UNIVERSAL FAMILIES OF RATIONAL TROPICAL CURVES

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ABSTRACT. We introduce the notion of families of n-marked smooth rational tropical curves over smooth tropical varieties and establish a one-to-one correspondence between (equivalence classes of) these families and morphisms from smooth tropical varieties into the moduli space of n-marked abstract rational tropical curves $\mathcal{M}_n$.

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1. INTRODUCTION

The moduli spaces $\mathcal{M}_n$ of n-marked abstract rational tropical curves have been well known for several years. An explicit description of the combinatorial structure of $\mathcal{M}_n$ and its embedding as a tropical fan can be found in [GKM]. However, so far the moduli spaces $\mathcal{M}_n$ have only been a parameter spaces, i.e. in bijection to the set of tropical curves. To further justify the nomenclature, we would like to equip them with a universal family. In classical geometry or category theory, such a universal family induces all possible families via pull-back along a unique morphism into $\mathcal{M}_n$. This paper gives a suitable definition of a family of tropical curves and proves that the forgetful map $\mathrm{ft} : \mathcal{M}_{n+1} \to \mathcal{M}_n$ is indeed a universal family.

After briefly recalling some known facts in section 2, we give a definition of families of smooth rational n-marked curves over smooth varieties in section 3. We show that the forgetful morphism is a family of curves and that we can assign a family of curves to each morphism of a smooth variety into $\mathcal{M}_n$.

In section 4 we establish an inverse operation, namely we prove that each family of n-marked curves also gives rise to a morphism into $\mathcal{M}_n$. This leads to our main theorem 4.5 which gives a bijection between equivalence classes of families of n-marked curves over a smooth variety B and morphisms $B \to \mathcal{M}_n$.

In the last section we prove that there is a bijective pseudo-morphism, a piecewise linear map respecting the balancing condition, between two equivalent families. In case the domain of one of the families is a smooth variety, this map is even an isomorphism.

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2. PRELIMINARIES AND NOTATIONS

In this section we quickly review some results on tropical intersection theory and the moduli space $\mathcal{M}_n$ of n-marked abstract rational tropical curves.

A tropical cycle X (in a vector space V containing a lattice Λ) is the equivalence class modulo refinement of a pure-dimensional rational polyhedral complex $\mathcal{X}$ in V which is weighted (i.e. each maximal polyhedron has an integer weight) and satisfies the balancing condition (defined in [AR, definition 2.6]). A tropical variety is a tropical cycle which has only positive weights. A representative $\mathcal{X}$ of a tropical cycle X is called a polyhedral structure of X. If X has a polyhedral structure $\mathcal{X}$ which is a fan, then we call X a fan cycle and $\mathcal{X}$ a fan structure of X. The support $|X|$ of a cycle X is the union of all maximal cells of non-zero weight in a polyhedral structure of X. More details can be found in [AR, section 2] which covers fan cycles, [AR, section 5] which introduces abstract cycles (which are more general than cycles in vector spaces), and [R, section 1.1] whose notation we follow in this article.

Matroid varieties $\mathrm{B}(M)$ constitute an important class of tropical varieties. They have a canonical fan structure $\mathcal{B}(M)$ which consists of cones

$$\langle \mathcal{F} \rangle := \left\{ \sum_{i=1}^{p} \lambda_i V_{F_i} : \lambda_1, \dots, \lambda_{p-1} \geq 0, \lambda_p \in \mathbb{R} \right\}$$

corresponding to chains $\mathcal{F} = (\emptyset \subsetneq F_1 \subsetneq \ldots \subsetneq \mathcal{F}_{p-1} \subsetneq F_p = E(M))$ of flats of a matroid M having ground set $E(M) := [n]$. Here $V_F = -\sum_{i \in F} e_i$, where $e_1, \dots, e_n$ form the standard basis of $\mathbb{R}^n$ and all maximal cones of $\mathcal{B}(M)$ have trivial weight 1. Note that matroid varieties naturally come with the lineality space $\mathbb{R} \cdot (1, \dots, 1)$. We refer to [FR, section 2] for more details about matroid varieties.

A tropical variety X is smooth if it is locally a matroid variety modulo lineality space $\mathrm{B}(M)/L$ (cf. [FR, section 6]). This means that for each point p in X, the star $\mathrm{Star}_X(p)$ (cf. [R, section 1.2.3]) is isomorphic to a matroid variety modulo lineality space. We should note that $\mathrm{Star}_X(p)$ is a tropical cycle whose support consists of vectors v such that $p + \epsilon v$ is in X for small (positive) ϵ. Recall that L_1^n denotes the curve in $\mathbb{R}^n$ which consists of edges $\mathbb{R}_{\leq 0} \cdot e_i,\ i = 0, 1, \dots, n$ (all having trivial weight 1), where $e_1, \dots, e_n$ form the standard basis of $\mathbb{R}^n$ and $e_0 = -(e_1 + \ldots + e_n)$. Then smooth curves are exactly the curves which are locally isomorphic to some L_1^n.

A main property of smooth varieties which will be crucial in the next section is that they admit an intersection product of cycles having the expected properties [FR, theorem 6.4]. Furthermore, if $f : X \to Y$ is a morphism of smooth varieties (that is a locally affine linear map), then we can pull back any cycle C in Y to obtain a cycle $f^*(C)$ in X [FR, definition 8.1]. In the case when Y is smooth, we can still pull back points of Y along f [F, remark 3.10]; this will be an essential ingredient to define families of curves in definition 3.1.

In [GKM, section 3] the authors map an n-marked rational curve to the vector whose entries are pairwise distances of its leaves and use this to give the moduli space $\mathcal{M}_n$ of n-marked abstract rational tropical curves the structure of a tropical fan of dimension $n-3$ in $Q_n := \mathbb{R}^{\binom{n}{2}}/\mathrm{Im}(\phi)$, where ϕ maps $x \in \mathbb{R}^n$ to $(x_i + x_j)_{i<j}$. The edges of $\mathcal{M}_n$ are generated by vectors $v_{I|n} := v_I$ (with $I \subsetneq [n], 1 < |I| < n-1$) corresponding to abstract curves with exactly one bounded edge of length 1 separating the leaves with labels in I from the leaves with labels in the complement of I. Furthermore, the relative interior of each k-dimensional cone of $\mathcal{M}_n$ corresponds to curves with exactly k bounded edges, whose combinatorial type (i.e. the graph without the metric) is the same. The forgetful map $\mathrm{ft}_0 := \mathrm{ft} : \mathcal{M}_{n+1} \to \mathcal{M}_n$ forgetting the 0-th marked end is the morphism of tropical fan cycles induced by the projection $\pi : \mathbb{R}^{\binom{n+1}{2}} \to \mathbb{R}^{\binom{n}{2}}$ [GKM, proposition 3.9]. Note that, in order to ease the notations, we equip $\mathcal{M}_{n+1}$ with the markings $0, 1, \dots, n$, when we consider the forgetful map.

It was shown in [FR, example 7.2] that $\mathcal{M}_n$ is even isomorphic to a matroid variety modulo lineality space and thus admits an intersection product of cycles: if $\mathrm{B}(K_{n-1})$ denotes the matroid variety corresponding to the matroid associated to the complete graph K_{n-1} on $n-1$ vertices, then $\mathcal{M}_n$ is isomorphic to $\mathrm{B}(K_{n-1})/L$, with $L = \mathbb{R} \cdot (1, \dots, 1)$. Note that the ground set of the matroid associated to K_{n-1} is the set of edges of K_{n-1}, whereas its flats are exactly the edges of vertex disjoint unions of complete subgraphs of K_{n-1}. In this setting the forgetful map is induced by the projection $\pi : \mathbb{R}^{\binom{n}{2}} \to \mathbb{R}^{\binom{n-1}{2}}$.

3. Families of curves

The aim of this section is to prove that every morphism from a smooth variety X to $\mathcal{M}_n$ gives rise to a family of curves. We start by defining families of curves over smooth varieties.

Definition 3.1 (Family of curves)**.** Let $n \geq 3$ and let B be a smooth tropical variety. A morphism $T \stackrel{g}{\to} B$ of tropical varieties is a *prefamily* of n-marked tropical curves if it satisfies the following conditions:

(1) For each point b in B the cycle $g^*(b)$ is a smooth rational tropical curve with exactly n unbounded edges (called the leaves of $g^*(b)$).
(2) For any point p in T, the induced linear map
$$\lambda_{g,p} : \text{Star}_T(p) \to \text{Star}_B(g(p))$$
is surjective.
(3) The linear part of g at any cell τ in (some and thus any polyhedral structure of) T induces a surjective map $\lambda_{g|\tau} : \Lambda_\tau \to \Lambda_{g(\tau)}$ on the corresponding lattices.

A *tropical marking* on a prefamily $T \stackrel{g}{\to} B$ is an open cover $\{U_\theta, \theta \in \Theta\}$ of B together with a set of affine linear integral maps $s_i^\theta : U_\theta \to T, i = 1, \ldots, n$, such that the following holds:

(1) For all $\theta \in \Theta, i = 1, \ldots, n$, we have $g \circ s_i^\theta = \text{id}_{U_\theta}$.
(2) For any $b \in U_\theta$ if $l_1, \ldots, l_n$ denote the leaves of the fiber $g^*(b)$, then for each $i \in [n]$ there exists exactly one $j \in [n]$, such that $s_j^\theta(b) \in l_i^\circ$ (where l_i° denotes the leaf without its vertex).
(3) For any $\theta \neq \zeta \in \Theta$ and $b \in U_\theta \cap U_\zeta$, the points $s_i^\theta(b)$ and $s_i^\zeta(b)$ mark the same leaf of $g^*(b)$ (though they do not have to coincide).

A *family* of n-marked tropical curves is then a prefamily with a marking.

We call two families $T \stackrel{g}{\to} B, T' \stackrel{g'}{\to} B$ *equivalent* if for any b in B the fibers $g^*(b), g'^*(b)$ are isomorphic as n-marked tropical curves.

