Intersection Theory on Linear Subvarieties of Toric Varieties

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We give a complete description of the cohomology ring $A^*(\overline{Z})$ of a compactification of a linear subvariety Z of a torus in a smooth toric variety whose fan Σ is supported on the tropicalization of Z. It turns out that cocycles on $\overline{Z}$ canonically correspond to Minkowski weights on Σ and that the cup product is described by the intersection product on the tropical matroid variety $\operatorname{Trop}(Z)$.

1 Introduction

In [FS97], Fulton and Sturmfels show that the cohomology ring of a complete toric variety X_Σ is isomorphic to the ring of Minkowski weights on the corresponding fan Σ. The product of two Minkowski weights was described by the so-called "fan displacement rule". Later it was shown in [Kat12] and [Rau08] that the product described by this rule is equal to the intersection product in tropical geometry introduced in [AR10]. Here, we prove an analogous result about completions of linear subvarieties of tori in toric varieties. Let T be a torus over an algebraically closed field K with constant valuation, and let M be its character lattice. By linear subvarieties of T we mean those varieties which are cut out by an ideal $I \trianglelefteq K[M]$ which is linear in $K[\mathbb{Z}^n]$ in the canonical sense after choosing an appropriate isomorphism $K[M] \cong K[\mathbb{Z}^n]$. We compactify Z in a smooth toric variety $X = X_\Sigma$ whose fan has support $|\Sigma| = |\operatorname{Trop}(Z)|$. This restriction on Σ is made to ensure that $\overline{Z}$ intersects the torus orbits of X properly. Our main theorem states that in this situation the cohomology ring $A^*(\overline{Z})$ can be described by the group $M^*(\Sigma)$ of Minkowski weights on Σ. Furthermore, the multiplication is induced by the intersection product on the tropical matroid variety $\operatorname{Trop}(Z)$, which has been introduced in [FR13] and [Sha13].

Theorem 1.1. *Let Z be a linear subvariety of the torus T and let Σ be a unimodular fan with $|\Sigma| = |\operatorname{Trop}(Z)|$. Then there is a canonical ring isomorphism $\mathcal{I}_Z : A^*(\overline{Z}) \to M^*(\Sigma)$, where the ring structure on $M^*(\Sigma)$ is induced by the tropical intersection product on* $\operatorname{Trop}(Z)$.

The proof of this theorem uses algebraic as well as tropical intersection theory. We begin Section 2 by reviewing the basic constructions of the tropical theory. Afterwards, we look at Minkowski weights in greater detail. The main result of this section is of technical nature, but will be very useful in the proof of Theorem 1.1. For its statement let Σ be a complete unimodular fan in $\mathbb{R}^n$ which has a subfan $\widetilde{\Sigma} \subseteq \Sigma$ that defines a matroid variety A after assigning weight 1 to all its maximal cones. Writing $Z^*(\mathbb{R}^n)$ and $Z^*(A)$ for the graded rings of tropical fan cycles in $\mathbb{R}^n$ and A, respectively, we get a map $Z^*(\mathbb{R}^n) \to Z^*(A)$ which assigns to $C \in Z^*(\mathbb{R}^n)$ the intersection product $C \cdot A$ in $\mathbb{R}^n$. Using piecewise polynomials, it easily follows from the results of [Fra13] that this map is a ring epimorphism. But as we are interested in Minkowski weights, the question arises if this still holds if we replace $Z^*(\mathbb{R}^n)$ by $M^*(\Sigma)$ and $Z^*(A)$ by $M^*(\widetilde{\Sigma})$. Theorem 2.2 provides an affirmative answer to this. As a direct consequence we obtain a new proof of the fact that $M^*(\widetilde{\Sigma})$ is a subring of $Z^*(A)$. To our knowledge, this result has first been proven in [Sha13] using tropical modifications, whereas our proof uses the interplay between intersection theory on toric varieties and tropical intersection theory.

In Section 3 we start with the examination of compactifications of linear subvarieties of a torus T. Given a linear subvariety $Z \subseteq T$, we will only consider compactifications in smooth toric varieties such that $\overline{Z}$ is proper and intersects all torus orbits properly. As shown in [Gub13], this happens if and only if the fan Σ corresponding to the toric variety has a subfan $\widetilde{\Sigma}$ which is supported on $\operatorname{Trop}(Z)$. Note that in contrast to the statement of Theorem 1.1, this condition allows fans with support strictly larger than that of $\operatorname{Trop}(Z)$. This gives us more flexibility in the choice of the ambient toric variety: in particular, we can choose it to be complete. However, the closure of Z does not meet the torus orbits corresponding to cones in Σ that are not contained in $\widetilde{\Sigma}$, so $\overline{Z}$ only depends on $\widetilde{\Sigma}$. Hence allowing larger fans does not make the statement more general.

Now given Σ, we will show that the orbit structure $X_\Sigma = \bigcup_{\sigma \in \Sigma} O(\sigma)$ induces a stratification $\overline{Z} = \bigcup_{\sigma \in \Sigma} \overline{Z} \cap O(\sigma)$ and that each stratum $\overline{Z} \cap O(\sigma)$ is a linear subvariety of the torus $O(\sigma)$. We use this stratification to define the morphism $\mathcal{I}_Z$: a cocycle $c \in A^k(\overline{Z})$ is mapped to the Minkowski weight $\sigma \mapsto \deg(c \cap [\overline{Z} \cap V(\sigma)])$, where $V(\sigma)$ denotes the closure of the torus orbit $O(\sigma)$. The morphism $\mathcal{I}_Z$ is analogous to the one constructed by Fulton and Sturmfels, and, in fact, we recover their isomorphism if we set $Z = T$. The injectivity of $\mathcal{I}_Z$ will follow from the fact that the Kronecker duality homomorphism $\mathcal{D}_Z : A^k(\overline{Z}) \to \operatorname{Hom}(A_k(\overline{Z}), \mathbb{Z})$, which assigns to $c \in A^k(\overline{Z})$ the morphism $\alpha \mapsto \deg(c \cap \alpha)$, is an isomorphism. We show this by proving that $\overline{Z}$ is what Totaro defines to be a linear variety in [Tot], a type of variety for which he proves that the Kronecker map always is an isomorphism. To prove the surjectivity of $\mathcal{I}_Z$, we embed X_Σ in a complete toric variety X_Δ and show that the pull-back map $A^*(X_\Delta) \to A^*(\overline{Z})$ corresponds to intersection with the tropical cycle $\operatorname{Trop}(Z)$ on the level of Minkowski weights. Then the statement follows from the fact that the resulting map $M^*(\Delta) \to M^*(\Sigma)$ is onto, which was our main result of Section 2.

I would like to thank Andreas Gathmann for many helpful discussions and comments.

