk} f = 2 = n-k$ in accordance with the statement of the lemma.

(b) For $\omega = e_1 \wedge e_2 + e_3 \wedge e_4$ we get

$$\begin{aligned} & f(a_1e_1+a_2e_2+a_3e_3+a_4e_4) \\ &\quad = (a_1e_1+a_2e_2+a_3e_3+a_4e_4) \wedge (e_1 \wedge e_2 + e_3 \wedge e_4) \\ &\quad = a_1 e_1 \wedge e_3 \wedge e_4 + a_2 e_2 \wedge e_3 \wedge e_4 + a_3 e_1 \wedge e_2 \wedge e_3 + a_4 e_1 \wedge e_2 \wedge e_4 \end{aligned}$$

instead, so that $\operatorname{rk} f = 4$. Hence Lemma 8.14 tells us that there is no way to write ω as a pure tensor $v_1 \wedge v_2$ for some vectors $v_1,v_2 \in K^4$.

Proof of Lemma 8.14. Let $v_1,\dots,v_r$ be a basis of $\operatorname{Ker} f$ (with $r = n - \operatorname{rk} f$), and extend it to a basis $v_1,\dots,v_n$ of K^n. By Example 8.8 (a) the alternating tensors $v_{i_1} \wedge \cdots \wedge v_{i_k}$ with $i_1 < \cdots < i_k$ then form a basis of $\Lambda^k K^n$, and so we can write

$$\omega = \sum_{i_1<\cdots<i_k} a_{i_1,\dots,i_k}\, v_{i_1} \wedge \cdots \wedge v_{i_k}$$

for suitable coefficients $a_{i_1,\dots,i_k} \in K$. Now for $i=1,\dots,r$ we know that $v_i \in \operatorname{Ker} f$, and thus

$$0 = v_i \wedge \omega = \sum_{i_1<\cdots<i_k} a_{i_1,\dots,i_k}\, v_i \wedge v_{i_1} \wedge \cdots \wedge v_{i_k}. \qquad (*)$$

Note that $v_i \wedge v_{i_1} \wedge \cdots \wedge v_{i_k} = 0$ if $i \in \{i_1,\dots,i_k\}$, and in the other cases these products are (up to sign) different basis vectors of $\Lambda^{k+1} K^n$. So the equation $(*)$ tells us that we must have $a_{i_1,\dots,i_k} = 0$ whenever $i \notin \{i_1,\dots,i_k\}$. As this holds for all $i=1,\dots,r$ we conclude that the coefficient $a_{i_1,\dots,i_k} = 0$ can only be non-zero if $\{1,\dots,r\} \subset \{i_1,\dots,i_k\}$.

But at least one of these coefficients has to be non-zero since $\omega \neq 0$ by assumption. This obviously requires that $r \leq k$, i. e. that $\operatorname{rk} f = n-r \geq n-k$. Moreover, if we have equality then only the coefficient $a_{1,\dots,k}$ can be non-zero, which means that ω is a scalar multiple of $v_1 \wedge \cdots \wedge v_k$.

Conversely, if $\omega = w_1 \wedge \cdots \wedge w_k$ for some (necessarily linearly independent) $w_1,\dots,w_k \in K^n$ then $w_1,\dots,w_k \in \operatorname{Ker} f$. Hence in this case $\dim \operatorname{Ker} f \geq k$, i. e. $\operatorname{rk} f \leq n-k$, and together with the above result $\operatorname{rk} f \geq n-k$ we have equality. □

Corollary 8.16 **($G(k,n)$ as a projective variety).** *With the Plücker embedding of Construction 8.12, the Grassmannian $G(k,n)$ is a closed subset of $\mathbb{P}^{\binom{n}{k}-1}$. In particular, it is a projective variety.*