Example 3.2.
- The morphism
$$\pi : L_1^n \times \mathbb{R} \to \mathbb{R},\ (x_1, \ldots, x_n, y) \mapsto y,$$
together with the trivial marking $y \mapsto (e_i, y),\ i = 0, 1, \ldots, n$, is a family of $(n+1)$-marked curves.
- Let e_1, e_2 be the standard basis of $\mathbb{R}^2$. We consider the tropical curves $X_1 := L_1^2$ and $X_2 := \mathbb{R} \cdot e_1 + \mathbb{R} \cdot e_2$. Let us consider the morphisms
$$\pi_i : L_1^n \times X_i \to \mathbb{R},\ (x_1, \ldots, x_n, y_1, y_2) \mapsto y_2.$$
Although $\pi_i^*(p) = L_1^n \times \{p\}$ for all points p in $\mathbb{R}$, π_i is not a family of curves: e.g. for $i \in \{1, 2\}$ and $p = ((0, \ldots, 0), (-1, 0)) \in L_1^n \times X_i$ the map
$$\lambda_{\pi_i,p} : \text{Star}_{L_1^n \times X_i}(p) \cong L_1^n \times \mathbb{R} \to \text{Star}_{\mathbb{R}}(0) \cong \mathbb{R}$$
is just the constant zero map. Geometrically, we see that the set-theoretic fiber $\pi_i^{-1}(0)$ is 2-dimensional. This illustrates the necessity of the second axiom on a prefamily which could be seen as a tropical flatness condition without which π, π_1, π_2 would be equivalent families with completely different domains $L_1^n \times \mathbb{R}, L_1^n \times X_1, L_1^n \times X_2$ (compare to section 5).

Remark 3.3. We will see later that for all cells τ in (a polyhedral structure of) T on which g is not injective, condition (3) on a prefamily follows from the other conditions (cf. lemma 4.8). We will need condition (3) on all cells τ (including those on which g is injective) to show that the locally affine linear map $B \to \mathcal{M}_n$ induced by the family $T \to B$ is an integer map and thus a tropical morphism (cf. definition 4.1, proposition 4.6).

It is clear from the definition that the support of the intersection-theoretic fiber of a point is contained in the set-theoretic fiber. We need the following two lemmas to prove that we actually have an equality if $g : T \to B$ is a prefamily of curves. That property will be crucial in sections 4 and 5.

Lemma 3.4. *Let $g : C \to C'$ be an affine linear surjective map of tropical cycles such that $\lambda_{g,p} : \text{Star}_C(p) \to \text{Star}_{C'}(g(p))$ is surjective for all points p in C. Then the following holds:*

- *Let $\mathcal{C}, \mathcal{C}'$ be polyhedral structures of C and C' such that $g(\tau) \in \mathcal{C}'$ for all $\tau \in \mathcal{C}$ (cf.* [R, lemma 1.3.4]*). For $\tau \in \mathcal{C}$ we have*
$$g(U(\tau)) = U(g(\tau)), \text{ where } U(\tau) := \bigcup_{\sigma \in \mathcal{C}: \sigma > \tau} \text{rel int}(\sigma).$$
In particular, g is an open map, i.e. maps open sets to open sets.
- *Let φ be a rational function on C'. Then the domain of non-linearity (cf.* [R, definition 1.2.1]*) of $\varphi \circ g$ is equal to the preimage of the domain of linearity of φ, i.e.*
$$|\varphi \circ g| = g^{-1}(|\varphi|).$$

Proof. The first part is obviously equivalent to the surjectivitiy condition on $\lambda_{g,p}$. Note that the set of all possible $U(\tau)$ for all possible polyhedral structures of C forms a topological basis of the standard euclidean topology on $|C|$. For the second part it suffices to prove that φ is locally linear at $p \in C'$ if and only if $\varphi \circ g$ is locally linear at some point $q \in g^{-1}(p)$. But this is already clear from the first part. □

Lemma 3.5. *Let M be a matroid of rank r on the set $[m]$. Let $L := \mathbb{R} \cdot (1, \ldots, 1)$. Then* $\max\{x_1, \ldots, x_m\}^{r-1} \cdot \text{B}(M) = L$.

Proof. We set $\varphi := \max\{x_1, \ldots, x_m\}$. It suffices to show by induction that $\varphi^k \cdot \mathcal{B}(M)$ consists exactly of the cones corresponding to chains of flats $\mathcal{F} := (\emptyset \subsetneq F_1 \ldots \subsetneq F_{r-k-1} \subsetneq E(M))$ with $\text{r}(F_i) = i$ (all of them having trivial weight 1): Let $\mathcal{G} := (\emptyset \subsetneq G_1 \ldots \subsetneq G_{r-k-2} \subsetneq G_{r-k-1} := E(M))$ be a chain of flats with $\text{r}(G_i) = i$ for $i \leq j$ and $\text{r}(G_i) = i + 1$ for $j + 1 \leq i \leq r - k - 2$. Note that φ is linear on the cones of $\mathcal{B}(M)$ and satisfies $\varphi(V_F) = -1$ if $F = E(M)$, and 0 otherwise. As
$$\sum_{F \text{ flat with } G_j \subsetneq F \subsetneq G_{j+1}} V_F = V_{G_{j+1}} + (|F \text{ flat with } G_j \subsetneq F \subsetneq G_{j+1}| - 1) \cdot V_{G_j},$$
the claim follows directly from the definition of intersecting with rational functions [AR, definition 3.4]. □

Lemma 3.6. *Let $g : T \to B$ be a morphism from a variety T to a smooth variety B which fulfils axiom (1) and (2) of a prefamily of curves. Then the support of the intersection-theoretic fiber over each point b in B agrees with the set-theoretic fiber, that means*
$$|g^*(b)| = g^{-1}(b).$$

Proof. Let b be a point in B and let p be a point in T with $g(p) = b$. As the intersection-theoretic computations are local, it suffices to show the claim for the induced morphism $\lambda_{g,p}$ on the respective stars; that means we can assume that g is linear, T is a fan cycle, B is a matroid variety modulo lineality space and $b = 0$. We choose convex rational functions φ_i such that $b = \varphi_1 \cdots \varphi_{\dim(B)} \cdot B$. This can be done by decomposing B into a cross product of matroid varieties modulo 1-dimensional lineality spaces (cf. [FR, section 2]) and then using lemma 3.5. We show by induction that $g^*\varphi_i \cdots g^*\varphi_{\dim(B)} \cdot T$ is a cycle having only positive weights and satisfying

$$|g^*\varphi_i \cdots g^*\varphi_{\dim(B)} \cdot T| = g^{-1}(|\varphi_i \cdots \varphi_{\dim(B)} \cdot B|),$$

which implies the claim because $g^*(b) = g^*\varphi_1 \cdots g^*\varphi_{\dim(B)} \cdot T$: Since $g^*\varphi_{i-1}$ is convex and $g^*\varphi_i \cdots g^*\varphi_{\dim(B)} \cdot T$ has only positive weights, it follows from [R, lemma 1.2.25] that

$$|g^*\varphi_{i-1} \cdot g^*\varphi_i \cdots g^*\varphi_{\dim(B)} \cdot T| = |(g^*\varphi_{i-1})_{||g^*\varphi_i \cdots g^*\varphi_{\dim(B)} \cdot T|}|,$$

where the right hand side is the domain of non-linearity of the restriction of the rational function $g^*\varphi_{i-1}$ to (the support of) $g^*\varphi_i \cdots g^*\varphi_{\dim(B)} \cdot T$. By induction hypothesis, this is equal to the domain of non-linearity

$$|(\varphi_{i-1} \circ g)_{|g^{-1}(|\varphi_i \cdots \varphi_{\dim(B)} \cdot B|)}|,$$

which by the second axiom of a prefamily and lemma 3.4 coincides with

$$g^{-1}(|\varphi_{i-1}{}_{||\varphi_i \cdots \varphi_{\dim(B)} \cdot B|}|) = g^{-1}(|\varphi_{i-1} \cdot \varphi_i \cdots \varphi_{\dim(B)} \cdot B|).$$

Note that our induction hypothesis (for stars around different points) and the locality of intersecting with rational functions (cf. [R, proposition 1.2.12]) ensure that the restriction of g to $g^*\varphi_i \cdots g^*\varphi_{\dim(B)} \cdot T$ satisfies the assumptions of lemma 3.4. □

Our next aim is to show that the forgetful map is a prefamily of n-marked curves. Therefore, we compute its fibers in the following proposition.

Proposition 3.7. *Let* $\mathrm{ft} : \mathcal{M}_{n+1} \to \mathcal{M}_n$ *be the forgetful map. Then for each point* p *in* $\mathcal{M}_n$ *the (intersection-theoretic) fiber* $\mathrm{ft}^*(p)$ *is a smooth rational curve having* n *unbounded edges. Furthermore, the support satisfies* $|\mathrm{ft}^*(p)| = \mathrm{ft}^{-1}(p)$.

Proof. We know from [R, proposition 2.1.21] that for each p in $\mathcal{M}_n$ there is a smooth rational irreducible curve C_p which has n unbounded ends and whose support $|C_p|$ is equal to the set-theoretic fiber $\mathrm{ft}^{-1}(p)$. (The edges of C_0 are $\mathbb{R}_{\geq 0} \cdot v_{0,i}$ with $i \in [n]$). As it is clear from the definition of the pull-back [FR, definition 8.1] that $\mathrm{ft}^*(p)$ is a curve satisfying $|\mathrm{ft}^*(p)| \subseteq \mathrm{ft}^{-1}(p)$, the irreducibility of C_p allows us to conclude that $\mathrm{ft}^*(p) = \lambda_p \cdot C_p$ for some integer λ_p. Since morphisms of matroid varieties (modulo lineality spaces) are compatible with rational equivalence [FR, remark 9.2], it follows from [FR, theorem 9.5] that $\mathrm{ft}^*(p)$ and $\mathrm{ft}^*(0)$ are rationally equivalent; thus $\lambda_p = \lambda_0$. So it suffices to show that $\lambda_0 = 1$. Using the isomorphism of [FR] mentioned in section 2 we have to compute the fiber over the origin of the projection $\pi : \mathrm{B}(K_n)/L \to \mathrm{B}(K_{n-1})/L$ which forgets the coordinates $x_{0,i}$. Note that we gave K_n and K_{n-1} the vertex sets $\{0, 1, \ldots, n-1\}$ and $\{1, \ldots, n-1\}$ respectively and that by abuse of notation we denoted both lineality spaces by L. By [FR, proposition 8.5] we have $\pi^*(0) = (\tilde{\pi}^*(L))/L$, where $\tilde{\pi} : \mathrm{B}(K_n) \to \mathrm{B}(K_{n-1})$ is the "naturally lifted" projection. It follows from lemma 3.5 that $\tilde{\pi}^*(L) = \varphi^{n-3} \cdot \mathrm{B}(K_n)$, where $\varphi := \max\{x_{i,j} : 0 < i < j \leq n-1\}$. It is easy to see that φ is linear on the cones of $\mathcal{B}(K_n)$ and that $\varphi(V_F) = -1$ if F corresponds to K_n or its complete subgraph on the vertex set $\{1, \ldots, n-1\}$, and $\varphi(V_F) = 0$ otherwise. A straightforward induction shows that the cone associated to $\mathcal{F} := (\emptyset \subsetneq F_1 \subsetneq \ldots \subsetneq F_{n-3-k} \subsetneq F \subsetneq E(K_n))$, where $\mathrm{r}(F_i) = i$ and F is the flat corresponding to $\{1, \ldots, n-1\}$, has weight 1 in $\varphi^k \cdot \mathcal{B}(K_n)$. Thus $\mathbb{R}_{\geq 0} \cdot v_{\{0,n\}}$ has weight 1 in $\mathrm{ft}^*(0)$ and it follows that $\lambda_0 = 1$ (as C_0 is irreducible and all its edges have weight 1).