2 Tropical Intersection Theory

The tropical objects occurring in this paper are tropical (fan) cycles in the sense of [AR10] on the one hand and Minkowski weights on the other. Both objects will always live in the real vector space $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ for a lattice N. *Tropical cycles* are equivalence classes of certain *weighted fans*, that is pure-dimensional fans with weights on their top-dimensional cones, where two weighted fans are called equivalent if and only if they have a common refinement which respects the weights. Note that when speaking of a "cone" we actually mean a rational polyhedral cone. The equivalence classes we consider are exactly those of *tropical fans*, which are those weighted fans satisfying the so-called *balancing condition*. To explain the balancing condition we first need the concept of lattice normal vectors. Suppose that τ is a codimension one face of a cone σ. If we write $N_\delta = N \cap \operatorname{Lin}(\delta)$ for a cone $\delta \subseteq N_\mathbb{R}$, then N_σ / N_τ is a one-dimensional lattice. The *lattice normal vector* $u_{\sigma,\tau}$ of σ with respect to τ is the unique generator of N_σ/N_τ which is contained in the image of σ in $(N_\sigma/N_\tau)_\mathbb{R} = \operatorname{Lin}(\sigma)/\operatorname{Lin}(\tau)$. Now if $\mathcal{A} = (\Sigma, \omega)$ is a weighted fan with underlying fan Σ and weight function ω, and for $k \in \mathbb{N}$ we denote by $\Sigma^{(k)}$ the codimension k cones of Σ (that is the cones of dimension $\dim(\Sigma) - k$), then $\mathcal{A}$ satisfies the balancing condition if and only if we have

$$\sum_{\sigma:\tau\le\sigma\in\Sigma^{(0)}} \omega(\sigma)u_{\sigma,\tau} = 0 \text{ in } N_{\mathbb{R}}/\operatorname{Lin}(\tau)$$

for every $\tau \in \Sigma^{(1)}$. The set of all tropical cycles of given codimension k in $N_{\mathbb{R}}$ can be given the structure of an Abelian group $Z^k(N_{\mathbb{R}})$ ([AR10, Lemma 2.14]).

If Σ is a fan, weighted fan or tropical cycle, then we can define its support $|\Sigma|$. For a fan this is just the union of all its cones. For a weighted fan it is the union of all maximal cones with nonzero weight. Finally, if Σ is a tropical cycle, then its support is defined to be the support of any representative (this is easily seen to be well-defined). In any of these cases we can define the subgroup $Z^k(\Sigma)$ of $Z^{k+\operatorname{codim}(\Sigma,N_{\mathbb{R}})}(N_{\mathbb{R}})$ consisting only of those cycles whose support is contained in that of Σ.

An important operation on fans is that of taking stars. For any cone $\sigma \subseteq N_{\mathbb{R}}$ let $N(\sigma) = N/N_{\sigma}$. If σ is a cone of a fan Σ, we define the *star* of Σ at σ to be the fan $\operatorname{Star}_{\Sigma}(\sigma)$ in $N(\sigma)_{\mathbb{R}}$ having as cones the images of the cones of Σ containing σ under the projection map $N_{\mathbb{R}} \to N(\sigma)_{\mathbb{R}}$. In case there are weights defined on Σ we equip $\operatorname{Star}_{\Sigma}(\sigma)$ with the induced weights. As the projection respects lattice normal vectors, the star will be tropical if Σ is.

Minkowski weights differ from cycles in that they do not allow refinements. To be more precise, given a pure-dimensional fan Σ, the set of codimension k Minkowski weights $M^k(\Sigma)$ is defined as the set of weight functions $c : \Sigma^{(k)} \to \mathbb{Z}$ making the codimension k skeleton of Σ a tropical fan. This is easily seen to be a subgroup of the group of functions $\Sigma^{(k)} \to \mathbb{Z}$. It is also evident that there is a canonical inclusion map $M^k(\Sigma) \hookrightarrow Z^k(\Sigma), c \mapsto [c]$.

A particularly nice class of tropical cycles are matroid varieties. For every loopfree matroid M on $E = \{1, \dots, n\}$ there is a pure-dimensional fan $\mathcal{B}(M)$ in $\mathbb{R}^n = \mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{R}$. Its cones are of the form

$$\langle \mathcal{F} \rangle = \operatorname{cone}\{v_{F_1}, \dots, v_{F_{k-1}}, v_E, -v_E\},$$

where $\mathcal{F} = \{\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k = E\}$ is a chain of flats in M, and $v_F = -\sum_{f\in F} e_f$ ($e_1, \dots, e_n$ being the standard basis of $\mathbb{R}^n$). Assigning weight 1 to all maximal cones of $\mathcal{B}(M)$ we obtain a tropical fan. These fans always have lineality space $\operatorname{Lin}\{v_E\}$. As we also want to consider fans without lineality space we define a *matroidal fan* to be any tropical fan isomorphic to $\mathcal{B}(M)/\operatorname{Lin}\{v_E\}$ for some loopfree matroid M. Here, by an isomorphism of two fans $\mathcal{A} \subseteq N_{\mathbb{R}}$ and $\mathcal{B} \subseteq K_{\mathbb{R}}$ we mean a linear map $N_{\mathbb{R}} \to K_{\mathbb{R}}$ which maps $|\mathcal{A}|$ bijectively onto $|\mathcal{B}|$ in a way compatible with the weights and which is induced by a lattice morphism $N \to K$ that maps the lattice $N_{\mathcal{A}} = \operatorname{Lin}|\mathcal{A}| \cap N$ isomorphically onto $K_{\mathcal{B}} = \operatorname{Lin}|\mathcal{B}| \cap K$. *Tropical matroid varieties* are then defined to be exactly those tropical cycles which are associated to matroidal fans.

Taking for the matroid M the uniform matroid of rank n on n elements yields a complete fan $\mathcal{B}(M)$. In particular, $\mathbb{R}^n$, or more generally $N_{\mathbb{R}}$ for any lattice N, equipped with the trivial weight 1, is a matroid variety.

The nice thing about matroid varieties is that they allow an intersection product. That is, for each matroid variety B, the graded Abelian group $Z^*(\mathrm{B}) = \bigoplus_{k\in\mathbb{N}} Z^k(\mathrm{B})$ can be given the structure of a graded commutative ring with unity B, which also has the property that the support of a product $C \cdot D$ is contained in $|C| \cap |D|$ ([FR13, Thm. 4.5], [Sha13, Prop. 3.13]).

Intersection products on matroid varieties also respect intersections with piecewise polynomials, which are somewhat the tropical analogues of equivariant cocycles on toric varieties. A piecewise polynomial of degree k on a cycle A is a continuous function $\varphi : |A| \to \mathbb{R}$ for which there exists a fan structure Σ on $|A|$ such that φ is given by a homogeneous polynomial of degree k with integer coefficients on every cone of Σ. The Abelian group of piecewise polynomials of degree k on A is denoted by $\mathrm{PP}^k(A)$. Allowing also polynomials with mixed degrees we obtain a graded ring $\mathrm{PP}^*(A) = \bigoplus_{k\in\mathbb{N}} \mathrm{PP}^k(A)$. In [Fra13, Prop. 2.24] it is shown that $Z^*(A)$ can be given the structure of a graded $\mathrm{PP}^*(A)$-module, making $Z^*(A)$ a $\mathrm{PP}^*(A)$-algebra in case A is a matroid variety. Given two cycles $A, B \in Z^*(N_\mathbb{R})$ with $|A| \subseteq |B|$, there is a natural restriction morphism $\mathrm{PP}^*(B) \to \mathrm{PP}^*(A)$. This induces a $\mathrm{PP}^*(B)$-module structure on $Z^*(A)$ and with this structure, the inclusion $Z^*(A) \to Z^*(B)$ (note that this is a graded morphism of degree $\mathrm{codim}(A, B)$) is a morphism of $\mathrm{PP}^*(B)$-modules.