Proof. As $G(n,n)$ is just a single point (and hence clearly a variety) we may assume that $k<n$. Then by construction a point $\omega \in \mathbb{P}^{\binom{n}{k}-1}$ lies in $G(k,n)$ if and only if it is the class of a pure tensor $v_1 \wedge \cdots \wedge v_k$. Lemma 8.14 shows that this is the case if and only if the rank of the linear map $f\colon K^n \to \Lambda^{k+1} K^n$, $v \mapsto v \wedge \omega$ is $n-k$. As we also know that the rank of this map is always at least

$n-k$, this condition can be checked by the vanishing of all $(n-k+1)\times(n-k+1)$ minors of the matrix corresponding to f. But these minors are polynomials in the entries of this matrix, and thus in the coordinates of ω. Hence we see that the condition for ω to be in $G(k,n)$ is closed. □

Example 8.17. By the proof of Corollary 8.16, the Grassmannian $G(2,4)$ is given by the vanishing of all sixteen 3×3 minors of a 4×4 matrix corresponding to a linear map $K^4 \to \Lambda^3 K^4$, i. e. it is a subset of $\mathbb{P}^{\binom{4}{2}-1} = \mathbb{P}^5$ given by 16 cubic equations.

As you might expect, this is by no means the simplest set of equations describing $G(2,4)$ — in fact, we will see in Exercise 8.22 (a) that a single quadratic equation suffices to cut out $G(2,4)$ from $\mathbb{P}^5$. Our proof of Corollary 8.16 is just the easiest way to show that $G(k,n)$ is a variety; it is not suitable in practice to find a nice description of $G(k,n)$ as a zero locus of simple equations.

However, there is another useful description of the Grassmannian in terms of affine patches, as we will see now. This will then also allow us to easily read off the dimension of $G(k,n)$ — which would be very hard to compute from its equations as in Corollary 8.16.

Construction 8.18 (Affine cover of the Grassmannian)**.** Let $U_0 \subset G(k,n) \subset \mathbb{P}^{\binom{n}{k}-1}$ be the affine open subset where the $e_1\wedge\cdots\wedge e_k$-coordinate is non-zero. Then by Remark 8.9 a linear subspace $L \in G(k,n)$ is in U_0 if and only if it is the row span of a $k\times n$ matrix of the form $(A\,|\,B)$ for an invertible $k\times k$ matrix A and an arbitrary $k\times(n-k)$ matrix B. Multiplying such a matrix by A^{-1} from the left, which does not change its row span, yields that U_0 is the image of the map

$$f\colon \mathbb{A}^{k(n-k)} = \mathrm{Mat}(k\times(n-k),K) \to U_0,$$
$$C \mapsto \text{the row span of } (E_k\,|\,C),$$

where $C = A^{-1}B$ in the notation above. It is clear that different matrices C lead to different row spans of $(E_k\,|\,C)$, so f is bijective. Moreover, as the maximal minors of $(E_k\,|\,C)$ are polynomial functions in the entries of C, we see that f is a morphism. Conversely, the (i,j)-entry of C can be reconstructed from $f(C)$ up to sign as the maximal minor of $(E_k\,|\,C)$ where we take all columns of E_k except the i-th, together with the j-th column of C. Hence f^{-1} is a morphism as well, showing that f is in fact an isomorphism.

In other words, we have seen that $U_0 \cong \mathbb{A}^{k(n-k)}$ is an affine *space* (and not just an affine *variety*, which is already clear from Proposition 7.2). As the same arguments also holds for all other affine patches where one of the Plücker coordinates is non-zero, we conclude that $G(k,n)$ can be covered by such affine spaces. In particular, it follows:

Corollary 8.19. *$G(k,n)$ is an irreducible variety of dimension $k(n-k)$.*

Proof. We have just seen in Construction 8.18 that $G(k,n)$ has an open cover by affine spaces $\mathbb{A}^{k(n-k)}$. As any two of these patches have a non-empty intersection (it is in fact easy to write down a $k\times n$ matrix such that any two given maximal minors are non-zero), the result follows from Exercises 2.21 (b) and 2.34 (a). □