□

Lemma 3.8. *For $n \geq 3$ and $v \in \mathcal{M}_{n+1}$, the map $\lambda_{\text{ft},v}$ is surjective, i.e. the forgetful map fulfils the second axiom of a family of tropical curves.*

Proof. Let τ be the minimal cell of $\mathcal{M}_{n+1}$ containing v and let C be the curve corresponding to the point v. Let w' be an element of $\text{Star}_{\mathcal{M}_n}(\text{ft}(v))$. Then w' comes from a curve which is obtained from the curve corresponding to $\text{ft}(v)$ by resolving some higher-valent vertices. If we resolve the same vertices in C, we get a curve C' corresponding to a point $v' \in \mathcal{M}_{n+1}$ such that $\text{ft}(v') = w'$. In particular, the combinatorial type of C' corresponds to a cell $\tau' \geq \tau$, so $v' \in \text{Star}_{\mathcal{M}_{n+1}}(v)$. □

The following corollary is a direct consequence of proposition 3.7 and lemma 3.8.

Corollary 3.9. *The forgetful map is a prefamily of n-marked tropical curves.*

We now want to define a marking on the forgetful map. To do that, we need a basis of the ambient space Q_n of $\mathcal{M}_n$. In [KM, section 2] the authors construct a generating set in the way that we will shortly describe and it is easy to see (e.g. by induction on n, using the forgetful map) that it becomes a basis if we remove an arbitrary element.

For any $k \in \{1, \ldots, n\}$, we set

$$V_{k,n} := V_k := \{v_I; k \notin I, |I| = 2\}.$$

For any $I_0 \subseteq \{1, \ldots, n\}$ with $v_{I_0} \in V_k$ we define

$$V_{k,n}^{I_0} := V_k^{I_0} := V_k \setminus \{v_{I_0}\}.$$

Lemma 3.10. *Let $v_I \in \mathcal{M}_n, I \subseteq [n]$ and assume that $k \notin I$. Then we have*

$$v_I = \begin{cases} \sum_{J \subseteq I, v_J \in V_k^{I_0}} v_J, & \text{if } I_0 \nsubseteq I \\ - \sum_{J \nsubseteq I, v_J \in V_k^{I_0}} v_J, & \text{otherwise} \end{cases}.$$

Proof. It was shown in [KM, lemma 2.4, lemma 2.7] that $\sum_{w \in V_k} w = 0$ and that $v_I = \sum_{v_S \in V_k, S \subseteq I} v_S$. This implies the above equation. □

For the following proposition, for each $i = 1, \ldots, n$ we fix an arbitrary $I_0(i)$ with $v_{I_0(i)} \in V_{i,n}$ and write $W_{i,n} := V_{i,n}^{I_0(i)}$ for simplicity.

Proposition 3.11. *There exists a tropical marking s_i^θ on the forgetful map, such that, as a marked curve, the fiber over each point p in $\mathcal{M}_n$ is exactly the curve represented by that point. In particular, $(\mathcal{M}_{n+1} \stackrel{\text{ft}}{\to} \mathcal{M}_n, s_i^\theta)$ is a family of n-marked rational tropical curves.*

Proof. Again, [R, proposition 2.1.21] tells us that the fiber over each point is exactly the curve represented by that point (without markings).

For $\alpha > 0$ we define

$$U_\alpha := \left\{ \sum_{v_I \in \mathcal{M}_n} \lambda_I v_I; \lambda_I \geq 0; \sum \lambda_I < \alpha \right\} \cap |\mathcal{M}_n|.$$

Clearly $\{U_\alpha, \alpha \in \mathbb{N}_{>0}\}$ is a cover of $\mathcal{M}_n$. Now pick any $\alpha \in \mathbb{N}_{>0}, i \in 1, \ldots, n$. We define

$$s_i^\alpha : U_\alpha \to \mathcal{M}_{n+1}, v \mapsto \alpha \cdot v_{\{0,i\}} + A_i(v),$$

where $A_i : Q_n \to Q_{n+1}$ is the linear map defined by $A_i(v_I) = v_{I|n+1}$ for all $v_I \in W_{i,n}$. (Note that in this proof the v_I represent curves with markings in $\{1, \ldots, n\}$ and thus live in Q_n, whereas the $v_{I|n+1}$ correspond to curves with markings in $\{0, 1, \ldots, n\}$ and thus live in Q_{n+1}.) We have to show that this defines indeed a map into $\mathcal{M}_{n+1}$ and that it is a tropical marking.

For this, choose any $v_I \in \mathcal{M}_n$ (we assume without restriction that $i \notin I$, since $v_I = v_{I^c}$). By lemma 3.10 we have

$$v_I = \begin{cases} \sum_{J \subseteq I, v_J \in W_{i,n}} v_J, \text{ if } I_0 \nsubseteq I \\ - \sum_{J \nsubseteq I, v_J \in W_{i,n}} v_J, \text{ otherwise} \end{cases},$$

and similarly in $\mathcal{M}_{n+1}$:

$$v_{I|n+1} = \begin{cases} \displaystyle\sum_{J \subseteq I, v_J \in W_{i,n+1}} v_J = \sum_{J \subseteq I, v_J \in W_{i,n}} v_{J|n+1}, & \text{if } I_0 \nsubseteq I \\ \displaystyle - \sum_{J \nsubseteq I, v_J \in W_{i,n+1}} v_J = - \sum_{J \nsubseteq I, v_J \in W_{i,n}} v_{J|n+1} - \sum_{j \neq 0,i} v_{\{0,j\}}, & \text{otherwise} \end{cases}$$

$$= \begin{cases} A_i(v_I), \text{ if } I_0 \nsubseteq I \\ A_i(v_I) + v_{\{0,i\}}, \text{ otherwise (since } \displaystyle\sum_{j=1}^{n} v_{\{0,j\}} = 0) \end{cases}.$$

Summarising we obtain for $\lambda \in [0, \alpha)$:

$$s_i^\alpha(\lambda v_I) = \begin{cases} \alpha v_{\{0,i\}} + \lambda v_{I|n+1}, & \text{if } I_0 \nsubseteq I \\ (\alpha - \lambda) v_{\{0,i\}} + \lambda v_{I|n+1}, & \text{otherwise} \end{cases}.$$

Now for an arbitrary $v = \sum \lambda_I v_I \in U_\alpha$ (where we can assume that all the v_I with $\lambda_I \neq 0$ lie in the same maximal cone in $\mathcal{M}_n$) we have

$$s_i^\alpha(v) = \sum \lambda_I v_{I|n+1} + \underbrace{(\alpha - \sum_{I_0 \subseteq I} \lambda_I)}_{>0} v_{\{0,i\}}.$$

In particular this is a vector in a leaf of the fiber of v (which as a set can be described as $\{\sum \lambda_I v_{I|n+1} + \gamma v_{\{0,i\}}, \gamma \geq 0\}$) and for different i this marks a different leaf. Also it is clear that for different α, α' and $v \in U_\alpha \cap U_{\alpha'}$, s_i^α and $s_i^{\alpha'}$ mark the same leaf. Hence the s_i^α define a tropical marking. □

We will now prove that any two markings on the forgetful map only differ by a permutation on $\{1, \ldots, n\}$.

Proposition 3.12. *For any two families of tropical curves of the form*

$$(\mathcal{M}_{n+1} \overset{ft_0}{\to} \mathcal{M}_n, (s_i^\theta)), (\mathcal{M}_{n+1} \overset{ft_0}{\to} \mathcal{M}_n, (r_i^\zeta)),$$

there exist isomorphisms $\phi : \mathcal{M}_n \to \mathcal{M}_n$ and $\psi : \mathcal{M}_{n+1} \to \mathcal{M}_{n+1}$, such that $ft_0 \circ \psi = \phi \circ ft_0$ and such that for any b in $\mathcal{M}_n$, ψ identifies equally marked leaves of $\mathrm{ft}_0^*(b)$ and $\mathrm{ft}_0^*(\phi(b))$ in the two families. Furthermore, ϕ, ψ are induced by permutations on the coordinates of $\mathbb{R}^{\binom{n}{2}}$ and $\mathbb{R}^{\binom{n+1}{2}}$ respectively.

Proof. We can assume without restriction that both markings $(s_i^\theta), (r_i^\theta)$ are defined on the same open subsets U_θ. Since they are tropical markings, if we choose θ such that $0 \in U_\theta$, we must have for all i that

$$s_i^\theta(0) = \lambda_i^\theta v_{\{0,\sigma_1(i)\}}; \; r_i^\theta(0) = \rho_i^\theta v_{\{0,\sigma_2(i)\}}$$

for some permutations $\sigma_1, \sigma_2 \in \mathrm{S}_n, \lambda_i^\theta, \rho_i^\theta > 0$. Note that by definition of a marking, σ_1, σ_2 are independent of the choice of θ.