Lemma 2.1. *Let $A, B \in Z^*(N_\mathbb{R})$ be matroid varieties with $|A| \subseteq |B|$, then the map*

$$\varphi : Z^*(B) \to Z^*(A) : C \mapsto C \cdot A,$$

which maps a cycle C to its intersection product in $Z^*(B)$ with A, is a surjective $\mathrm{PP}^*(B)$-algebra homomorphism.

Proof. The fact that φ is a morphism of $\mathrm{PP}^*(B)$-modules follows directly from the compatibility properties stated above. To prove that φ also respects the ring structures we first note that the structure maps $\mathrm{PP}^*(A) \to Z^*(A)$ and $\mathrm{PP}^*(B) \to Z^*(B)$ are surjective by [Fra13, Remark 3.2]. Also, as a direct consequence of [Fra13, Prop. 2.8], the restriction morphism $\mathrm{PP}^*(B) \to \mathrm{PP}^*(A)$ is surjective. This shows that both, $Z^*(A)$ and $Z^*(B)$ are homomorphic images of $\mathrm{PP}^*(B)$. Together with the fact that the unity B of $Z^*(B)$ is mapped A, the unity of $Z^*(A)$, this proves the lemma. □

The following Theorem is an adaption of Lemma 2.1 to the world of Minkowski weights. It suffices to look at the case where $B = N_\mathbb{R}$ and A is a matroid variety contained in $N_\mathbb{R}$. To make a sensible statement about Minkowski weights, we have to fix a fan structure on $N_\mathbb{R}$ that respects A. More precisely, we treat complete unimodular fans Σ having a subfan $\widetilde{\Sigma}$ with support equal to $|A|$. Now we can replace $Z^*(B)$ by $M^*(\Sigma)$ and $Z^*(A)$ by $M^*(\widetilde{\Sigma})$ in the above Lemma and are left with the question why intersection with A induces a map $M^*(\Sigma) \to M^*(\widetilde{\Sigma})$, that is why the intersection of two cycles represented by Minkowski weights on Σ is again represented by a Minkowski weight on Σ. But this is a well-known statement: Fulton and Sturmfels have shown in [FS97] that if Σ is a complete fan in $N_\mathbb{R}$, there is a natural ring structure on $M^*(\Sigma)$, the product being described by the so-called "fan displacement rule", and later it was shown in [Kat12, Thm. 4.4] and [Rau08, Thm. 1.9] that the fan displacement rule on $M^*(\Sigma)$ is induced by the intersection product on $Z^*(N_\mathbb{R})$.

Theorem 2.2. *Let $A \in Z^*(N_\mathbb{R})$ be a matroid variety and let Σ be a complete unimodular fan having a subfan $\widetilde{\Sigma} \subseteq \Sigma$ with $|\widetilde{\Sigma}| = |A|$. Then the map*

$$M^*(\Sigma) \to M^*(\widetilde{\Sigma}), c \mapsto c \cdot A$$

is surjective.

Proof. To prove surjectivity we use methods from algebraic geometry, especially toric geometry and intersection theory. After dividing by the lineality space of Σ we can assume that Σ consists of strongly convex cones. In this way, we ensure that the toric variety X_Σ (over the complex numbers say) associated to Σ has the correct dimension. Let $c \in M^k(\widetilde{\Sigma})$. By Lemma 2.1 there is a cycle $B \in Z^k(N_\mathbb{R})$ such that $[c] = B \cdot A$ (remember that $[c]$ denotes the tropical cycle associated to c). There is a complete fan Δ refining Σ, such that $B = [b]$ for a Minkowski weight $b \in M^k(\Delta)$. Let $\pi : X_\Delta \to X_\Sigma$ be the toric morphism induced by the identity on N, and let s be the Minkowski weight on Σ with $[s] = A$. Identifying Minkowski weights and cocycles on complete toric varieties, π induces a morphism $\pi^* : M^k(\Sigma) \to M^k(\Delta)$. By applying the projection formula, one easily sees that π^* is nothing but the refinement of Minkowski weights. Then the equation $[b] \cdot A = [c]$ translates into

$$b \cup \pi^*(s) = \pi^*(c), \tag{1}$$

where we write the intersection product as "cup"-product to emphasize that we think of Minkowski weights as cocycles on toric varieties. Since X_Σ is smooth and π is proper there is the Gysin push-forward $\pi_* : M^k(\Delta) \to M^k(\Sigma)$ (see [Ful98, pp. 328-329]). Applying it to Equation 1 we obtain

$$\pi_*(b) \cup s = \pi_*(b \cup \pi^*(s)) = \pi_*\pi^*(c),$$

where the first equality follows from [FM81, (G_4) (i), p. 26]. By [FM81, (G_3) (ii), p. 26] (with $g = \mathrm{id}$ and $\theta = [\pi]$) we have $\pi_*\pi^*(c) = \pi_*[\pi] \cup c$. So if we show that $\pi_*[\pi] = 1$ we get $\pi_*(b) \cdot A = \pi_*(b) \cup s = c$ and are done. We have

$$\pi_*[\pi] \cap [X_\Sigma] = \pi_*([\pi] \cap [X_\Sigma])$$

by the definition of the push-forward of bivariant classes. By [Ful98, Ex. 17.4.3 (c)] and [Ful98, Cor. 8.1.3] we have $[\pi] \cap [X_\Sigma] = [X_\Delta]$. Hence

$$\pi_*[\pi] \cap [X_\Sigma] = \pi_*[X_\Delta] = [X_\Sigma]$$

and we are done by Poincaré duality ([Ful98, Cor. 17.4]). ☐

Now we can easily show that the tropical intersection product on a matroid variety induces a ring structure on the group of Minkowski weights on any unimodular fan with the same support. This statement also follows from [Sha13, Prop. 3.13], where it has been proved using the purely tropical method of tropical modifications.

Corollary 2.3. *Let Δ be a pure-dimensional unimodular fan in $N_\mathbb{R}$ which defines a matroid variety A if we assign weight 1 to all of its maximal cones. Then the group of Minkowski weights $M^*(\Delta)$ can be given a ring structure which is compatible with the intersection product on $Z^*(A)$.*

Proof. By [Ewa96, Thm. 2.8, p. 75] and [CLS11, Thm. 11.1.9] there is a complete unimodular fan Σ with $\Delta \subseteq \Sigma$. Now consider the commutative diagram

$$\begin{array}{ccc} M^*(\Sigma) & \hookrightarrow & Z^*(N_\mathbb{R}) \\ \downarrow {\scriptstyle \_ \cdot A} & & \downarrow {\scriptstyle \_ \cdot A} \\ M^*(\Delta) & \hookrightarrow & Z^*(A). \end{array}$$

All groups except $M^*(\Delta)$ occurring in this diagram have a ring structure and the morphisms at the top and on the right are ring homomorphisms. The one on the right even is surjective by Lemma 2.1 and so is the map on the left by Theorem 2.2. Together with the fact that the horizontal maps are injective it follows that $M^*(\Delta)$ is a subring of $Z^*(A)$. □

Let us end this section with a result which is not directly related to tropical intersection theory but rather to the concept of tropicalization. There are several approaches of how to define the tropicalization of a subvariety of a torus and it has been a major result in tropical geometry that they are all equivalent. Our working definition of tropicalization, however, is nonstandard but instead chosen in a way that makes the connection to toric intersection theory evident.