Remark 8.20. The argument of Construction 8.18 also shows an alternative description of the Grassmannian: It is the space of all full-rank $k\times n$ matrices modulo row transformations. As we know that every such matrix is equivalent modulo row transformations to a unique matrix in reduced row echelon form, we can also think of $G(k,n)$ as the set of full-rank $k\times n$ matrices in such a form. For example, in the case $k=1$ and $n=2$ (when $G(1,2)=\mathbb{P}^1$ by Example 8.13 (a)) the full-rank 1×2 matrices in reduced row echelon form are

$$\begin{aligned} &(\,1\;\;*\,) \quad \text{corresponding to } \mathbb{A}^1 \subset \mathbb{P}^1 \\ \text{and}\quad &(\,0\;\;1\,) \quad \text{corresponding to } \infty \in \mathbb{P}^1 \end{aligned}$$

as in the homogeneous coordinates of $\mathbb{P}^1$.

The affine cover of Construction 8.18 can also be used to show the following symmetry property of the Grassmannians.

Proposition 8.21. *For all* $0 \le k \le n$ *we have* $G(k,n) \cong G(n-k,n)$.

Proof. There is an obvious well-defined set-theoretic bijection $f\colon G(k,n) \to G(n-k,n)$ that sends a k-dimensional linear subspace L of K^n to its "orthogonal" complement

$$L^\perp = \{x \in K^n : \langle x,y\rangle = 0 \text{ for all } y \in L\},$$

where $\langle x,y\rangle = \sum_{i=1}^n x_i y_i$ denotes the standard bilinear form. It remains to be shown that f (and analogously f^{-1}) is a morphism. By Lemma 4.6, we can do this on the affine coordinates of Construction 8.18. So let $L \in G(k,n)$ be described as the subspace spanned by the rows of a matrix $(E_k \,|\, C)$, where the entries of $C \in \operatorname{Mat}(k \times (n-k), K)$ are the affine coordinates of L. As

$$(E_k \,|\, C) \cdot \begin{pmatrix} -C \\ E_{n-k} \end{pmatrix} = 0,$$

we see that $L^\perp$ is the subspace spanned by the rows of $(-C^\mathsf{T} \,|\, E_{n-k})$. But the maximal minors of this matrix, i. e. the Plücker coordinates of $L^\perp$, are clearly polynomials in the entries of C, and thus we conclude that f is a morphism. □

Exercise 8.22. Let $G(2,4) \subset \mathbb{P}^5$ be the Grassmannian of lines in $\mathbb{P}^3$ (or of 2-dimensional linear subspaces of K^4). We denote the homogeneous Plücker coordinates of $G(2,4)$ in $\mathbb{P}^5$ by $x_{i,j}$ for $1 \le i < j \le 4$. Show:

(a) $G(2,4) = V(x_{1,2}x_{3,4} - x_{1,3}x_{2,4} + x_{1,4}x_{2,3})$.

(b) Let $L \subset \mathbb{P}^3$ be an arbitrary line. Show that the set of lines in $\mathbb{P}^3$ that intersect L, considered as a subset of $G(2,4) \subset \mathbb{P}^5$, is the zero locus of a homogeneous linear polynomial.

How many lines in $\mathbb{P}^3$ would you expect to intersect four general given lines?

Exercise 8.23. Show that the following sets are projective varieties:

(a) the *incidence correspondence*

$$\{(L,a) \in G(k,n) \times \mathbb{P}^{n-1} : L \subset \mathbb{P}^{n-1} \text{ a } (k-1)\text{-dimensional linear subspace and } a \in L\};$$

(b) the *join* of two disjoint varieties $X, Y \subset \mathbb{P}^n$, i. e. the union in $\mathbb{P}^n$ of all lines intersecting both X and Y.