We can extend σ_1, σ_2 to bijections $\bar\sigma_1, \bar\sigma_2$ on $\{0, 1, \ldots, n\}$ by setting $\bar\sigma_1(0) = \bar\sigma_2(0) = 0$. These bijections induce automorphisms of $\mathbb{R}^{\binom{n+1}{2}}$ and $\mathbb{R}^{\binom{n}{2}}$ given by

$$e_{\{i,j\}} \mapsto e_{\{(\bar\sigma_2 \circ \bar\sigma_1^{-1})(i), \bar\sigma_2 \circ \bar\sigma_1^{-1})(j)\}},$$

which map Im(ϕ) to Im(ϕ) and thus give rise to automorphisms

$$\psi : \mathcal{M}_{n+1} \to \mathcal{M}_{n+1}, \quad \phi : \mathcal{M}_n \to \mathcal{M}_n.$$

Obviously $ft_0 \circ \phi = \psi \circ ft_0$ (since the 0-mark which is discarded by ft_0 is not affected by σ_1, σ_2). We will now prove compatibility with markings for ray vectors v_I:

Let $v_I \in U_\zeta \subseteq |\mathcal{M}_n|$ with $i \notin I$ and assume $\phi^{-1}(v_I) = v_{(\sigma_1 \circ \sigma_2^{-1})(I)} \in U_\theta \subseteq |\mathcal{M}_n|$. Then we have

$$r_i^\zeta(v_I) = v_{I|n+1} + \lambda \cdot v_{\{0,\sigma_2(i)\}}$$

for some λ and

$$\begin{aligned}(\psi \circ s_i^\theta \circ \phi^{-1})(v_I) &= (\psi \circ s_i^\theta)(v_{(\sigma_1 \circ \sigma_2^{-1})(I)}) \\ &= \phi(v_{(\sigma_1 \circ \sigma_2^{-1})(I)|n+1} + \rho \cdot v_{\{0,\sigma_1(i)\}}) \text{ for some } \rho \\ &= v_{(\sigma_2 \circ \sigma_1^{-1} \circ \sigma_1 \circ \sigma_2^{-1})(I)|n+1} + \rho \cdot v_{\{0,(\sigma_2 \circ \sigma_1^{-1} \circ \sigma_1)(i)\}} \\ &= v_{I|n+1} + \rho \cdot v_{\{0,\sigma_2(i)\}}\end{aligned}$$

which lies on the same leaf as $r_i^\zeta(v_I)$. For an arbitrary vector $v = \sum \alpha_I v_I$ the same argument can be applied by linearity of ϕ. □

As mentioned earlier we want to assign a family of n-marked curves to each morphism from a smooth cycle to $\mathcal{M}_n$. Therefore, we need the following definition.

Definition 3.13. Let X be a smooth variety and $f : X \to \mathcal{M}_n$ a morphism. We define X^f to be the pull-back of the diagonal $\Delta_{\mathcal{M}_n}$ along the morphism $(f \times \text{ft})$, i.e.

$$X^f := (f \times \text{ft})^*(\Delta_{\mathcal{M}_n}) \in \text{Z}_{\dim X+1}(X \times \mathcal{M}_{n+1}).$$

Note that X^f is well-defined by [FR, definition 8.1] because $X \times \mathcal{M}_{n+1}$ and $\mathcal{M}_n \times \mathcal{M}_n$ are smooth tropical varieties (which follows from the fact that cross products of matroid varieties (modulo lineality spaces) are again matroid varieties (modulo lineality spaces) [FR, lemma 2.1, remark 5.3]).

In order to show that the projection from X^f to X is a prefamily of n-marked curves we compute its fibers in the following proposition.

Proposition 3.14. *Let $\pi_X : X^f \to X$ be the projection to X. Then $\pi_X^*(p) = \{p\} \times \text{ft}^*(f(p))$ for each p in X. In particular, the fiber over each point is a smooth rational curve with n leaves.*

Proof. In this proof by abuse of notation $\pi_X, \pi_{\mathcal{M}_{n+1}}, \pi_{X \times \mathcal{M}_{n+1}}$ denote projections from a product of $X, \mathcal{M}_n, \mathcal{M}_{n+1}$ to the respective cycle. Let $\varphi \in \text{C}^{\dim X}(X)$ be the (uniquely defined) cocycle such that $\varphi \cdot X = \{p\}$ [F, definitions 2.17, 2.20, corollary 3.8]. By the projection formula and commutativity of intersection products [F, proposition 2.24] we have

$$\pi_X^*(p) = \pi_X^* \varphi \cdot X^f = (\pi_{X \times \mathcal{M}_{n+1}})_* \Gamma_{f \times \text{ft}} \cdot (\{p\} \times \mathcal{M}_{n+1} \times \Delta_{\mathcal{M}_n}).$$

Since we know by [FR, theorem 6.4(9) and lemma 8.4(1)] that

$$\{p\} \times \mathcal{M}_{n+1} \times \Delta_{\mathcal{M}_n} = (\{p\} \times \mathcal{M}_{n+1} \times \mathcal{M}_n \times \mathcal{M}_n) \cdot (X \times \mathcal{M}_{n+1} \times \Delta_{\mathcal{M}_n})$$

and $\Gamma_f \cdot (\{p\} \times \mathcal{M}_n) = \{(p, f(p))\}$, the above is equal to

$$\{p\} \times (\pi_{\mathcal{M}_{n+1}})_*((\Gamma_{\text{ft}} \times \{f(p)\}) \cdot (\mathcal{M}_{n+1} \times \Delta_{\mathcal{M}_n})).$$

Now it follows in an analogous way from [FR, theorem 6.4(9) and lemma 8.4(2)] that the latter equals

$$\begin{aligned} & \{p\} \times (\pi_{\mathcal{M}_{n+1}})_*(\Gamma_{(\text{ft},\text{ft})} \cdot (\mathcal{M}_{n+1} \times \mathcal{M}_n \times \{f(p)\})) \\ = \; & \{p\} \times (\pi_{\mathcal{M}_{n+1}})_*(\Gamma_{\text{ft}} \cdot (\mathcal{M}_{n+1} \times \{f(p)\})) \\ = \; & \{p\} \times \text{ft}^*(f(p)).\end{aligned}$$

□

Remark 3.15. The support of X^f satisfies

$$\left|X^f\right| = (f \times \mathrm{ft})^{-1}(|\Delta_{\mathcal{M}_n}|) = \{(x,y) \in X \times \mathcal{M}_{n+1} : f(x) = \mathrm{ft}(y)\}.$$

Here, one implication follows from definition of the pull-back, whereas the other is a direct consequence of proposition 3.14 together with the equality of intersection-theoretic and set-theoretic fibers of the forgetful map (proposition 3.7).

In order to conclude that $\pi_X : X^f \to X$ is a prefamily we need to prove that it satisfies the second axiom of a prefamily and that the cycle X^f is a tropical variety (i.e. has only positive weights). It is obvious that it fulfils the last condition.

Lemma 3.16. *The projection morphism $\pi_X : X^f \to X$ fulfils the second prefamily axiom.*

Proof. By remark 3.15, we can consider X^f to be equipped with the polyhedral structure

$$\mathcal{X}^f := \{\tau \times_f \sigma; \tau \in \mathcal{X}, \sigma \in \mathcal{M}\},$$

where $\mathcal{X}$ is a polyhedral structure on X, $\mathcal{M}$ is the standard polyhedral structure on $\mathcal{M}_{n+1}$ and

$$\tau \times_f \sigma := \{(x,y) \in \tau \times \sigma : f(x) = \mathrm{ft}(y)\}$$

is the set-theoretic fiber-product of τ and σ. Now let p be in some cell $\tau \times_f \sigma$, $q' \in \tau'$ for some $\tau' \geq \tau$. Consider $f(q')$ as an element of $\mathrm{Star}_{\mathcal{M}_n}(f(p))$. By lemma 3.8, it has a preimage v' under the forgetful map in some $\sigma' \geq \sigma$; so the point (q', v') is in $\mathrm{Star}_{X^f}(p)$ (and is obviously mapped to q' by π_X). □

Lemma 3.17. *All maximal cells of X^f have trivial weight 1. In particular, X^f is a tropical variety.*

Proof. Let $\mathcal{X}^f, \mathcal{X}$ be polyhedral structures of X^f, X considered in the proof of the previous lemma. If $\dim(\tau) = \dim(\pi_X(\tau)) + 1$, then we observe that

$$\{\sigma \in \mathcal{X}^f : \sigma > \tau\} \to \{\alpha \in \mathcal{X} : \alpha > \pi_X(\tau)\},\ \sigma \mapsto \pi_X(\sigma)$$

is a bijection. Since π_X maps normal vectors relative to τ to normal vectors relative to $\pi_X(\tau)$, the local irreducibility and the connectedness in codimension one of X (cf. [FR, lemma 2.4]) allow us to conclude that there is a $\lambda \in \mathbb{Z}$ such that the weight functions of X^f, X satisfy

$$\omega_{X^f}(\sigma) = \lambda \cdot \omega_X(\pi_X(\sigma)) \text{ for all maximal } \sigma \in \mathcal{X}^f.$$

Now let τ be an edge in $\mathcal{X}^f$ mapped to a point $p \in \mathcal{X}$ by π_X. After finding rational functions whose product (locally) cuts out the point p from X, it follows from the definitions of pulling-back and intersecting with rational functions that $1 = \omega_{g^*(p)}(\tau) = \lambda$, which finishes the proof. □

The following corollary is an immediate consequence of proposition 3.14 and lemmas 3.17 and 3.16.

Corollary 3.18. *For each morphism of smooth varieties $X \xrightarrow{f} \mathcal{M}_n$, we obtain a family of n-marked rational curves as*

$$(X^f \xrightarrow{\pi_X} X, t_i^\alpha),$$

where $t_i^\alpha : f^{-1}(U_\alpha) \to X^f, x \mapsto (x, s_i^\alpha \circ f(x))$ (and s_i^α is the marking on the universal family we defined above).

4. THE FIBER MORPHISM

We now want to construct a morphism into $\mathcal{M}_n$ for a given family $T \xrightarrow{g} B$ (we will omit the marking to make the notation more concise). It is actually already clear what this map should look like: It should map each b in B to the point in $\mathcal{M}_n$ that represents the fiber over b. For the pull-back family X^f defined above this gives us back the map f. For an arbitrary family however, it is not even clear that it is a morphism. In fact, we will only show that it is a so-called *pseudo-morphism* and then use the fact that B is smooth to deduce that it is a morphism.

Definition 4.1 (The fiber morphism)**.** For a family $T \xrightarrow{g} B$ we define a map

$$d_g : B \to \mathbb{R}^{\binom{n}{2}} : b \mapsto (\text{dist}_{k,l}(g^*(b)))_{k<l},$$

where the length of the path from leaf k to leaf l on the fiber is determined in the following way: The length of a bounded edge $E = \text{conv}\{p,q\}$ is defined to be the positive real number α, such that $q = p + \alpha \cdot v$, where v is the primitive lattice vector generating that edge.

We define $\varphi_g := q_n \circ d_g : B \to \mathcal{M}_n$, where $q_n : \mathbb{R}^{\binom{n}{2}} \to \mathbb{R}^{\binom{n}{2}}/\text{Im}(\phi)$ is the quotient map and ϕ maps $x \in \mathbb{R}^n$ to $(x_i + x_j)_{i<j}$.