Proposition and Definition 2.4. *Let K be an algebraically closed field with constant valuation and let Z be a subvariety of an algebraic torus T over K. Let N be the lattice of one parameter subgroups of T and let Σ be a complete unimodular fan in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ such that the closure of Z in the toric variety X_Σ intersects all orbits properly. Then the tropical cycle associated to the Minkowski weight corresponding to the cycle $[\overline{Z}]$ in $A^*(X_\Sigma)$ is independent of the choice of Σ and called the* tropicalization $\operatorname{Trop}(Z)$ *of Z.*

Proof. The fact that the tropical cycle associated to $[\overline{Z}]$ does not depend on Σ has been proven in [ST08, Lemma 3.2] in the special case of tropical compactifications and later in [KP11, Lemma 2.3] in the general case. The existence of such a Σ follows from [Tev07, Thm. 1.2]. □

Note that it also follows from [KP11, Lemma 2.3] that our definition of tropicalization is equivalent to the standard ones which use non-Archimedian amoebas or initial degenerations.

3 Linear Subvarieties of Tori

Let us first make precise what we mean when speaking of a linear subvariety Z of a torus T. If T is equal to $\operatorname{Spec} K[\mathbb{Z}^n] = \operatorname{Spec} K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ for some n, it would be natural to call a polynomial $f \in K[M]$ linear if it is of the form $f = a_0 + \sum_i a_i x_i$ for some coefficients $a_i \in K$. As the character lattice M of T is always isomorphic to $\mathbb{Z}^n$ for some n, the assumption $T = \operatorname{Spec} K[\mathbb{Z}^n]$ is not really a restriction, and we could use this for our definition of linear by calling a subvariety $Z \subseteq T$ linear if its vanishing ideal is generated by linear polynomials. However, if we do so there will arise some difficulties: If Σ is a fan in $N_{\mathbb{R}}$, where N is the lattice dual to M, and we take the closure $\overline{Z}$ of a linear subvariety Z in the toric variety X_Σ, then we would like say (and in fact will prove in Proposition 3.5) that the subvariety $\overline{Z} \cap O(\sigma)$ of the torus $O(\sigma)$ is linear again. But the coordinates on $\mathbb{Z}^n$ do not induce natural coordinates on the character lattice $M(\sigma) = M \cap \sigma^\perp$ of $O(\sigma)$ so that it is unclear what linear should mean in this context. This problem does not occur if we instead go with the following definition.

Definition 3.1. Let M be a lattice and $B = \{e_1, \dots, e_n\}$ a basis for M. We call $f \in K[M]$ *linear with respect to B* if it is of the form

$$f = a_0 + \sum_{i=1}^{n} a_i \chi^{e_i}.$$

for some $a_i \in K$, where χ^m denotes the basis vector of $K[M]$ corresponding to $m \in M$. An ideal $I \trianglelefteq K[M]$ is said to be *linear with respect to B* if it is generated by polynomials which are linear with respect to B. We will call I *linear* if it is linear with respect to some basis. A subvariety $Z \subseteq T$ will be called linear (w.r.t. some basis B), if its vanishing ideal in $K[M]$ is.

We wish to describe the Chow cohomology $A^*(\overline{Z})$ of the closure of a linear variety $Z \subset T$ in a toric variety $X = X_\Sigma$ as Minkowski weights on the fan Σ. In analogy to the result of Fulton and Sturmfels, the isomorphism between Chow cocycles and Minkowski weights should use the stratification of X_Σ into torus orbits and assign to a cocycle $c \in A^k(\overline{Z})$ the Minkowski weight with multiplicity $\deg(c \cap [\overline{\overline{Z} \cap O(\sigma)}])$ on a cone $\sigma \in \Sigma$ with $\dim(\overline{Z} \cap O(\sigma)) = k$. But this construction cannot make sense for arbitrary Σ. The two things that can go wrong are first, that $\overline{Z}$ could be non-complete so that the degree in the formula above is not defined. By [Tev07, Prop 2.3], this happens if and only if the tropicalization of Z is not contained in the support of Σ. The second problem is that Minkowski weights on Σ only assign weights to cones of a fixed dimension. So the cones σ with $\dim(\overline{\overline{Z} \cap O(\sigma)}) = k$ should all have the same dimension and, furthermore, $\overline{Z} \cap O(\sigma)$ should be pure-dimensional. We see that our approach only makes sense if $\overline{Z}$ intersects all torus orbits properly. It is a direct consequence of the results proved in [Gub13, Chapter 14] that this is the case if and only if for each $\sigma \in \Sigma$ we either have $\operatorname{relint}(\sigma) \cap |\operatorname{Trop}(Z)| = \emptyset$ or $\sigma \subseteq |\operatorname{Trop}(Z)|$. These considerations lead to the following definition.

Definition 3.2. Let Z be a subvariety of the torus T. A fan Σ in $N_\mathbb{R}$ is called *admissible* (for Z) if it is unimodular, and $|\operatorname{Trop}(Z)|$ is a union of cones of Σ. If Σ is admissible for Z, we denote by $\widetilde{\Sigma} = \{\sigma \in \Sigma \mid \sigma \subseteq |\operatorname{Trop}(Z)|\}$ the subfan of Σ consisting of all cones contained in the tropicalization of Z. Note that the cones of $\widetilde{\Sigma}$ are exactly those cones of Σ with $\overline{Z} \cap O(\sigma) \neq \emptyset$ ([Tev07, Lemma 2.2]).

As we have just seen, admissible fans are exactly those which are unimodular and for which the closure $\overline{Z}$ in the corresponding toric variety is complete and intersects all torus orbits properly. The following result is an easy consequence of this.

Lemma 3.3. *Let $Z \subseteq T$ be a subvariety of the torus, Σ an admissible fan for Z and $\sigma \in \Sigma$ a cone in Σ. Then we have the set-theoretic equality*

$$\overline{\overline{Z} \cap O(\sigma)} = \overline{Z} \cap V(\sigma)$$

in the toric variety $X = X_\Sigma$, where $V(\sigma)$ denotes the closure of the torus orbit $O(\sigma)$ corresponding to σ.

Proof. Let Y be an irreducible component of $\overline{Z} \cap V(\sigma)$ and let $\tau \in \Sigma$ be a cone which is maximal with the property that $\sigma \leq \tau$ and $Y \subseteq V(\tau)$. Then Y has nonempty intersection with the orbit $O(\tau)$. Because $O(\tau)$ is open in $V(\tau)$, the set $Y \cap O(\tau)$ is open in Y, showing that Y has dimension at most $\dim(Z) - \dim(\tau)$. On the other hand, because X is smooth, every component of $\overline{Z} \cap V(\sigma)$ has dimension at least $\dim(Z) - \dim(\sigma)$. We conclude that $\sigma = \tau$, showing that every component of $\overline{Z} \cap V(\sigma)$ intersects $O(\sigma)$. The equality we want to prove follows. ☐

The result of the preceding lemma is very important for us and will be used at various places in the remaining part of this paper. Unfortunately, it is only a set-theoretic statement and neglects potentially arising intersection multiplicities. As we do not want to bother with this, we will only consider varieties in this paper. In particular, expressions like $\overline{Z} \cap V(\sigma)$ will denote the subvariety of X_Σ whose underlying set is the set-theoretic intersection of $\overline{Z}$ and $V(\sigma)$.