As mentioned above, we will not be able to prove directly that φ_g is a morphism. But we can show that, in addition to being piecewise linear, it respects the balancing equations of B. Let us make this precise:

Definition 4.2 (Pseudo-morphism)**.** A map $f : X \to Y$ of tropical varieties is called a *pseudo-morphism* if there is a polyhedral structure $\mathcal{X}$ of X such that:

(1) $f_{|\tau}$ is integral affine linear for each $\tau \in \mathcal{X}$
(2) f respects the balancing equations of X, i.e. for each $\tau \in \mathcal{X}^{(\text{codim } 1)}$ if $\bar{f}$ denotes the induced piecewise affine linear map on $\text{Star}_X(\tau)$ (cf. [R, section 1.2.3]), we have
$$\sum_{\sigma>\tau} \omega_X(\sigma)\bar{f}(u_{\sigma/\tau}) = 0 \in V/V_{f(\tau)}.$$
More precisely, if we choose a $v_\sigma \in \sigma$ for each $\sigma > \tau$ and $p_0, ..., p_d \in \tau$ a basis of V_τ, such that $\overline{v_\sigma - p_0} = u_{\sigma/\tau}$ and $\sum_{\sigma>\tau} \omega_X(\sigma)(v_\sigma - p_0) = \sum_{i=1}^d \alpha_i(p_i - p_0)$ with $\alpha_1, ...\alpha_d \in \mathbb{R}$, then
$$\sum_{\sigma>\tau} \omega_X(\sigma)(f(v_\sigma) - f(p_0)) = \sum_{i=1}^{d} \alpha_i(f(p_i) - f(p_0)).$$
Note that it suffices to check this condition for a single choice of $v_\sigma, p_0, ...p_d$, since any other choice would only differ by elements from V_τ, on which f is affine linear. It is also clear that f satisfies the above properties on any refinement of $\mathcal{X}$ if it does so for $\mathcal{X}$.

As for a morphism, we denote by $\lambda_{f|\tau}$ the linear part of f on τ.

Proposition 4.3. *Let X be a smooth tropical variety, Y any tropical variety and $f : X \to Y$ a pseudo-morphism. Then f is already a morphism.*

Proof. It suffices to prove the claim for piecewise linear pseudo-morphisms $f : \text{B}(M) \to Y$ from matroid varieties to fan cycles because being a morphism is a local property and we can lift any pseudo-morphism $\text{B}(M)/L \to Y$ to a pseudo-morphism $\text{B}(M) \to Y$. By deleting parallel elements we can assume that one element subsets of the ground set $E(M)$ are flats of M. It is easy to see that f must be a pseudo-morphism with respect to the fan structure $\mathcal{B}(M)$. Now we show by induction on the rank of the flats that for all flats F we have $f(V_F) = \sum_{i\in F} f(V_{\{i\}})$. As the vectors $V_{\{i\}}$ are linearly independent this implies the claim. Let F be a flat of rank r. We choose a chain of flats of the form $\mathcal{F} = (\emptyset \subsetneq F_1 \subsetneq \ldots \subsetneq F_{r-2} \subsetneq F \subsetneq F_{r+1} \subsetneq \ldots \subsetneq F_{\mathrm{r}(M)} = E(M))$, with $\mathrm{r}(F_i) = i$. The fact that f is a pseudo-morphism translates the balancing condition around the facet $\mathcal{F}$ in $\mathcal{B}(M)$ into

$$\sum_{F_{r-2}\subsetneq G\subsetneq F \text{ flat}} f(V_G) = f(V_F) + (|\{G : F_{r-2} \subsetneq G \subsetneq F \text{ flat }\}| - 1)\cdot f(V_{F_{r-2}}).$$

Now the induction hypothesis implies the claim. □

Proposition 4.4. *For any family $T \xrightarrow{g} B$, the map $\varphi_g : B \to \mathcal{M}_n$ is a pseudo-morphism.*

Before we give a proof of this proposition we use it to prove our main theorem.

Theorem 4.5. *For any smooth variety B, we have a bijection*

$$\left\{ \begin{array}{c} \text{Families } (T \xrightarrow{g} B, r_i^\theta) \\ \text{of } n\text{-marked tropical curves} \\ \text{modulo equiv.} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{Morphisms} \\ f: B \to \mathcal{M}_n \end{array} \right\}$$

$$(T \xrightarrow{g} B, r_i^\theta) \mapsto \varphi_g$$

$$(B^f \xrightarrow{\pi_B} B, (\mathrm{id}\times(s_i^\alpha\circ f))) \mapsfrom f,$$

where $\varphi_g : B \to \mathcal{M}_n$ is the morphism constructed in definition 4.1, B^f is the tropical subvariety of $B\times\mathcal{M}_{n+1}$ introduced in definition 3.13, $\pi_B : B^f \to B$ is the projection to B, and $s_i^\alpha, i = 1,\ldots,n$ is the tropical marking of the forgetful map described in proposition 3.11.

Proof. We have already shown in corollary 3.18 and proposition 4.4 that these maps are well-defined. It is obvious that they are inverse to each other. □

The rest of this section is dedicated to proving proposition 4.4. For all the following proofs, we will assume that $\mathcal{T}$ and $\mathcal{B}$ are polyhedral structures of T and B satisfying $\mathcal{B} = \{g(\sigma), \sigma \in \mathcal{T}\}$. This is possible by [R, lemma 1.3.4].

Proposition 4.6. *The map d_g of definition 4.1 is integral affine linear on each $\tau \in \mathcal{B}$.*

Proof. We first show that d_g is affine linear on each cell: Since $\tau \in \mathcal{B}$ is closed and convex, it suffices to show that d_g is affine linear on any line segment conv$\{b, b'\} \subseteq \tau$, where $b \in \tau$ and $b' \in$ rel int(τ).

Denote by $G_\tau := \{\sigma \in \mathcal{T} : g(\sigma) = \tau\}$ and choose any $\sigma \in G_\tau$. If $\dim\sigma = \dim\tau$, then $g_{|\sigma}$ is injective and the preimage of b and b', respectively, is a point. If $\dim\sigma = \dim\tau + 1$, then, since we have chosen b' from the interior of τ, there must be a $c' \in$ rel int(σ), such that $g(c') = b'$. As $\dim\ker g_{|V_\sigma} = 1$, the preimage $C_{b'} := g_{|\sigma}^{-1}(b')$ is a (possibly unbounded) line segment. The fiber $C_b := g_{|\sigma}^{-1}(b)$ is either a parallel line segment or a point.

For now we assume both fibers to be bounded. We claim that for each such σ the map d_σ : conv$(\{b, b'\}) \to \mathbb{R}$ which assigns to each $b_\lambda := b + \lambda(b' - b), \lambda \in [0, 1]$ the length of the fiber $g_{|\sigma}^{-1}(b_\lambda)$ is affine linear. The map d_g will then be a sum of these maps. First we argue that the endpoints of the fibers $C_b, C_{b'}$ must lie in the same faces of σ: Denote by q_1, q_2 the endpoints of $C_{b'}$, lying in faces $\sigma_1, \sigma_2 < \sigma$, so $C_{b'} =$ conv$(\{q_1, q_2\}); q_1 \in \sigma_1, q_2 \in \sigma_2$. Then $g(\sigma_i) \subseteq g(\sigma) = \tau$ and $b' \in g(\sigma_i) \cap$ rel int(τ). Hence $g(\sigma_i) = \tau$ and there must be $p_1 \in \sigma_1, p_2 \in \sigma_2$ which map to b. Hence, since they lie in proper faces, they must be the endpoints of C_b and we conclude:

$$C_b = \text{conv}(\{p_1, p_2\}); p_1 \in \sigma_1, p_2 \in \sigma_2.$$

It immediately follows that

$$C_{b_\lambda} = \text{conv}(\{\underbrace{p_1 + \lambda(q_1 - p_1)}_{\in\sigma_1}, \underbrace{p_2 + \lambda(q_2 - p_2)}_{\in\sigma_2}\}) \text{ for all } \lambda \in [0,1].$$

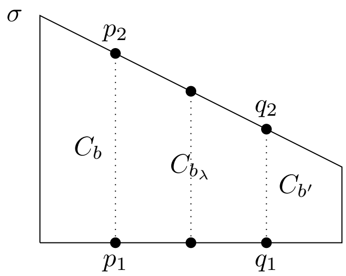

FIGURE 4.1. An illustration of the fibers $C_b, C_{b'}$ and C_{b_λ}

Denote by v the primitive vector generating the kernel of $g_{|V_\sigma}$. Then

$$(q_2 - q_1) = \alpha \cdot v, (p_2 - p_1) = \beta \cdot v$$

for some $\alpha, \beta \in \mathbb{R}$. Now the length of a fiber C_{b_λ} is determined by the difference of its endpoints

$$\begin{aligned}(p_2 + \lambda(q_2 - p_2)) - (p_1 + \lambda(q_1 - p_1)) &= (p_2 - p_1) + \lambda((q_2 - q_1) - (p_2 - p_1)) \\ &= v \cdot (\beta + \lambda \cdot (\alpha - \beta)).\end{aligned}$$

Hence we have

$$d_\sigma(b_\lambda) = \beta + \lambda \cdot (\alpha - \beta),$$

which is an affine linear map.

We also have to consider the case that one fiber is unbounded (i.e. a subset of a leaf). In this case there is no length to consider; we only have to show that C_b is unbounded if and only if $C_{b'}$ is. We have already proven that every endpoint of $C_{b'}$ induces an endpoint of C_b in the same face. Hence, if C_b is unbounded, i.e. has only one or no endpoint, so does $C_{b'}$. For the other direction, assume $C_{b'}$ has only one endpoint q and let p be any point in C_b. We can rewrite this as

$$C_{b'} = \{q + \alpha \cdot v; \alpha \geq 0\} \subseteq \sigma.$$

Since σ is convex, we have

$$\begin{aligned}\sigma &\ni (1 - \lambda) \cdot p + \lambda(q + \alpha \cdot v) \\ &= ((1 - \lambda) \cdot p + \lambda q) + \alpha \cdot \lambda \cdot v \in C_{b_\lambda} \\ &\text{for all } \lambda \in [0,1], \alpha \geq 0.\end{aligned}$$

In particular, C_{b_λ} is unbounded for all $\lambda > 0$.

Since g is continuous, $g_{|\sigma}^{-1}(\text{conv}(\{b, b'\}))$ must be a closed set. Hence C_b must be unbounded as well.

For both the bounded and unbounded case, this description of the fibers also gives us an affine linear map $C_{b_\rho} \to C_{b_\lambda}$ for all $\lambda \leq \rho \in [0,1]$. If $\rho, \lambda > 0$, this map is even bijective (since both fibers are line segments). We can glue together all these maps for each $\sigma \in G_\tau$ to obtain a homeomorphism $t_{\rho,\lambda} : g^{-1}(b_\rho) \to g^{-1}(b_\lambda)$ which is an affine linear map on each edge. If $\lambda = 0, \rho > 0$, we still obtain a map $t_{\rho,\lambda}$ which might contract certain edges to a point.

We can furthermore assume that there exists a $\theta \in \Theta$, such that $b_\lambda, b_\rho \in U_\theta$ (otherwise cover conv($\{b_\lambda, b_\rho\}$) with finitely many U_θ and use induction). Now affine linearity of s_i^θ implies that the leaves which are identified under $t_{\lambda,\rho}$ are also marked by the same s_i. In other words, $g^{-1}(b_\lambda), g^{-1}(b_\rho)$ have the same combinatorial type if $\lambda, \rho > 0$. If $\lambda = 0$, then $C_{b_\lambda} = C_b$ either has the same combinatorial type as C_{b_ρ} or is obtained by contracting some edges of the latter curve.

Denote by $G_{b_\lambda}(k,l)$ the set of all cones in G_σ of dimension $(\dim \tau + 1)$, such that $g_{|\sigma}^{-1}(b_\lambda)$ is contained in the path from k to l in the curve $g^{-1}(b_\lambda)$. Then we have

$$\text{dist}_{k,l}(g^{-1}(b_\lambda)) = \sum_{\sigma \in G_{b_\lambda}(k,l)} d_\sigma(b_\lambda).$$

Since we know that d_σ is affine linear, it suffices to show that $G_{b_\lambda}(k,l) = G_{b_\rho}(k,l)$ for all $\lambda, \rho \in [0,1]$, which immediately follows from the fact that the map $t_{\lambda,\rho}$ identifies equally marked leaves and hence edges lying on the same path.

It remains to show that d_g is an integral map: We want to show that for $b, b' \in \tau$ (of dimension k), such that $b - b' \in \Lambda_\tau$, we have $d_g(b') - d_g(b) \in \mathbb{Z}^{\binom{n}{2}}$. Note that the lattice elements in $\mathcal{M}_n$ are exactly the points representing curves with integer edge lengths, so φ_g will be an integer map as well. Choose σ, such that the fiber of b' in σ is a bounded line segment. We have already shown that we have two endpoints p, q of both fibers lying in the same face $\sigma' < \sigma$, hence in the same hypersurface of V_σ which is defined by an integral equation

$$h(x) = \alpha;\ h \in \Lambda_\sigma^\vee, \alpha \in \mathbb{R}.$$

By surjectivity of $\bar{\lambda}_{g|\tau} : \Lambda_\sigma \to \Lambda_\tau$, we have

$$\Lambda_\sigma \cong \Lambda_\tau \times \langle v \rangle_\mathbb{Z}$$

for some primitive integral vector v (which generates $\ker \lambda_{g_\tau}$).

Under this isomorphism we write the coordinates of p, q and h as

$$\begin{aligned} p &= (p_1, \dots, p_k, p_v) \\ q &= (q_1, \dots, q_k, q_v) \\ h(x_1, \dots, x_k, x_v) &= h_1 x_1 + \dots + h_k x_k + h_v x_v, \end{aligned}$$

where $p_i - q_i \in \mathbb{Z}$ for $i = 1, \dots, k$, $h_j \in \mathbb{Z}$ for all j and $h_v \neq 0$ (since otherwise λ_g would not be injective on the corresponding hypersurface). Now the identity $h(p-q) = 0$ transforms into

$$\begin{aligned} 0 &= \sum_{i=1}^k (q_i - p_i) h_i + (q_v - p_v) h_v \\ &= \underbrace{\sum_{i=1}^k (b' - b)_i h_i}_{\in \mathbb{Z}} + (q_v - p_v) \underbrace{h_v}_{\in \mathbb{Z}}. \end{aligned}$$

Hence $q_v - p_v \in \mathbb{Q}$ and $q - p \in \Lambda_\sigma \otimes_\mathbb{Z} \mathbb{Q}$.

So there exists a minimal $k \in \mathbb{N}$, such that $k \cdot (q-p) \in \Lambda_\sigma$. In particular, $k \cdot (q-p)$ is primitive. Assume $k > 1$. Then $\bar{\lambda}_g(k \cdot (q-p)) = k \cdot (b' - b)$. By surjectivity of $\bar{\lambda}_g$, there exists an $a \in \Lambda_{\sigma'}$, such that $\bar{\lambda}_g(a) = b' - b$. This implies $\bar{\lambda}_g(k \cdot a) = \bar{\lambda}_g(k \cdot (q-p))$. Since $\bar{\lambda}_g$ is injective on $\Lambda_{\sigma'}$, we must have $k \cdot a = k \cdot (q-p)$, which is a contradiction, since the latter is primitive. Hence $k = 1$ and $q - p \in \Lambda_\sigma$.

Finally we obtain

$$\Lambda_\sigma \ni (q' - p') - (q - p) = (d_\sigma(b') - d_\sigma(b)) \cdot v.$$

Hence, since v is primitive, $d_\sigma(b')-d_\sigma(b) \in \mathbb{Z}$ and the same follows for $d_g(b')-d_g(b)$. $\square$

The first part of the preceding proof also gives us the following result as a byproduct, which boils down to saying that fibers over the interior of a cell have the same combinatorial type:

Corollary 4.7. *For each $\tau \in \mathcal{B}, b \in \tau, b' \in \text{rel int}(\tau)$, there exists a piecewise linear, continuous and surjective map $t_{b',b} : g^*(b') \to g^*(b)$ for which the following holds:*

(1) *If $b, b' \in \text{rel int}(\tau)$, then $t_{b',b}$ is a homeomorphism.*
(2) *If $l_i(b), l_i(b')$ denote the i-th leafs of the respective fiber, then*
$$t_{b',b}(l_i(b')) = l_i(b).$$
(3) *On each edge e of $g^*(b')$, $t_{b',b}$ is affine linear and e is either mapped bijectively onto its image or to a single vertex. In particular, vertices are mapped to vertices.*
(4) *If e_1, e_2 are two different edges of $g^*(b')$, then*
$$|t_{b',b}(e_1) \cap t_{b',b}(e_2)| \leq 1.$$
(5) *For each $\sigma \in G_\tau$ we have*
$$t_{b',b}(|g^*(b)| \cap \sigma) \subseteq \sigma.$$

In fact the part of the proof of proposition 4.6 which implies corollary 4.7 does not use the last condition on a prefamily; therefore we can use it to prove the following lemma.

Lemma 4.8. *Let $g : T \to B$ (with B smooth) be a morphism of tropical varieties which satisfies conditions (1) and (2) on a prefamily. Then*
$$\Pi : \{\sigma \in \mathcal{T} : \sigma > \tau\} \to \{\alpha \in \mathcal{B} : \alpha > g(\tau)\},\ \sigma \mapsto g(\sigma)$$
is a bijection if $\tau \in \mathcal{T}$ is a cell on which g is not injective. In this case we have furthermore that $\lambda_{g|\tau} : \Lambda_\tau \to \Lambda_{g(\tau)}$ is surjective. Moreover, all maximal cells in $\mathcal{T}$ have trivial weight 1.

Proof. As in the proof of lemma 3.6 we can assume that g is a linear function and that $\mathcal{T}, \mathcal{B}$ are fan structures of the fan cycle T and the matroid variety modulo lineality space B such that $g(\tau) \in \mathcal{B}$ for all cones $\tau \in \mathcal{T}$.

For surjectivitiy of Π, let $\alpha > g(\tau)$. Choose elements $p \in \text{rel int}(g(\tau)), q \in \text{rel int}(\alpha)$. By corollary 4.7, $t_{q,p}^{-1}(g^*(p) \cap \tau)$ is a line segment. Let σ be any cone containing an infinite subset of this. In particular, $g(\sigma) = \alpha$. Then we can use the last statement of 4.7 to see that we must have $\sigma > \tau$.

For injectivity, assume that $g(\sigma_1) = g(\sigma_2) = \alpha > g(\tau)$ for two distinct $\sigma_i > \tau$. Then $t_{q,p}(|g^*(q)| \cap \sigma_i) = |g^*(p)| \cap \tau$ for $i = 1, 2$, which is a contradiction to the fourth statement of 4.7.

As B is locally irreducible and connected in codimension 1 (cf. [FR, lemma 2.4]) the above bijection implies that there is an integer λ such that $\omega_{\mathcal{T}}(\sigma) = \lambda \cdot \omega_{\mathcal{B}}(g(\sigma))$ for all maximal cells in $\sigma \in \mathcal{T}$. For the last part, we thus need to show that $\lambda = 1$ and that $g(v_{\sigma/\tau}) = v_{g(\sigma)/g(\tau)}$ if g is not injective on τ, i.e. g maps normal vectors to normal vectors. It is clear that $g(v_{\sigma/\tau})$ is a multiple of $v_{g(\sigma)/g(\tau)}$; as B is a matroid fan, it follows that $g(v_{\sigma/\tau}) = \lambda_\tau \cdot v_{g(\sigma)/g(\tau)}$ for some $\lambda_\tau \in \mathbb{Z}_{>0}$ which does not depend on σ. Let $\varphi_1 \ldots, \varphi_{\dim(B)}$ be rational functions with $\varphi_1 \cdots \varphi_{\dim(B)} \cdot B = \{0\}$ (cf. proof of lemma 3.6). Comparing the weight formulas for intersection products of $\omega_{\varphi_1 \cdots \varphi_{\dim(B)} \cdot B}(\{0\})$ and $\omega_{g^*\varphi_1 \cdots g^*\varphi_{\dim(B)} \cdot T}(\tau)$ for an edge $\tau \in \mathcal{T}$, we see that $\lambda = 1$ and $\lambda_\beta = 1$ for all cones $\beta \geq \tau$. $\square$

Before we can prove that φ_g is a pseudo-morphism, we need to fix a few notations:

Notation 4.9.