We recall that a stratification of a variety X is a finite decomposition $X = \coprod S_i$ such that each stratum S_i is locally closed, and the "boundary" $\overline{S_i} \setminus S_i$ is a union of strata of lower dimension. The following corollary will show that every admissible compactification of a subvariety $Z \subseteq T$ has a natural stratification. Afterwards, we will show that in case Z is linear, the strata of this stratification are, in fact, linear subspaces of tori.

Corollary 3.4. *Let Σ be an admissible fan for a subvariety $Z \subseteq T$. Then the closure of Z in X_Σ is stratified by the subsets $\{\overline{Z} \cap O(\sigma) \mid \sigma \in \widetilde{\Sigma}\}$. For $1 \leq k \leq n$, the strata of dimension k in this stratification are exactly those with $\sigma \in \widetilde{\Sigma}^{(k)}$.*

Proof. We have seen that the closures of Z in toric varieties corresponding to admissible fans intersects all orbits properly. Together with the fact that $\dim(\mathrm{Trop}(Z)) = \dim(Z)$, this proves the statement about dimensions. The rest follows from Lemma 3.3. ☐

Proposition 3.5. *Let $Z \subseteq T$ be linear and Σ an admissible fan. Then for each cone $\sigma \in \widetilde{\Sigma}$ the intersection $\overline{Z} \cap O(\sigma)$ is a linear subvariety of the torus $O(\sigma)$.*

Proof. We prove the statement by induction on the dimension of σ. If $\dim(\sigma) = 0$, then $\overline{Z} \cap O(\sigma) = Z$ is linear by assumption. Hence we can suppose $\dim(\sigma) > 0$. In this case we find a face $\tau \in \widetilde{\Sigma}$ of σ of dimension $\dim(\tau) = \dim(\sigma) - 1$. By induction we know that $\overline{Z} \cap O(\tau)$ is linear, and by Lemma 3.3 we have $\overline{Z} \cap O(\sigma) = \overline{\overline{Z} \cap O(\tau)} \cap O(\sigma)$. By [Gub13, Prop. 14.3], the support of the tropicalization of $\overline{Z} \cap O(\tau)$ is equal to that of $\mathrm{Star}_{\widetilde{\Sigma}}(\tau)$, showing that $\mathrm{Star}_\Sigma(\tau)$, the fan corresponding to the toric variety $V(\tau)$, is admissible for $\overline{Z} \cap O(\tau)$. We see that by considering the toric variety $V(\tau)$ we can assume that $\dim(\sigma) = 1$.

Let $u \in \sigma$ be a generator of σ. To show that $\overline{Z} \cap O(\sigma)$ is linear we will use initial ideals with respect to u. The initial form $\mathrm{in}_u(f)$ of a polynomial $f \in K[M]$ with respect to u is the sum of all terms of f corresponding to monomials with minimal u-weight. That is, if we write $f = \sum_i a_i \chi^{m_i}$ as a finite sum with distinct m_i and all a_i nonzero, then $\mathrm{in}_u(f)$ is the sum of all terms $a_i \chi^{m_i}$ for which $\langle m_i, u \rangle$ is minimal. Given an ideal $I \trianglelefteq K[M]$ we now define $\mathrm{in}_u(I) = \langle \mathrm{in}_u(f) \mid f \in I \rangle$. It is immediate that if we choose coordinates on M, our definition coincides with the usual definition used in the theory of standard bases (except maybe by taking minima instead of maxima) and therefore all techniques developed there apply in our setting.

The torus orbit $O(\sigma)$ is contained in the affine toric variety $U_\sigma = \mathrm{Spec}\, K[\sigma^\vee \cap M]$. If Z is given by the ideal $I \trianglelefteq K[M]$, the intersection $I \cap K[\sigma^\vee \cap M]$ is the vanishing ideal of the closure of Z in U_σ. The orbit $O(\sigma)$ is embedded into U_σ via the morphism

$$\varphi : K[\sigma^\vee \cap M] \to K[M(\sigma)], \chi^u \mapsto \begin{cases} \chi^u & \text{, if } u \in \sigma^\perp \\ 0 & \text{, else.} \end{cases}$$

It follows that the ideal of $\overline{Z} \cap O(\sigma)$ in the coordinate ring of $O(\sigma)$ is given by the radical of $I_\sigma = \varphi(I \cap K[\sigma^\vee \cap M])$. We claim that I_σ is equal to $\mathrm{in}_u(I) \cap K[M(\sigma)]$. Whenever the image $\varphi(f)$ of an $f \in K[\sigma^\vee \cap M]$ is nonzero, the u-weight of f must be zero. Hence $\mathrm{in}_u(f)$ is equal to the sum of terms of f with u-weight zero, which are exactly those which do not go to zero when applying φ. It follows that $\varphi(f) = \mathrm{in}_u(f) \in \mathrm{in}_u(I) \cap K[M(\sigma)]$. Conversely, if $0 \neq g \in \mathrm{in}_u(I) \cap K[M(\sigma)]$, then g is u-homogeneous and therefore there exists some $f \in I$ with $g = \mathrm{in}_u(f)$. With the same argument as before we get $f \in K[\sigma^\vee \cap M]$ and $g = \mathrm{in}_u(f) = \varphi(f) \in I_\sigma$.

Let B be a basis with respect to which I is linear. It is a consequence of [MT07, Prop. 1.4.4] that the initial ideal $\mathrm{in}_u(I)$ is again linear with respect to B, say generated by the linear polynomials $f_1, \dots, f_k$. Because $\mathrm{in}_u(I)$ is u-homogeneous, we can assume all of the f_i to be u-homogeneous. Let $\mathcal{M}$ be the set of all $m \in M$ such that the monomial χ^m occurs in one of the f_i. We define an equivalence relation on $\mathcal{M}$ by letting $m \sim m'$ if and only if $\langle m, u\rangle = \langle m', u\rangle$, and for each equivalence class $x \in \mathcal{M}/\sim$ we choose a representative $r(x) \in x$. As each f_i is u-homogeneous, all of the exponents occurring in f_i are equivalent, hence there is a well defined equivalence class $[f_i] \in \mathcal{M}/\sim$ with generator $r(i) = r([f_i])$. This choice of $r(i)$ ensures that $\chi^{-r(i)} f_i \in K[M(\sigma)]$. Hence, $\mathrm{in}_u(I)$ is generated in $K[M(\sigma)]$, namely by the polynomials $\chi^{-r(1)} f_1, \dots, \chi^{-r(k)} f_k$, and since $K[M]$ is free as $K[M(\sigma)]$-module, the ideal $\mathrm{in}_u(I) \cap K[M(\sigma)]$ is generated by these polynomials as well. Let $B' = \{a - r(x) \mid x \in \mathcal{M}/\sim,\ a \in x \setminus \{r(x)\}\}$. This is certainly a subset of $M(\sigma)$, and if we add the elements $\{r(x) \mid x \in \mathcal{M}/\sim,\ 0 \notin x\}$ we obtain a basis of the sublattice of M generated by the elements in $\mathcal{M}$. Since this sublattice is saturated, we can also complete B' to a basis $\widetilde{B}$ of $M(\sigma)$. By construction, the generators for $\mathrm{in}_u(I) \cap K[M(\sigma)]$ from above are all linear with respect to this basis, showing that $\overline{Z} \cap O(\sigma)$ is linear. ☐

Before we start to consider the Chow groups and the Chow cohomology of linear subvarieties of tori, let us state the following immediate consequence of the preceding proposition, which will be of great importance in the proof of Theorem 3.11.

Corollary 3.6. *Let Z be a linear subvariety of the torus T, and $\mathcal{A} = (\Sigma, \omega)$ a unimodular tropical fan representing the cycle $\mathrm{Trop}(Z)$. Furthermore, let $\sigma \in \Sigma$. Then the tropicalization of the subvariety $\overline{Z} \cap O(\sigma)$ of the torus $O(\sigma)$ is represented by $\mathrm{Star}_{\mathcal{A}}(\sigma)$.*

Proof. By [Gub13, Prop. 14.3], the underlying sets of $\mathrm{Trop}(\overline{Z} \cap O(\sigma))$ and $\mathrm{Star}_{\mathcal{A}}(\sigma)$ are equal, so we just need to show that the weights coincide. But Z is linear by assumption and $\overline{Z} \cap O(\sigma)$ by Proposition 3.5, hence all weights are 1 in both cases. ☐

Our first result on the intersection theory of linear subvarieties of toric varieties will give us generators for the Chow groups.

Corollary 3.7. *Let $Z \subseteq T$ be linear, Σ an admissible fan for Z, and $1 \leq k \leq \dim(Z)$. Then the k-th Chow group of the closure of Z in X_Σ is generated by the cycles $[\overline{Z} \cap V(\sigma)]$ for $\sigma \in \widetilde{\Sigma}^{(k)}$.*

Proof. By Proposition 3.5 we know that all strata of the canonical stratification of $\overline{Z}$ introduced in Corollary 3.4 are linear subvarieties of some torus. In particular, only their top-dimensional Chow groups are nonzero. This reduces the statement to a well-known result about Chow groups of stratified varieties ([EH13, Prop. 1.19]). ☐

Now we know generators of the Chow groups of $\overline{Z}$, but we do not know any relations between them. The next lemma will be the essential ingredient to change this.

Lemma 3.8. *Let $I \trianglelefteq K[M]$ be a linear ideal and Z the corresponding subvariety of the torus. Furthermore, let Σ be an admissible fan for Z, let $\sigma \in \widetilde{\Sigma}$ be a ray in Σ with primitive generator u_σ, and let $m \in M$. Then $\chi^m \in K[M]$ defines a rational function on the closure of Z in X_Σ and*

$$\operatorname{ord}_{\overline{Z} \cap V(\sigma)}(\chi^m) = \langle m, u_\sigma \rangle.$$

In particular, we have $\operatorname{div}(\chi^m) = \sum_\sigma \langle m, u_\sigma \rangle [\overline{Z} \cap V(\sigma)]$*, where the sum is taken over all rays of* $\widetilde{\Sigma}$.

Proof. The character χ^m clearly defines a rational function on $\overline{Z}$. To prove the formula we can assume that $X_\Sigma = U_\sigma$ in which case we have $\overline{Z} \cap V(\sigma) = \overline{Z} \cap O(\sigma)$. Let $\varphi : K[\sigma^\vee \cap M] \to K[M(\sigma)]$ be the morphism corresponding to the closed embedding $O(\sigma) \to U_\sigma$. In the course of the proof of Proposition 3.5 we showed that the image $\varphi(I \cap K[\sigma^\vee \cap M])$ is prime in $K[M(\sigma)]$ (it is linear with respect to some basis). Therefore, its preimage $\ker \varphi + I \cap K[\sigma^\vee \cap M]$ in $K[\sigma^\vee \cap M]$ is prime, too. Choose a basis $e_1^*, e_2^*, \ldots, e_n^*$ of N with dual basis $e_1, \ldots, e_n \in M$ such that $e_1^* = u_\sigma$. With these coordinates, $K[\sigma^\vee \cap M] = K[\chi^{e_1}, \chi^{\pm e_2}, \ldots, \chi^{\pm e_n}]$ and $\ker(\varphi) = \langle \chi^{e_1} \rangle$. Consequently, the ideal in the coordinate ring $K[\sigma^\vee \cap M]/I \cap K[\sigma^\vee \cap M]$ of $\overline{Z}$ cutting out $\overline{Z} \cap O(\sigma)$ is generated by χ^{e_1}. Hence, $\mathcal{O}_{\overline{Z} \cap O(\sigma), \overline{Z}}$ is a discrete valuation ring whose maximal ideal is generated by χ^{e_1}. Writing

$$m = \langle m, u_\sigma \rangle e_1 + \sum_{i=2}^{n} \langle m, e_i^* \rangle e_i$$

and noting that χ^{e_i} is a unit in $\mathcal{O}_{\overline{Z} \cap O(\sigma), \overline{Z}}$ for $i \geq 2$ we obtain $\operatorname{ord}_{\overline{Z} \cap V(\sigma)}(\chi^m) = \langle m, u_\sigma \rangle$.

Because χ^m is an invertible regular function on Z, the divisor $\operatorname{div}(\chi^m)$ is a linear combination of prime divisors contained in $\overline{Z} \setminus Z$. But this is exactly the union of the varieties $\overline{Z} \cap V(\sigma)$ for rays $\sigma \in \widetilde{\Sigma}$, which yields the "in particular" statement. □

Corollary 3.7 and Lemma 3.8 show us that there is a strong analogy between the Chow groups of toric varieties and those of admissible compactifications of linear subvarieties of tori. The problem is that our description of the Chow groups in the latter case is incomplete: we do know generators and we do know the relations induced by the characters of the various tori $O(\sigma)$, but we do not know whether or not all relations are of this form. We will only obtain an answer to this question up to torsion in Corollary 3.12 as a consequence of our results on Chow cohomology.