- Let $\tau \in \mathcal{B}^{(\text{codim } 1)}$. Choose $p_0, p_1, \ldots, p_d \in$ rel int(τ), such that $\{p_i - p_0; i = 1, \ldots, d\}$ is a basis of V_τ. Furthermore, for each $\sigma > \tau$, choose a point $v_\sigma \in$ rel int(σ), such that $v_\sigma - p_0$ is a representative of $u_{\sigma/\tau}$. We can assume that this is possible, since there always exists a $v_\sigma \in$ rel int$(\sigma), q_\sigma \in \mathbb{Q}$, such that $v_\sigma - p_0 = q_\sigma \cdot u_{\sigma/\tau}$ modulo V_τ. We can then make our choice such that $q_\sigma = q_{\sigma'} =: q$ for all $\sigma, \sigma' > \tau$, so

$$\sum_{\sigma > \tau} \omega_B(\sigma) \cdot u_{\sigma/\tau} = \frac{1}{q} \sum_{\sigma > \tau} \omega_B(\sigma)(v_\sigma - p_0).$$

  Hence the left hand side is in V_τ if and only if the right hand side is.
  So we obtain that

$$\sum_{\sigma > \tau} \omega_B(\sigma)(v_\sigma - p_0) = \sum_{j=1}^{d} \alpha_j (p_j - p_0)$$

  for some $\alpha_i \in \mathbb{R}$.
- Corollary 4.7 justifies the following definitions:
  - For $k, l \in [n]$, denote by $q_1(k,l), \ldots, q_r(k,l) \in T$ the vertices of the fiber $g^*(p_0)$ which lie on the path from k to l (Actually, r also depends on the choice of k and l, but we will omit that to make notations simpler). Where k and l are clear from the context, we will also write $q_1, \ldots, q_r$.
  - The fiber of p_j has the same combinatorial type as $g^*(p_0)$, so for $j = 1, \ldots, d$, denote by $q_i^{(j)}, i = 1, \ldots, r$ the i-th vertex in the fiber of p_j (Again, this actually depends on k, l).
  - Let $\sigma > \tau$. The preimage of $q_i(k,l)$ under t_{v_σ, p_0} contains a certain number of vertices lying on the path from k to l, the first and last of which we denote by $q_{i,k}^\sigma$ and $q_{i,l}^\sigma$ respectively.
  - Let $w_i, i = 1, \ldots, r-1$ be the primitive direction vector of the bounded edge from q_i to q_{i+1}. We define the lengths $e_i, e_i^{(j)}, e_i^\sigma > 0$ of the corresponding edges via:

$$\begin{aligned} q_{i+1} &= q_i + e_i \cdot w_i, \\ q_{i+1}^{(j)} &= q_i^{(j)} + e_i^{(j)} \cdot w_i, \\ q_{i+1,k}^\sigma &= q_{i,l}^\sigma + e_i^\sigma \cdot w_i. \end{aligned}$$

  - In addition we fix $w_0 := -v_k, w_r := v_l$, where v_k and v_l are the primitive direction vectors of the leaves marked k and l.
  - For $i = 1, \ldots, r$, denote by $e_{i,t}^\sigma(k,l), t = 1, \ldots, r(i,k,l,\sigma)$ the length of the edges on the path from $q_{i,k}^\sigma$ to $q_{i,l}^\sigma$.
- We define

$$\Delta_{k,l}^i := \sum_{\sigma > \tau} \omega(\sigma)(e_i^\sigma - e_i) - \sum_{j=1}^{d} \alpha_j (e_i^{(j)} - e_i);\ i = 1, \ldots, r-1,$$

$$d_{k,l}^i := \sum_{\sigma > \tau} \omega(\sigma) \left(\sum_{t=1}^{r(i,k,l,\sigma)} e_{i,t}^\sigma(k,l) \right);\ i = 1, \ldots, r.$$

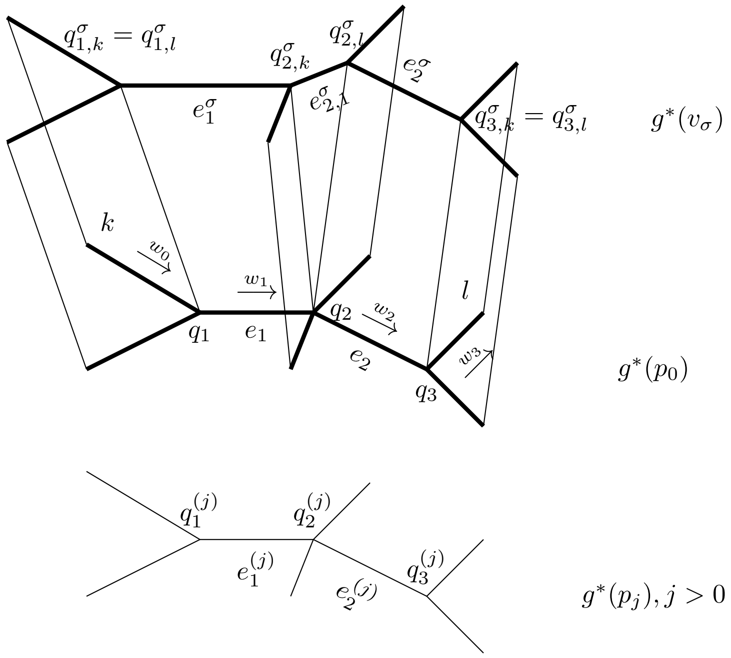

FIGURE 4.2. An illustration of the chosen notation

Summing up over all length differences at each vertex and edge and exchanging sums gives us the following equation:

$$\delta_{k,l}(\tau) := \sum_{\sigma > \tau} \omega(\sigma)(\text{dist}_{k,l}(v_\sigma) - \text{dist}_{k,l}(p_0)) - \sum_{j=1}^{d} \alpha_j (\text{dist}_{k,l}(p_j) - \text{dist}_{k,l}(p_0))$$
$$= \sum_{i=1}^{r-1} (d_{k,l}^i + \Delta_{k,l}^i) + d_{k,l}^r. \tag{4.1}$$

Remark 4.10. To prove that φ_g is a pseudo-morphism, we need to show that $(\delta_{k,l})_{k<l} \in \text{Im}(\Phi_n)$, i.e. it is 0 in $\mathcal{M}_n$. The idea for the proof is the following: A cell ρ' that maps non-injectively onto some $\tau \in \mathcal{B}$ (and thus carries edges of the fibers of the p_i) is a codimension one cell in T. We will show that the vertices of the fibers in the surrounding maximal cones can be used to express the balancing condition of ρ', such that the coefficients coincide with the balancing equation of τ (lemma 4.11). However, $\dim \rho' = \dim \tau + 1$, so we have an additional generator w_i of $V_{\rho'}$ (that generates the kernel of $g_{|\rho'}$). We will then show that the quantities $\Delta_{k,l}^i$ and $d_{k,l}^i$ we defined above can be expressed in terms of the coordinates of the balancing equation in this element w_i (lemma 4.13). These expressions will then yield $\delta_{k,l}$ as an alternating sum where everything except the w_i-coefficients of the vertices at the leaves k and l cancels out.

Lemma 4.11. *For each $k \neq l \in [n]$, each $i = 1, \ldots, r$, there exist $\xi_i(k,l), \chi_i(k,l) \in \mathbb{R}$, such that*

$$\sum_{j=1}^{d} \alpha_j (q_i^{(j)} - q_i) = \sum_{\sigma > \tau} \omega(\sigma)(q_{i,l}^\sigma - q_i) + \xi_i(k,l) \cdot w_i, \tag{4.2}$$

$$\sum_{j=1}^{d} \alpha_j (q_i^{(j)} - q_i) = \sum_{\sigma > \tau} \omega(\sigma)(q_{i,k}^\sigma - q_i) + \chi_i(k,l) \cdot w_{i-1}. \tag{4.3}$$

Proof. By corollary 4.7, $q_i, q_i^{(1)}, \dots, q_i^{(d)}$ are all contained in the relative interior of the same minimal cone $\rho \in G_\tau$. Since the q_i are vertices, $\dim \rho = \dim \tau$, since otherwise, the kernel of $g_{|V_\rho}$ would be spanned by all edges emanating from q_i and thus have a dimension higher than 1.

Now let $G_\tau \ni \rho' > \rho$ be the adjacent cone, such that the kernel of $g_{|V_{\rho'}}$ is spanned by w_i (i.e. ρ' contains (part of) the i-th edge). By lemma 4.8, there is a bijection

$$\Pi : \{\sigma' > \rho'\} \to \{\sigma > \tau\}; \sigma' \mapsto g(\sigma').$$

Since $\bar{\lambda}_g$ is surjective, we have the following isomorphisms:

$$\begin{aligned} \Lambda_{\sigma'} &\cong \Lambda_{g(\sigma')} \times \langle w_i \rangle \text{ for all } \sigma' > \rho', \\ \Lambda_{\rho'} &\cong \Lambda_\tau \times \langle w_i \rangle \\ \implies \Lambda_{\sigma'} / \Lambda_{\rho'} &\cong \Lambda_{g(\sigma')} / \Lambda_\tau. \end{aligned}$$

Since $t_{v_\sigma, p_j}(q_{i,l}^\sigma) = q_i^{(j)}, t_{v_\sigma, p_0}(q_{i,l}^\sigma) = q_i$ and both maps preserve polyhedra, all these vertices are contained in a common polyhedron which must be a face of $\sigma' := \Pi^{-1}(\sigma)$. Hence $q_{i,l}^\sigma - q_i$ is a representative of $u_{\sigma'/\rho'} = (u_{\sigma/\tau}, 0)$. This implies

$$\sum_{\sigma > \tau} \omega(\sigma)(q_{i,l}^\sigma - q_i) \in V_{\rho'}.$$

We also have

$$\sum_{j=1}^d \alpha_j (q_i^{(j)} - q_i) \in V_\rho \subseteq V_{\rho'}.$$

and since both are mapped to the same element $\sum_{\sigma > \tau} \omega(\sigma)(v_\sigma - p_0) = \sum_{j=1}^d \alpha_j (p_j - p_0)$ under g, they can only differ by an element from $\ker g_{|V_{\rho'}} = \langle w_i \rangle$, which implies the first equation. Exchanging k and l gives the second equation. □

Remark 4.12. It is obvious from the equations themselves, that $\chi_1(k,l) = \chi_1(k)$ actually only depends on k (since $w_0 = v_k$ is the same for all l). Similarly, ξ_r only depends on l and if we reverse the path direction, we find that

$$\chi_1(k) = \chi_1(k,l) = -\xi_r(l,k).$$

Lemma 4.13. *For each $k \neq l \in [n]$ we have*

$$\begin{aligned} \Delta_{k,l}^i &= \xi_i - \chi_{i+1} \text{ for all } i = 1, \dots, r-1, \\ d_{k,l}^i &= \chi_i - \xi_i \text{ for all } i = 1, \dots, r. \end{aligned}$$

Proof. If we subtract equation (4.2) from (4.3) for $i+1$, we obtain

$$\begin{aligned} &\sum_{j=1}^d \alpha_j (\underbrace{(q_{i+1}^{(j)} - q_i^{(j)}) - (q_{i+1} - q_i)}_{=(e_i^{(j)} - e_i) \cdot w_i}) \\ = &\sum_{\sigma > \tau} \omega(\sigma)(\underbrace{(q_{i+1,k}^\sigma - q_{i,l}^\sigma) - (q_{i+1} - q_i)}_{=(e_i^\sigma - e_i) \cdot w_i}) + (\chi_{i+1} - \xi_i) \cdot w_i. \end{aligned}$$