To describe the Chow cohomology of an admissible compactification $\overline{Z}$ in a toric variety X_Σ we proceed similarly to Fulton and Sturmfels in [FS97]. Given a cocycle $c \in A^k(\overline{Z})$, we will first apply the Kronecker duality homomorphism $\mathcal{D}_Z : A^k(\overline{Z}) \to \operatorname{Hom}(A_k(\overline{Z}), \mathbb{Z})$ which assigns to c the morphism mapping $\alpha \in A_k(\overline{Z})$ to $\deg(c \cap \alpha)$. The Chow group $A_k(\overline{Z})$ is generated by the cycles $[\overline{Z} \cap V(\sigma)]$ for $\sigma \in \widetilde{\Sigma}^{(k)}$, hence $\mathcal{D}_Z(c)$ is uniquely determined by its images on those cycles. Let us denote the induced map $\widetilde{\Sigma}^{(k)} \to \mathbb{Z}$ by $\mathcal{I}_Z(c)$. For every $\tau \in \widetilde{\Sigma}^{(k+1)}$, the subvariety $\overline{Z} \cap V(\tau)$ is an admissible compactification of the linear variety $\overline{Z} \cap O(\tau)$ by Lemma 3.3, Proposition 3.5, and Corollary 3.6. Consequently, we know that

$$\sum_{\sigma : \tau \leq \sigma \in \widetilde{\Sigma}^{(k)}} \langle m, u_{\sigma,\tau} \rangle [\overline{Z} \cap V(\sigma)] = 0 \text{ in } A_k(\overline{Z})$$

for all $m \in M(\tau)$ by Lemma 3.8 (remember that $u_{\sigma,\tau}$ denotes the lattice normal vector of σ with respect to τ). We conclude that $\mathcal{I}_Z(c)$ actually is a Minkowski weight on $\widetilde{\Sigma}$, that is an element in $M^k(\widetilde{\Sigma})$. This shows that our construction really yields a morphism of Abelian groups

$$\mathcal{I}_Z : A^*(\overline{Z}) \to M^*(\widetilde{\Sigma})$$

analogous to the one constructed by Fulton and Sturmfels for complete toric varieties, and, in fact, we recover the isomorphism of [FS97] if we take $Z = T$. But we do not know yet that the image $\mathcal{I}_Z(c)$ of a cocycle $c \in A^*(\overline{Z})$ completely determines c. This will be ensured by the next proposition.

Proposition 3.9. *Let Σ be admissible for a linear subvariety $Z \subseteq T$. Then the Kronecker duality map $\mathcal{D}_Z$ is an isomorphism. In particular, $\mathcal{I}_Z$ is injective.*

Proof. The statement basically follows from the result [Tot, Thm. 2] in Totaro's paper, which says that the Kronecker duality map is an isomorphism for all complete varieties of a certain type. The type of varieties considered by Totaro is that of what he calls linear varieties. The class of varieties which are linear in his sense is defined recursively. We do not want to go into the details here. In our setting it is sufficient to note that varieties stratified by linear varieties are again linear and that the complement of a union of affine subspaces in an ambient affine space is linear (see [Tot, p. 5]). By Corollary 3.4, $\overline{Z}$ is stratified by $\{\overline{Z} \cap O(\sigma) \mid \sigma \in \widetilde{\Sigma}\}$. Each of the strata $\overline{Z} \cap O(\sigma)$ is cut out by equations which are linear in some coordinates and is hence isomorphic to the intersection of an affine subspace L of some $\mathbb{A}^k$ with the k-dimensional torus $T' \subseteq \mathbb{A}^k$. This can also be considered as the complement of a finite union of affine subspaces of L and therefore it is linear in the sense of Totaro. Consequently, $\overline{Z}$, too, is linear in Totaro's sense. Applying Totaro's theorem we see that $\mathcal{D}_Z$ is an isomorphism. The "in particular" statement then follows directly from the construction of $\mathcal{I}_Z$. □

Corollary 3.10. *With the same requirements as in the preceding proposition we have $A^0(\overline{Z}) \cong \mathbb{Z}$.*

Proof. Consider the composite morphism $A^0(\overline{Z}) \xrightarrow{\mathcal{I}_Z} M^0(\widetilde{\Sigma}) \hookrightarrow Z^0(\widetilde{\Sigma})$, which is one-to-one by what we just saw. We have $|\widetilde{\Sigma}| = |\operatorname{Trop}(Z)|$ and the cycle $\operatorname{Trop}(Z)$ is, as it is a matroid variety, irreducible by [FR13, Lemma 2.4]. Therefore, $Z^0(\widetilde{\Sigma})$ is freely generated by $\operatorname{Trop}(Z)$. The tropical cycle $\operatorname{Trop}(Z)$ is represented by the Minkowski weight in $M^0(\widetilde{\Sigma})$ having weight 1 on all maximal cones. Noting that this is exactly the image of $1 \in A^0(\overline{Z})$ under $\mathcal{I}_Z$ finishes the proof. □

Now that we know that $\mathcal{I}_Z$ embeds the Chow cohomology of $\overline{Z}$ into the group of Minkowski weights on $\widetilde{\Sigma}$, our next goal is to show that $\mathcal{I}_Z$ even is an isomorphism. One of the key ingredients for this is the following theorem.

Theorem 3.11. *Let Σ be a complete fan which is admissible for a linear subvariety $Z \subset T$, and let $X = X_\Sigma$ be its associated complete toric variety. Then the diagram*

$$\begin{array}{ccc} A^*(X) & \xrightarrow{\mathcal{I}_T} & M^*(\Sigma) \\ \Big\downarrow i^* & & \Big\downarrow \_\cdot \operatorname{Trop}(Z) \\ A^*(\overline{Z}) & \xrightarrow{\mathcal{I}_Z} & M^*(\widetilde{\Sigma}), \end{array}$$

where the vertical map on the right assigns to a Minkowski weight $c \in M^*(\Sigma)$ the Minkowski weight representing the intersection cycle $[c] \cdot \operatorname{Trop}(Z)$, is commutative.

Proof. Because all maps involved are graded, it is sufficient to show the commutativity for homogeneous elements. Let d be the dimension of Z, then for $k > d$ the k-th components of the vertical maps are both zero and hence the diagram commutes in degrees greater d. Now assume that $c \in A^d(X)$. Let $t \in A^{n-d}(X)$ be the cocycle on X corresponding to $[\overline{Z}]$ by Poincaré duality, that is the unique cocycle with $t \cap [X] = [\overline{Z}]$. Then its associated tropical cycle $[\mathcal{I}_X(t)] = \operatorname{Trop}(Z)$ is the tropicalization of Z (see Definition 2.4). Identifying both, $M^n(\Sigma)$ and $M^d(\widetilde{\Sigma})$, with $\mathbb{Z}$, we get

$$\begin{aligned}\mathcal{I}_Z(i^*c) = \deg(i^*c \cap [\overline{Z}]) &= \deg(c \cap [\overline{Z}]) = \\ &= \deg((c \cup t) \cap [X]) = \mathcal{I}_T(c \cup t) = \mathcal{I}_T(c) \cdot \operatorname{Trop}(Z),\end{aligned}$$

where the second equality uses the projection formula, and the last equality uses that the ring structure of the Chow cohomology of complete toric varieties is compatible with tropical intersection products.

Now assume $c \in A^k(X)$ for some $k < d$ and let $\sigma \in \widetilde{\Sigma}^{(k)}$. By definition, the weight of $\mathcal{I}_Z(i^*c)$ at σ is equal to $\deg(i^*c \cap [\overline{Z} \cap V(\tau)])$. Denoting the inclusion $V(\sigma) \to X$ by j, this is equal to $\deg(j^*c \cap [\overline{Z} \cap V(\tau)])$ by the projection formula. By Lemma 3.3, Proposition 3.5, and Corollary 3.6, we know that $\overline{Z} \cap V(\sigma)$ is an admissible compactification of the linear variety $\overline{Z} \cap O(\sigma)$. So if we denote the inclusion $\overline{Z} \cap V(\sigma) \to V(\sigma)$ by κ, then the $k = d$ case applied to j^*c yields

$$\begin{aligned}\deg(j^*c \cap [\overline{Z} \cap V(\sigma)]) &= \deg(\kappa^*(j^*c) \cap [\overline{Z} \cap V(\sigma)]) \\ &= \mathcal{I}_{\overline{Z} \cap O(\sigma)}(\kappa^*(j^*c)) \\ &= \mathcal{I}_{O(\sigma)}(j^*c) \cdot \operatorname{Trop}(\overline{Z} \cap O(\sigma)),\end{aligned}$$

where the first equality again follows by the projection formula and in the last two expressions we identify $M^k(\operatorname{Star}_{\widetilde{\Sigma}}(\sigma))$ with $\mathbb{Z}$. Using the projection formula one easily sees that $\mathcal{I}_{O(\sigma)}(j^*c)$ is equal to $\operatorname{Star}_{\mathcal{I}_T(c)}(\sigma)$, and by Corollary 3.6 we know that $\operatorname{Trop}(\overline{Z} \cap O(\sigma)) = \operatorname{Star}_{\operatorname{Trop}(Z)}(\sigma)$. The locality of the intersection product ([Rau08, Lemma 1.2]) then implies that the intersection product of $\operatorname{Star}_{\mathcal{I}_T(c)}(\sigma)$ and $\operatorname{Star}_{\operatorname{Trop}(Z)}(\sigma)$ is equal to the weight of $\mathcal{I}_T(c) \cdot \operatorname{Trop}(Z)$ at σ, which is exactly what we wanted to show. □

Now we are able to prove our main result.

Proof of Theorem 1.1. We have already constructed the morphism $\mathcal{I}_Z$ and have seen that it is injective in Proposition 3.9. To see that it is also surjective, let Δ be a complete unimodular fan containing Σ (which exists by [Ewa96, Thm. 2.8, p. 75] and [CLS11, Thm. 11.1.9]). Because $|\Sigma| = |\operatorname{Trop}(Z)|$, the closure of Z in X_Δ is equal to that in X_Σ. Let $i : \overline{Z} \to X_\Delta$ denote the inclusion map and consider the diagram

$$\begin{array}{ccc} A^*(X_\Delta) & \xrightarrow{\mathcal{I}_T} & M^*(\Delta) \\ \Big\downarrow{\scriptstyle i^*} & & \Big\downarrow{\scriptstyle \_\cdot \operatorname{Trop}(Z)} \\ A^*(\overline{Z}) & \xrightarrow{\mathcal{I}_Z} & M^*(\Sigma) \end{array}$$

which is commutative by Theorem 3.11. Since Z is linear, the fan Σ has support equal to that of a matroid variety. We have seen in Theorem 2.2 and Corollary 2.3 that in this situation the intersection product on $Z^*(\operatorname{Trop}(Z))$ induces a ring structure on $M^*(\Sigma)$ and that the morphism on the right is a surjective ring homomorphism. We also know that the upper horizontal map $\mathcal{I}_T$ is an isomorphism by [FS97, Thm 2.1]. Together, these facts imply that $\mathcal{I}_Z$ is a ring isomorphism. □

Corollary 3.12. *Let $k \in \mathbb{N}$, and Z and Σ as in Theorem 1.1. Then the k-th Chow group $A_k(\overline{Z})_\mathbb{Q} = A_k(\overline{Z}) \otimes_\mathbb{Z} \mathbb{Q}$ with rational coefficients is canonically isomorphic to the quotient of the $\mathbb{Q}$-vector space with basis $\{e_\sigma \mid \sigma \in \Sigma^{(k)}\}$ by the subspace spanned by the elements*

$$\sum_{\sigma:\tau \leq \sigma \in \Sigma^{(k)}} \langle m, u_{\sigma,\tau} \rangle e_\sigma$$

for $\tau \in \Sigma^{(k+1)}$ and $m \in M(\sigma)$, where $u_{\sigma,\tau}$ denotes the lattice normal vector of σ with respect to τ.

Proof. Denoting the vector space just described by $V = \operatorname{Lin}\{e_\sigma \mid \sigma \in \Sigma\}$, the dual V^* of V clearly is canonically isomorphic to $M^k(\Sigma)_\mathbb{Q}$. This in turn is canonically isomorphic to $A^k(\overline{Z})_\mathbb{Q}$ by Theorem 1.1. It follows from Proposition 3.9 that this is isomorphic to $(A_k(\overline{Z})_\mathbb{Q})^*$. Dualizing the composite isomorphism $V^* \cong (A_k(\overline{Z})_\mathbb{Q})^*$ finishes the proof. □

As a final result we will prove that the intersection ring of $\operatorname{Trop}(Z)$ is isomorphic to the direct limit of the cohomology rings of admissible compactifications of Z.

Corollary 3.13. *Let Z be a linear subvariety of the torus T and let $\mathcal{D}$ be the directed set of unimodular fans in $N_\mathbb{R}$ with support equal to $|\operatorname{Trop}(Z)|$. Then we have $\varinjlim A^*(\overline{Z}) \cong Z^*(\operatorname{Trop}(Z))$.*

Proof. As an immediate consequence of the fact that every fan has a unimodular refinement, we get the equality $\varinjlim M^*(\Sigma) \cong Z^*(\operatorname{Trop}(Z))$. Since we have $M^*(\Sigma) \cong A^*(\overline{Z}^\Sigma)$ for all $\Sigma \in \mathcal{D}$ by Theorem 1.1, it is only left to show that whenever $\Sigma \in \mathcal{D}$ refines $\Delta \in \mathcal{D}$ the diagram

$$\begin{array}{ccc} A^*(\overline{Z}) & \xrightarrow{\mathcal{I}_Z} & M^*(\Delta) \\ \Big\downarrow{\scriptstyle i^*} & & \Big\downarrow \\ A^*(\overline{Z}') & \xrightarrow{\mathcal{I}_Z} & M^*(\Sigma), \end{array}$$

where $\overline{Z}$ and $\overline{Z}'$ are the closures of Z in X_Δ and X_Σ, respectively, $i : \overline{Z}' \to \overline{Z}$ is the morphism induced by the identity on N, and the vertical arrow on the right is the refinement of Minkowski weights, is commutative. This boils down to showing that whenever $\sigma \in \Sigma$

and $\tau \in \Delta$ are two cones of the same codimension such that $\sigma \subseteq \tau$, then $i_*([\overline{Z}' \cap V(\sigma)]) = [\overline{Z} \cap V(\tau)]$. In this situation, we have $N_\sigma = N_\tau$ and i is induced by the toric morphism $V(\sigma) \to V(\tau)$ coming from the identity on $N(\sigma) = N(\tau)$. As $\overline{Z} \cap V(\sigma)$ is the closure of $\overline{Z} \cap O(\sigma)$ by Lemma 3.3, it is sufficient to show that the push-forward of $[\overline{Z}' \cap O(\sigma)]$ under the induced map $j : O(\sigma) \to O(\tau)$ is equal to $[\overline{Z} \cap O(\tau)]$. But this is clear because j is an isomorphism, $j(\overline{Z}' \cap O(\sigma)) \subseteq \overline{Z} \cap O(\tau)$, and $\overline{Z} \cap O(\sigma)$ and $\overline{Z}' \cap O(\tau)$ are both irreducible of the same dimension. □

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