Factoring out w_i we obtain

$$0 = \Delta_{k,l}^i - \xi_i + \chi_{i+1}.$$

For the second equation let $i \in \{1, \dots, r\}$ be arbitrary. Since $g^*(p_0)$ is a smooth curve, it is locally at q_i isomorphic to $L_1^{\text{val}(q_i)}$. Denote by $z_1, \dots, z_s$ the direction vectors of the outgoing edges, w.l.o.g. $z_1 = -w_{i-1}, z_s = w_i$. Now each edge E in the preimage of q_i under t_{v_σ, p_0} induces a partition of the set $\{1, \dots, s\} = I_E \dot\cup I_E^c$ such that $x, y \in \{1, \dots, s\}$ are contained in the same set if and only if the path from z_x to z_y does not pass through E

(i.e. we separate the z_i "on one side of E" from the others). It is easy to see that, due to the balancing condition of the curve, the direction vector of E must be

$$w_E = \pm \sum_{x \in I_E} z_x = \mp \sum_{y \in I_E^c} z_y,$$

depending on the choice of orientation (one can, for example, see this by induction on the number of edges). Now assume E lies on the path from k to l (i.e. in $t^{-1}_{v_\sigma,p_0}(q_i)$ it lies on the path from $q^\sigma_{i,k}$ to $q^\sigma_{i,l}$). Choose I_E, such that $1 \notin I_E \ni s$, i.e. w_E points towards l. Denote by $E_1^\sigma, \ldots, E^\sigma_{r(i,k,l,\sigma)}$ the sequence of edges from $q^\sigma_{i,k}$ to $q^\sigma_{i,l}$. Subtracting equation

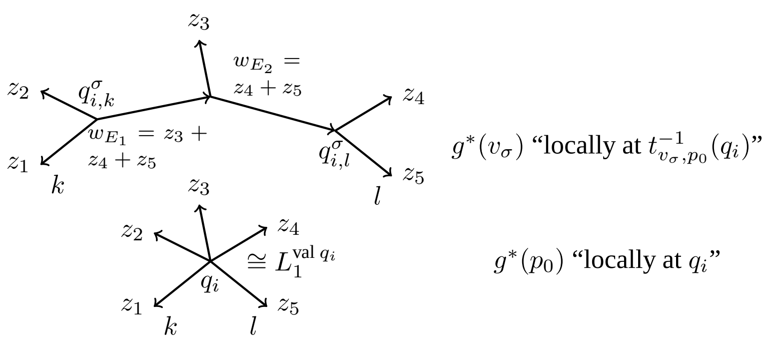

FIGURE 4.3. The direction vector of an edge is determined by the z_i lying "behind" it.

(4.2) from (4.3) for the same i, we obtain

$$\begin{aligned}
0 &= \sum_{\sigma > \tau} \omega(\sigma)(q^\sigma_{i,l} - q^\sigma_{i,k}) + \xi_i \cdot w_i - \chi_i \cdot w_{i-1} \\
&= \sum_{\sigma > \tau} \omega(\sigma) \left(\sum_{t=1}^{r(i,k,l,\sigma)} e^\sigma_{i,t} \cdot w_{E_t} \right) + \xi_i \cdot z_s + \chi_i \cdot z_1 \\
&= z_s \cdot \left(\sum_{\sigma>\tau} \omega(\sigma) \left(\sum_{t=1}^{r} e^\sigma_{i,t} \right) \right) + \underbrace{\sum_{\sigma>\tau} \omega(\sigma) \left(\sum_{t=1}^{r} e^\sigma_{i,t} \left(\sum_{x \in I_{E_t} \setminus \{s\}} z_x \right) \right)}_{:=R, \text{ contains neither } z_1 \text{ nor } z_s} \\
&\quad + \xi_i \cdot z_s + \chi_i \cdot z_1 \\
&= z_s \cdot (d^i_{k,l} + \xi_i) - \chi_i \left(\sum_{x \neq 1} z_x \right) + R.
\end{aligned}$$

Since z_1 does no longer appear in this equation and $\{z_x, x \neq 1\}$ is linearly independent by smoothness, the coefficient of z_s must be 0:

$$0 = d^i_{k,l} + \xi_i - \chi_i.$$

□

Proof of theorem 4.4. By equation (4.1) and lemma 4.13 we have

$$\begin{aligned}\delta_{k,l}(\tau) &= \sum_{i=1}^{r-1}(d_{k,l}^i + \Delta_{k,l}^i) + d_{k,l}^r \\ &= \chi_1(k,l) - \xi_r(k,l) \\ &\overset{4.12}{=} \chi_1(k,l) + \chi_1(l,k) \\ &\overset{4.12}{=} \chi_1(k) + \chi_1(l).\end{aligned}$$

Hence

$$(\delta_{k,l}(\tau))_{k<l} = \Phi_n((\chi_1(r))_{r=1,\dots,n}).$$

□

5. Equivalence of families

In the classical case, two families $T \xrightarrow{g} B, T' \xrightarrow{g'} B$ are equivalent if there is an isomorphism $\psi : T \to T'$ that commutes with the morphisms and markings. Such an isomorphism hence automatically induces isomorphisms between the fibers $g^*(p)$ and $g'^*(p)$ of a point p in B.

In fact, the last statement already uniquely fixes the map ψ, so for any two equivalent families of n-marked tropical curves we obtain a bijective map $T \to T'$ that commutes with g, g' and the markings by identifying the fibers over each point p (which are isomorphic by definition). We would like to see if this map is in fact a morphism. Again, we will only be able to show that it is a pseudo-morphism and since in general we can not assume T to be smooth, we cannot give a stronger statement.

Definition 5.1. Let $T \xrightarrow{g} B, T' \xrightarrow{g'} B$ be two equivalent families of n-marked tropical curves. Now for each point p in B there is a unique isomorphism (of tropical curves)

$$\psi_p : g^*(p) \to g'^*(p)$$

(i.e. it identifies equally marked leaves and is linear of slope 1 on each edge). We define a map

$$\begin{aligned}\psi : T &\to T' \\ t &\mapsto \psi_{g(t)}(t).\end{aligned}$$

Theorem 5.2. *The map ψ is a bijective pseudo-morphism whose inverse is also a pseudo-morphism. In particular, if T or T' is smooth, ψ is an isomorphism.*

Proof. Since the construction of ψ is symmetric, it is clear that the inverse of ψ is a pseudo-morphism if ψ itself is one. Also, by proposition 4.3, it is an isomorphism if any of T or T' is smooth.

First, we prove that ψ is piecewise integral affine linear: Let $\tau \in \mathcal{T}$ and choose $t \in \tau, t' \in$ rel int(τ). Again, it suffices to show that ψ is affine linear on the line segment conv$\{t, t'\}$.

By corollary 4.7, t and t' lie on edges of the corresponding fibers which have the same direction vector w. Select vertices p, p' of these edges, such that $t = p + \alpha \cdot w, t' = p' + \alpha' \cdot w$ for $\alpha, \alpha' \geq 0$.

Denote by $q := \psi(p), q' := \psi(p')$ and let ξ be the direction vector of the corresponding edge in T'. Hence

$$\begin{aligned}\psi(t) &= \psi(p + \alpha \cdot w) = q + \alpha \cdot \xi \\ \psi(t') &= \psi(p' + \alpha' \cdot w) = q' + \alpha' \cdot \xi\end{aligned}$$

and using the fact that any convex combination of p and p' must by 4.7 again be a vertex, it follows that

$$\begin{aligned}\psi(t+\gamma(t'-t)) &= \psi((p+\gamma(p'-p))+w\cdot(\alpha+\gamma(\alpha'-\alpha))) \\ &= (q+\gamma(q'-q))+\xi\cdot(\alpha+\gamma(\alpha'-\alpha)) \\ &= \psi(t)+\gamma(\psi(t')-\psi(t))\end{aligned}$$

for any $\gamma \in [0,1]$. Hence ψ is affine linear. Using the fact that it has slope 1 on each edge of a fiber and that $g' \circ \psi = g$, it is easy to see that it respects the lattice.

It remains to see that ψ is a pseudo-morphism, so let τ be a codimension one cell of T. We distinguish two cases:

- $g_{|\tau}$ is injective: Then $g(\tau)$ is a maximal cell of B, so the adjacent maximal cells $\sigma > \tau$ are also mapped to $g(\tau)$. So if we take a point $p \in \text{rel int}(\tau)$, the normal vectors $v_{\sigma/\tau} - p$ correspond to normal vectors of the edges of the fiber $g^*(g(p))$ adjacent to p (after proper refinement). Since the fiber is smooth, these add up to 0 and by definition of ψ, so do their images $\psi(v_{\sigma/\tau}) - \psi(p)$.
- $g_{|\tau}$ is not injective: Hence the fiber in τ over a generic point $p_0 \in g(\tau)$ is contained in the m-th edge on the path from some leaf k to some leaf l (it doesn't really matter, which one). Choose $p_0, \dots, p_d, v_\sigma$ in $g(\tau)$ and its adjacent cells $g(\sigma), \sigma > \tau$ as defined in 4.9. We now use the shorthand notation q_0, q_j, q_σ for the m-th vertex point of the fibers of p_0, p_j and v_σ. Now lemma 4.11 tells us that $q_\sigma - q_0$ is actually a normal vector of σ with respect to τ and that its balancing equation reads
$$\sum_{\sigma>\tau} \omega(\sigma)(q_\sigma - q_0) = \sum_{j=1}^{d} \alpha_j (q_j - q_0) - \xi_m^T(k,l) \cdot w_m.$$
Now the image of q_0 under ψ is by definition the m-th nodal point of the fiber $g'^*(p_0)$, so we also get
$$\sum_{\sigma>\tau} \omega(\sigma)(\psi(q_\sigma) - \psi(q_0)) = \sum_{j=1}^{d} \alpha_j (\psi(q_j) - \psi(q_0)) - \xi_m^{T'}(k,l) \cdot \psi(w_m).$$
Hence, to prove that ψ is a pseudo-morphism, it remains to show that $\xi_m^{T'}(k,l) = \xi_m^T(k,l)$.
  By the proof of proposition 4.4, we know that
$$\delta_{k,l}(\tau) = \Phi_n((\chi_1^T(k))_{k=1,\dots,n}) = \Phi_n((\chi_1^{T'}(k))_{k=1,\dots,n}).$$
Since the left side is independent on the choice of family by definition (it is defined only in terms of lengths of fibers) and Φ_n is injective, we must have $\chi_1^T(k) = \chi_1^{T'}(k)$ for any k. Using the fact that $d_{k,l}^i$ and $\Delta_{k,l}^i$ are also independent of the choice of family and applying lemma 4.13 inductively, we finally see that
$$\chi_i^T(k,l) = \chi_i^{T'}(k,l) \text{ and } \xi_i^T(k,l) = \xi_i^{T'}(k,l)$$
for any possible i, k, l.

□

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